AN ASYMPTOTIC HOMOGENIZATION FORMULA FOR COMPLEX PERMITTIVITY AND ITS APPLICATION

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Abstract. The $\mathbb{R}$-linear boundary value problem in a multiply connected domain on a flat torus is considered. This problem is closely related to the Riemann-Hilbert problem on analytic functions. The considered problem arises in the homogenization procedure of random media with complex constants which express the permittivity of components. A new asymptotic formula for the effective permittivity tensor is derived. The formula contains location of inclusions in symbolic form. The application of the derived formula to investigation of the morphology of the tumor cells in disordered biological media is discussed.

1. Introduction

Boundary value problems for differential operators with periodic fast oscillatory coefficients have applications in mechanics of composites and porous media [16, 17]. According to the homogenization procedure [3] an averaged equation with constant coefficients has to be contracted. These constants form the effective tensor used by scientists and engineers to estimate the macroscopic properties of regular and random dispersed media. Such an estimation creates a large knowledgebase showing the importance of metamaterial formalism to study different biological problems, in particular, to recognize glioma areas in brain tissue biopsies [7, 15, 18].

Plane double periodic problems can be considered in the equivalent statement on a flat torus represented by a parallelogram with glued opposite sides. Therefore, the plane homogenization problem can be considered as a problem for a multiply connected domain D on torus for differential operators having different form in different components, D and D_k, on torus. Here, D_k $(k = 1, 2, \dots, N)$ denote smooth non-overlapping domains, closures of which complement D to the whole torus. The fundamental parallelogram is called in applications by Representative Volume Element (RVE) with inclusions D_k $(k = 1, 2, \dots, N)$. The theory of analytical RVE (aRVE) was summarized in [6] for Laplace's equation on a flat torus. Asymptotic formulas were derived for random 2D two-phase composites with circular inclusions D_k when permittivity (conductivity) of components is real. The paper [12] extends these asymptotic formulas to inclusions D_k of other piecewise smooth shapes.

Metamaterial formalism developed in [7, 15] requires formulas similar to [6, 12] but for complex components. The present paper fills this gap of the

aRVE theory and extends the previously derived asymptotic formulas to the complex component media. The generalized alternating method of Schwarz first proposed by S.G. Mikhlin [9] and developed in [6, 12] is applied to solve the corresponding boundary value problem. The results can be applied to highly disordered biological medium in order to describe the macroscopic features for various concentrations of the healthy and tumor cells per RVE.

2. Multi-phase composites

Introduce the complex variable $z = x_1 + ix_2$ on the plane. Consider the unit square periodicity cell Q with N inclusions D_k bounded by the smooth curve $L_k = \partial D_k$ as shown in Figure 1. Let a_k denote the complex

