QUANTIZATION OF FIELDS OF ARBITRARY SPIN

by

W. COX

University of Aston in Birmingham

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SUMMARY.

The search for relativistic field equations for elementary particles of arbitrary spin is a long standing problem of Quantum Field Theory. Although much work has been done on relativistic field theories describing fields with various mass-spin spectra, little has been done on picking out those theories which are quantizable, that is, those for which a particle interpretation exists which is consistent with the basic postulates of quantum theory.

In this thesis we examine theories based on the relativistic field equation

$$(L_\mu \partial^\mu + \chi)\psi = 0.$$

The condition for quantization is expressed in terms of certain positive definiteness requirements on the eigenvectors corresponding to the non-zero eigenvalues of L_0. By restricting the discussion to a specific type of theory, in which spin states are not repeated, these positive definiteness requirements are expressed in terms of certain simple trace conditions. A systematic procedure for finding representations of L_0 for quantizable theories is given, and illustrated in the case of spin 0,1,2 fields. In this procedure use is made of graphs depicting the representations of the Lorentz Group according to which ψ transforms in a relativistic field theory. Some simple results of graph theory are applied to develop the theory and also to discuss the properties of the L_0 matrix. Further possibilities of this graphical approach are briefly discussed.

NOTATION AND CONVENTIONS.

We work throughout in natural units, such that $\hbar = c = 1$.

x^1, x^2, x^3, x^0 are the four coordinates of a point in space time, and we take the metric tensor

$$\begin{aligned} g_{\mu\nu} &= 0 && \mu \neq \nu \\ &= +1 && \mu = \nu = i = 1,2,3 \\ &= -1 && \mu = \nu = 0 \end{aligned}$$

∂_μ is used to denote $\partial/\partial x^\mu$.

Greek subscripts and superscripts range over 0,1,2,3 while Latin ones range over 1,2,3. The summation convention is used where appropriate.

The word representation is used in two ways. It is used to refer to representations of the Lorentz group in some vector space, and also to refer to representations of certain matrices with respect to particular bases. The meaning will be clear from the context, so no confusion should arise.

TABLE OF CONTENTS

TABLE OF CONTENTS (contd.)

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CHAPTER 1

HISTORICAL SURVEY OF HIGHER SPIN FIELD THEORIES.

1) Historical Survey of Higher Spin Field Theories.

1.1) The Dirac-Fierz-Pauli Theory

The first theory of higher spin particles to receive attention was proposed by Dirac[1] in 1936. This is a single particle theory, and is basically a generalization of the spinor form of the relativistic electron equation . The wave function is a symmetric spinor of appropriate rank, each component of which satisfies a Klein-Gordon equation ensuring unique mass. Also the spinor satisfies a set of subsidiary equations which reduces the number of independent components of the wave function and ensures unique spin.

In 1939 Fierz[2] developed the field theory for Dirac's equations, constructing an energy-momentum tensor and charge-current vector. The total energy (total charge) was shown to be positive definite for integral (half odd integral) spin fields, so the field theory could be quantized in the usual way, with the usual spin-statistics connection. Fierz also showed that if interaction with the electromagnetic field was introduced by the gauge invariant minimal replacement

$$p_\mu \rightarrow p_\mu - i e A_\mu$$

then the subsidiary conditions of Dirac's theory became inconsistent. This problem was soon overcome by Fierz and Pauli[3], who put the theory into a Lagrangian form and introduced "auxiliary field variables" in addition to the basic fields. As a result of the Euler-Lagrange equation these auxiliary fields vanished in the free field case, but became non-zero when interactions were introduced. The auxiliary fields were chosen so that the field equations were

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consistent in the presence of interactions, and so that the number of degrees of freedom of the free field system is the same with and without the interactions. This last condition was necessary to avoid singularities occurring in the free field solutions when the electromagnetic field was turned on. Unfortunately, there is a great deal of arbitrariness in the