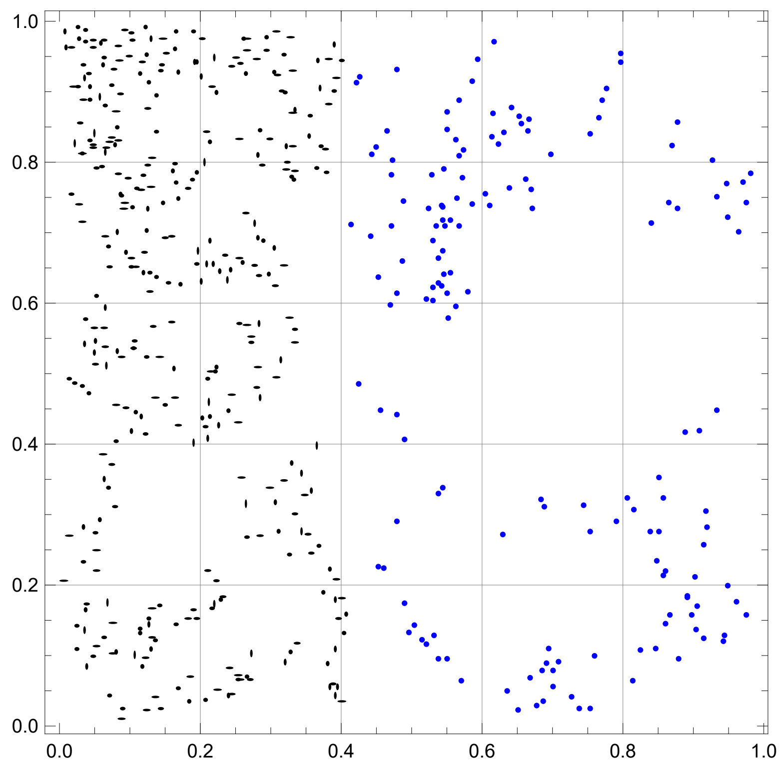

Figure 1. The centers of inclusions a_k are taken from the real observations [7] and the following simulations of their shapes. We take $N_1 = 356$ unidirectional elliptic inclusions of the same area $\pi A_k B_k = \pi 10^{-5}$ (black) and $N_2 = 159$ disks (blue) of radius 0.004. The semi-axes of ellipses A_k and B_k are randomly chosen in such a way that the values A_k satisfy the uniform distribution on the segments $(0.0016, 0.0064)$ and $B_k = 10^{-5} A_k^{-1}$.

coordinate of the gravitational center of D_k, and $r_k = \max_{D_k} |z - a_k|$ the generalized radius of D_k. We consider dispersed composites [12], when the closed domains $(D_k \cup L_k)$ for $k = 1, 2, \ldots, N$ are mutually disjoint and

$$r_k + r_m < |a_k - a_m|, \quad \text{for } k \neq m \quad (k, m = 1, 2, \ldots, N). \tag{1}$$

The concentration of inclusions has the form $f = \sum_{k=1}^{N} |D_k|$. The host domain is denoted by D. The polygon curve ∂Q and the curves L_k are oriented in the counterclockwise direction, hence, $\partial D = \partial Q - \sum_{k=1}^{N} L_k$. Let the permittivity of host is normalized to unity and the permittivity of kth inclusion be a complex number $\varepsilon_k = \varepsilon'_k + i\varepsilon''_k$, where $\varepsilon'_k = \text{Re } \varepsilon_k$ and $\varepsilon''_k = \text{Im } \varepsilon_k$. One can consider ε_k as the ratio of the permittivity of the kth inclusion to the permittivity of matrix, where the dimension permittivities can be complex. From mathematical point of view we assign complex numbers to the considered domains. Thus, we get the pairs (D_k, ε_k) for $k = 1, 2, \ldots, N$ and the pair $(D, 1)$. One can assume that the constants ε_k take the values from a set J which contains less than N elements. Let $\#J = M$, i.e., the composite is $(M+1)-$phases and $\varepsilon_k = \varepsilon^{(j)}$, if $j = 1, 2, \ldots, M$. An example of two-phases medium is displayed in Figure 1 where $M = 2$ and $N = N_1 + N_2$.

Let $u = u' + iu''$ and $u_k = u'_k + iu''_k$ denote the complex potentials in D and D_k, respectively, where for instance $u' = \text{Re } u$ and $u'' = \text{Im } u$ in D. The complex functions u and u_k satisfy Laplace's equation in the corresponding domains and continuously differentiable in their closures.

The perfect contact between the components is expressed by equations [6]

$$u(t) = u_k(t), \quad \frac{\partial u}{\partial \mathbf{n}}(t) = \varepsilon_k \frac{\partial u_k}{\partial \mathbf{n}}(t), \quad t \in \partial D_k \quad (k = 1, 2, \ldots, N), \tag{2}$$

where the normal derivative $\frac{\partial}{\partial \mathbf{n}}$ to L_k is used.

Following the homogenization theory [3] we have to consider a composite in the plane torus topology. The function $u(t)$ satisfies the normalized jump conditions per unit periodicity cell Q

$$u(z+1) - u(z) = 1, \quad u(z+i) - u(z) = 0. \tag{3}$$

The conditions (3) mean that the external complex flux $\mathbf{q}_0 = (-1, 0)$ is applied. Here, the components of $\mathbf{q}_0$ are complex numbers. More precisely, for instance the first relation (3) can be written in the real form as follows

$$\text{Re } u(z+1) - \text{Re } u(z) = 1, \quad \text{Im } u(z+1) - \text{Im } u(z) = 0. \tag{4}$$

Two complex relations (2) can be written in the extended real form

$$\begin{aligned} u'(t) = u'_k(t), &\quad \frac{\partial u'}{\partial \mathbf{n}}(t) = \varepsilon'_k \frac{\partial u'_k}{\partial \mathbf{n}}(t) - \varepsilon''_k \frac{\partial u''_k}{\partial \mathbf{n}}(t), \\ u''(t) = u''_k(t), &\quad \frac{\partial u''}{\partial \mathbf{n}}(t) = \varepsilon''_k \frac{\partial u'_k}{\partial \mathbf{n}}(t) + \varepsilon'_k \frac{\partial u''_k}{\partial \mathbf{n}}(t), \\ &\quad t \in L_k \quad (k = 1, 2, \ldots, N). \end{aligned} \tag{5}$$

The problem (5) can be reduced to an $\mathbb{R}$-linear problem [6, Chapter 1]. This problem refers to boundary value problem of complex analysis [5, 14, 4].

Introduce the non-degenerate real matrix

$$\boldsymbol{\alpha}_k = \begin{pmatrix} \varepsilon_k' & -\varepsilon_k'' \\ \varepsilon_k'' & \varepsilon_k' \end{pmatrix} \tag{6}$$

and the vector-functions $\varphi(z)$ and $\varphi_k(z)$ analytic in D and D_k, respectively. The considered harmonic and analytic functions are related by equations

$$\varphi(z) = \begin{pmatrix} u'(z) + iv'(z) \\ u''(z) + iv''(z) \end{pmatrix}, \quad z \in D \tag{7}$$

and

$$\varphi_k(z) = \frac{1}{2}(I + \boldsymbol{\alpha}_k) \begin{pmatrix} u_k'(z) + iv_k'(z) \\ u_k''(z) + iv_k''(z) \end{pmatrix}, \quad z \in D_k, \tag{8}$$

where $v'(z)$, $v''(z)$ and $v_k'(z)$, $v_k''(z)$ denote the imaginary parts of the vector-functions $\varphi(z)$ and $\varphi_k(z)$, respectively. The complex flux is defined by means of the derivatives $\psi(z) = \varphi'(z)$ and $\psi_k(z) = \varphi_k'(z)$, hence,

$$\psi(z) = \begin{pmatrix} \frac{\partial u'}{\partial x_1} - i\frac{\partial u'}{\partial x_2} \\ \\ \frac{\partial u''}{\partial x_1} - i\frac{\partial u''}{\partial x_2} \end{pmatrix}, \quad z \in D, \tag{9}$$

and

$$\psi_k(z) \equiv \begin{pmatrix} \psi_{1k}(z) \\ \\ \psi_{2k}(z) \end{pmatrix} = \frac{1}{2}(I + \boldsymbol{\alpha}_k) \begin{pmatrix} \frac{\partial u_k'}{\partial x_1} - i\frac{\partial u_k'}{\partial x_2} \\ \\ \frac{\partial u_k''}{\partial x_1} - i\frac{\partial u_k''}{\partial x_2} \end{pmatrix}, \quad z \in D_k. \tag{10}$$

The boundary behavior of analytic vector-functions is conditioned by their integral Cauchy representations. Following [5, 14] we consider the classic space of the Hölder continuous functions. The vector-function $\psi(z)$, $\psi_k(z)$ are analytic in D, D_k and Hölder continuous in the closures of the considered domains. The vector-function $\varphi(z)$, $\varphi_k(z)$ are analytic in D, D_k and their derivatives in the closures of the considered domains satisfy the Hölder condition. Introduce the space $\mathcal{H}_k^+$ of vector-functions analytic in D_k whose derivatives satisfy the Hölder condition in $\overline{D_k} := D_k \cup L_k$. In the next section, we use the space $\mathcal{H}^+ = \cup_{k=1}^N \mathcal{H}_k^+$ of vector-functions determined in the closure of D^+, where $D^+ = \cup_{k=1}^N D_k$ is the union of non-overlapping domains. The norm of $h = (h_1, h_2) \in \mathcal{H}^+$ can be introduced through the norm of the corresponding scalar functions $\|h\| = (\|h_1\|^2 + \|h_2\|^2)^{\frac{1}{2}}$, where

$$\|h_j\| = (\|h_j\|_{C^1} + \|h_j\|_{H^\gamma})^{\frac{1}{2}} \quad (j = 1, 2). \tag{11}$$

The norms in the space of continuously differentiable functions C^1 and of Hölder continuous functions H^γ $(0 < \gamma \leq 1)$ are introduced in the following standard way

$$\|h_j\|_{C^1} = \max_{k=1,2,\ldots,N} \left(\sup_{z \in \overline{D_k}} |h_j(z)| + \sup_{z \in \overline{D_k}} |h_j'(z)| \right) \tag{12}$$

and

$$\|h_j\|_{H^\gamma} = \max_{k=1,2,\dots,N} \sup_{\substack{z_1,z_2\in\overline{D_k}\\ z_1\neq z_2}} \frac{|h_j(z_1)-h_j(z_2)|}{|z_1-z_2|^\gamma}. \tag{13}$$

The space $\mathcal{H}^+$ is a Banach space [10].

We will use the contrast matrices

$$\boldsymbol{\varrho}_k = -(I-\boldsymbol{\alpha}_k)(I+\boldsymbol{\alpha}_k)^{-1} = \frac{1}{|1+\varepsilon_k|^2}\begin{pmatrix} |\varepsilon_k|^2-1 & 2\varepsilon_k'' \\ -2\varepsilon_k'' & |\boldsymbol{\varepsilon}_k|^2-1 \end{pmatrix}. \tag{14}$$

One can check that the eigenvalues of the matrix $\boldsymbol{\varrho}_k$ are conjugated numbers

$$\varrho_k = \frac{\varepsilon_k-1}{\varepsilon_k+1}, \quad \overline{\varrho_k} = \frac{\overline{\varepsilon_k}-1}{\overline{\varepsilon_k}+1}. \tag{15}$$

The conditions (5) can be written in the form of vector-matrix $\mathbb{R}$-linear problem

$$\varphi(t) = \varphi_k(t) - \boldsymbol{\varrho}_k\overline{\varphi_k(t)}, \quad t\in L_k \quad (k=1,2,\dots,N). \tag{16}$$

We arrive at the following boundary value problem. It is required to find the vector-functions $\varphi(z)$ and $\varphi_k(z)$ analytic in D and D_k, respectively, with the boundary behavior described above. The vector-functions $\varphi(z)$ and $\varphi_k(z)$ satisfy the $\mathbb{R}$-linear condition (16) and the jump conditions

$$\varphi(z+1)-\varphi(z) = \xi_1 + id_1, \quad \varphi(z+i)-\varphi(z) = \xi_2 + id_2, \tag{17}$$

where ξ_1 and ξ_2 are given constant vectors which model the external flux. For instance, the vectors

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{18}$$

determine the external flux parallel to the real axis. The undetermined real constant vectors d_1 and d_2 can be found during solution to the problem (16)-(17).

Let $\mathbf{n}(t)$ denote the outward unit normal vector to D_k expressed in terms of the complex function defined on L_k. The problem (16)-(17) can be written in terms of the vector-functions (9)-(10) [6]

$$\psi(t) = \psi_k(t) - \overline{\mathbf{n}^2(t)}\boldsymbol{\varrho}_k\overline{\psi_k(t)}, \quad t\in L_k \quad (k=1,2,\dots,N), \tag{19}$$

$$\psi(z+1) = \psi(z), \quad \psi(z+i) = \psi(z). \tag{20}$$

The general solution of the problem (19)-(20) is a linear combination $\psi(z) = \xi_1\psi^{(1)}(z) + \xi_2\psi^{(2)}(z)$ with arbitrary real vectors ξ_1 and ξ_2 [11].

Remark. The $\mathbb{R}$-linear problem is stated in a classic space [5, 14]. Following [4] it can be stated in a Sobolev-type space and for other types of differential equations [8].

3. Schwarz's method

The generalized alternating method of Schwarz [9] was developed and applied in [6, 12] to solve the scalar $\mathbb{R}$-linear problem. It is obtained from (16)-(17) by the assumption that the harmonic functions $u(z)$ and $u_k(z)$ are real. The vector-matrix $\mathbb{R}$-linear problem (16)-(17) can be studied by the same method by use of the matrix formalism.

Following [12] we use the Eisenstein functions $E_1(z)$ and $E_2(z) = -E_1'(z)$ related to the Weierstrass elliptic functions $E_1(z) = \zeta(z) - \pi z$ and $E_2(z) = \wp(z) + \pi$. Consider the matrix norm $\|\boldsymbol{\varrho}_k\| = \max_{l,m=1,2} |\varrho_{k,lm}|$ where $\varrho_{k,lm}$ denote the (l,m)th element of the matrix $\boldsymbol{\varrho}_k$. We consider the lower order approximation in $\varrho = \max_{k=1,2,\ldots,N} \|\boldsymbol{\varrho}_k\|$ assuming that ϱ is a sufficiently small number. The scalar $\mathbb{R}$-linear problem was reduced to a system of integral equations [12]. The same arguments can be applied to the vector-matrix $\mathbb{R}$-linear problem (16)-(17). The system of integral equation up to an additive constant vector becomes

$$\varphi_k(z) = \sum_{m=1}^{N} \frac{\boldsymbol{\varrho}_m}{2\pi i} \int_{L_m} \overline{\varphi_m(t)} E_1(t-z)\, \mathrm{d}t + \begin{pmatrix} z \\ 0 \end{pmatrix}, \ z \in D_k \ (k = 1, 2, \ldots, N). \tag{21}$$

The integral equations are considered in the Banach space $\mathcal{H}^+$ introduced in the previous section. The integral operator from the right part of (21) is bounded in $\mathcal{H}^+$ [10]. When the vector functions $\varphi_k(z)$ are found the complex potential in D is calculated by the integral

$$\varphi(z) = \sum_{m=1}^{N} \frac{\boldsymbol{\varrho}_m}{2\pi i} \int_{L_m} \overline{\varphi_m(t)} E_1(t-z)\, \mathrm{d}t + \begin{pmatrix} z \\ 0 \end{pmatrix}, \ z \in D. \tag{22}$$

Following the generalize alternating method of Schwarz [9, 12] we find the first order iteration for the system (21)

$$\varphi_k^{(1)}(z) = \left(\sum_{m=1}^{N} \frac{1}{2\pi i} \int_{L_m} \bar{t}\, E_1(t-z)\, \mathrm{d}t\, \boldsymbol{\varrho}_m \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \end{pmatrix}, \ z \in D_k. \tag{23}$$

The corresponding derivative $\psi_k^{(1)}(z) = \frac{d}{dz}\varphi_k^{(1)}(z)$ becomes

$$\psi_k^{(1)}(z) = \left(\sum_{m=1}^{N} \boldsymbol{\varrho}_m F_{mk}(z) + I \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ z \in D_k, \tag{24}$$

where

$$F_{mk}(z) = \frac{dI_{mk}}{dz}(z) = \frac{1}{2\pi i} \int_{L_m} \bar{t}\, E_2(t-z)\, \mathrm{d}t = \frac{1}{\pi} \int_{D_m} E_2(t-z)\, \mathrm{d}x_1 \mathrm{d}x_2. \tag{25}$$

Here, the complex Green formula $\int_G \frac{\partial w}{\partial \bar{z}}\, \mathrm{d}x_1 \mathrm{d}x_2 = \frac{1}{2i} \int_{\partial G} w\, \mathrm{d}t$ for a function $w(z, \bar{z})$ continuously differentiable in a smooth closed domain G is used. Estimate the integrals (25) in two different cases.

i) Let $m \neq k$. The Taylor approximate formula $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$ with $x_0 = a_m - a_k$ and $\Delta x = (t - a_m) - (z - a_k)$ yields

$$E_2(t-z) \approx E_2(a_m - a_k) - 2E_3(a_m - a_k)[(t-a_m) - (z-a_k)]. \tag{26}$$

The approximation (26) is used due to the inequalities $|t - a_m| \leq r_m$ $(t \in L_m)$, $|z - a_k| \leq r_k$ $(z \in D_k)$ and (1). Substitute (26) into (25)

$$F_{mk}(z) \approx \frac{1}{\pi}[E_2(a_m - a_k) + 2E_3(a_m - a_k)(z - a_k)]|D_m| - \frac{2}{\pi}E_3(a_m - a_k)s_{1m}, \tag{27}$$

where s_{qm} denotes the complex static moment of order q of the domain D_m

$$s_{qm} = \int_{D_m} (t - a_m)^q \, \mathrm{d}x_1 \mathrm{d}x_2 = \frac{1}{2i}\int_{L_m} \overline{(t - a_m)}(t - a_m)^q \mathrm{d}t, \; q = 0, 1, \ldots. \tag{28}$$

ii) Let $m = k$. Estimate the integrals (25) using the approximate formula [12]

$$E_2(t - z) \approx \frac{1}{(t - z)^2} + \pi, \tag{29}$$

Substitute (29) into (25)

$$F_{kk}(z) \approx J_k'(z) + |D_k|, \tag{30}$$

where

$$J_k(z) = \frac{1}{2\pi i}\int_{L_k} \frac{\bar{t}}{t - z}\mathrm{d}t, \quad z \in D_k. \tag{31}$$

3.1. **Effective permittivity tensor.** The main value used in the macroscopic behavior of media is the transverse effective permittivity tensor [7, 13]

$$\boldsymbol{\varepsilon}_\perp = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}. \tag{32}$$

The permittivity tensor $\boldsymbol{\varepsilon}_\perp$ is symmetric. It can be expressed through the complex gradient ∇u_k, hence, through the vector-functions (10). We have

$$\nabla u_k \equiv \begin{pmatrix} \frac{\partial u_k'}{\partial x_1} + i\frac{\partial u_k''}{\partial x_1} \\ \\ \frac{\partial u_k'}{\partial x_2} + i\frac{\partial u_k''}{\partial x_2} \end{pmatrix} = \frac{2}{\varepsilon_k + 1}\begin{pmatrix} \mathrm{Re}\;\psi_{1k} + i\mathrm{Re}\;\psi_{2k} \\ \\ -\mathrm{Im}\;\psi_{1k} - i\mathrm{Im}\;\psi_{2k}, \end{pmatrix}. \tag{33}$$

The following formula was derived in [13, 12]

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\sum_{k=1}^{N} \varrho_k \int_{D_k} \begin{pmatrix} \mathrm{Re}\;\psi_{1k} + i\mathrm{Re}\;\psi_{2k} \\ -\mathrm{Im}\;\psi_{1k} - i\mathrm{Im}\;\psi_{2k} \end{pmatrix} \mathrm{d}x_1 \mathrm{d}x_2, \tag{34}$$

where ϱ_k has the form (15). Analogous formula takes place for the component $\varepsilon_{22} + i\varepsilon_{12}$, where $\varepsilon_{12} = \varepsilon_{21}$.

The first order approximation $\psi_k^{(1)}(z) = \frac{d}{dz}\varphi_k^{(1)}(z)$ can be found from (24)

$$\psi_k^{(1)}(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{m=1}^{N} \frac{1}{|1 + \varepsilon_m|^2} \begin{pmatrix} |\varepsilon_m|^2 - 1 \\ 2\varepsilon_m'' \end{pmatrix} F_{mk}(z), \; z \in D_k. \tag{35}$$

Find the vector

$$\left(\begin{array}{c} \text{Re } \psi_{1k} + i\text{Re } \psi_{2k} \\ -\text{Im } \psi_{1k} - i\text{Im } \psi_{2k} \end{array} \right) = \left(\begin{array}{c} 1 \\ 0 \end{array} \right) + \sum_{m=1}^{N} \varrho_m \left(\begin{array}{c} \text{Re } F_{mk} \\ -\text{Im } F_{mk} \end{array} \right) \tag{36}$$

and substitute the result into (34)

$$\begin{aligned} \left(\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{21} \end{array} \right) &= (1 + 2\textstyle\sum_{k=1}^{N} \varrho_k |D_k|) \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \\ &+2\textstyle\sum_{k=1}^{N} \varrho_k \sum_{m=1}^{N} \varrho_m \int_{D_k} \left(\begin{array}{c} \text{Re } F_{mk} \\ -\text{Im } F_{mk} \end{array} \right) \mathrm{d}x_1 \mathrm{d}x_2. \end{aligned} \tag{37}$$

The integrals from (37) can be estimated by application of the approximations (27) and (30). Consider again two different cases.

i) Let $m \neq k$. Then, (27) yields

$$\begin{aligned} &\textstyle\int_{D_k} F_{mk}(z)\, \mathrm{d}x_1 \mathrm{d}x_2 \approx \frac{1}{\pi} E_2(a_m - a_k) |D_k||D_m| \\ &-\tfrac{2}{\pi} E_3(a_m - a_k)[s_{1m}|D_k| - s_{1k}|D_m|] := q_{mk}. \end{aligned} \tag{38}$$

This formula (38) can be written in terms of the Weierstrass functions

$$\int_{D_k} F_{mk}(z)\, \mathrm{d}x_1 \mathrm{d}x_2 \approx |D_k||D_m| + \tag{39}$$

$$\frac{1}{\pi} \wp(a_m - a_k)|D_k||D_m| + \frac{1}{\pi} \wp'(a_m - a_k)(s_{1m}|D_k| - s_{1k}|D_m|).$$

ii) Let $m = k$. Then, equation (30) has to be used in the estimation

$$\int_{D_k} F_{kk}(z)\, \mathrm{d}x_1 \mathrm{d}x_2 \approx \mathcal{J}_k + |D_k|^2 \tag{40}$$

where

$$\mathcal{J}_k = \int_{D_k} J_k'(z)\, \mathrm{d}x_1 \mathrm{d}x_2. \tag{41}$$

Substituting the approximation (38) and (40) into (37) we obtain

$$\begin{aligned} \left(\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{21} \end{array} \right) &= (1 + 2\textstyle\sum_{k=1}^{N} \varrho_k |D_k| + 2\sum_{k=1}^{N} \sum_{m=1}^{N} \varrho_k \varrho_m |D_k||D_m|) \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \\ &+2\textstyle\sum_{k=1}^{N} \varrho_k^2 \left(\begin{array}{c} \text{Re } \mathcal{J}_k \\ \\ -\text{Im } \mathcal{J}_k \end{array} \right) + 2\sum_{k=1}^{N} \sum_{m=1}^{N} \varrho_k \varrho_m \left(\begin{array}{c} \text{Re } q_{mk} \\ \\ -\text{Im } q_{mk} \end{array} \right). \end{aligned} \tag{42}$$

Here, it is assumed that $q_{kk} = 0$ in accordance with the case ii).

Introduce the weighted mean $\langle \varrho \rangle = \sum_{k=1}^{N} \varrho_k |D_k|$. Then, (42) can be written in the form

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{21} \end{pmatrix} = (1 + 2\langle \varrho \rangle + 2\langle \varrho \rangle^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$+2\sum_{k=1}^{N} \varrho_k^2 \begin{pmatrix} \text{Re } \mathcal{J}_k \\ \\ -\text{Im } \mathcal{J}_k \end{pmatrix} + 2\sum_{k,m=1}^{N} \varrho_k \varrho_m \begin{pmatrix} \text{Re } q_{mk} \\ \\ -\text{Im } q_{mk} \end{pmatrix}. \tag{43}$$

Similar to (43) we calculate the vector $\begin{pmatrix} \varepsilon_{12} \\ -\varepsilon_{22} \end{pmatrix}$. Its value can be obtained from (43) by rotation of the points a_k about 90^0. It is equivalent to replacement of a_k by ia_k. The Weierstrass functions for the square unit periodicity cell satisfy the relations

$$\wp(iz) = -\wp(z), \quad \wp'(iz) = i\wp'(z). \tag{44}$$

The integral $\mathcal{J}_k$ after the rotation becomes $\overline{\mathcal{J}_k}$ [12]. Then, (43) yields

$$\begin{pmatrix} \varepsilon_{22} \\ -\varepsilon_{12} \end{pmatrix} = (1 + 2\langle \varrho \rangle + 2\langle \varrho \rangle^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$+2\sum_{k=1}^{N} \varrho_k^2 \begin{pmatrix} \text{Re } \mathcal{J}_k \\ \\ \text{Im } \mathcal{J}_k \end{pmatrix} + 2\sum_{k,m=1}^{N} \varrho_k \varrho_m \begin{pmatrix} \text{Re } q_{mk}^* \\ \\ -\text{Im } q_{mk}^* \end{pmatrix}, \tag{45}$$

where

$$q_{mk}^* = -\frac{1}{\pi}\wp(a_m - a_k)|D_k||D_m| + \frac{i}{\pi}\wp'(a_m - a_k)[s_{1m}|D_k| - s_{1k}|D_m|]. \tag{46}$$

The vector equations (43) and (45) can be ultimately written in the extended form

$$\boldsymbol{\varepsilon}_\perp = (1 + 2\langle \varrho \rangle + 2\langle \varrho \rangle^2)I + 2\sum_{k=1}^{N} \varrho_k^2 \begin{pmatrix} \text{Re } \mathcal{J}_k & -\text{ Im } \mathcal{J}_k \\ -\text{Im } \mathcal{J}_k & \text{Re } \mathcal{J}_k \end{pmatrix}$$
$$+\frac{2}{\pi}\sum_{k,m=1}^{N} \varrho_k \varrho_m |D_k||D_m| \begin{pmatrix} \text{Re } \wp(a_m - a_k) & -\text{ Im } \wp(a_m - a_k) \\ -\text{Im } \wp(a_m - a_k) & -\text{ Re } \wp(a_m - a_k) \end{pmatrix}$$
$$+\frac{2}{\pi}\sum_{k,m=1}^{N} \varrho_k \varrho_m (s_{1m}|D_k| - s_{1k}|D_m|)\times$$
$$\begin{pmatrix} \text{Re } \wp'(a_m - a_k) & -\text{ Im } \wp'(a_m - a_k) \\ -\text{Im } \wp'(a_m - a_k) & -\text{ Re } \wp'(a_m - a_k) \end{pmatrix} + O(f^3\varrho^3). \tag{47}$$

The dimension analysis of the above formula in concentration $f = \sum_{k=1}^{N} |D_k|$ is performed following [12].

4. Conclusion and discussion

The main result of the present work consists in the derivation of the analytical asymptotic formula (47) for the effective permittivity tensor. Many authors declare that such a formula is virtually impossible and perform numerical computations for some special geometries. One can meet other types of declaration in literature when a pure numerical procedure is called by "exact solution". A discussion concerning "exact" and exact solution can be found in [1, 2]. The dependence of $\boldsymbol{\varepsilon}_\perp$ on the frequency ω is the main suggested criterion to detect the tumor cells and their fraction [7]. It can be obtained after substitution of the dependencies $\varepsilon_k = \varepsilon_k(\omega)$ into the formula (15) for $\varrho_k = \varrho_k(\omega)$, next substituted into (47). Cancerous (glioma) cells are modeled by elliptic inclusion and noncancerous (neuron) cells by disks as displayed in Figure 1. In the considered two-phase medium, the dependencies of permittivity ε_1 of glioma and ε_2 of neuron on ω [7] and their different shapes can allow to investigate the impact of the tumor cells morphology on the tensor $\boldsymbol{\varepsilon}_\perp$.

5. Acknowledgement

This research, V. Mityushev, Zh. Zhunussova and K. Dosmagulova, is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08856381). Tatjana Gric was supported by the European Unions Horizon 2020 research and innovation programme under the Marie Sklodowska Curie grant agreement No 713694.

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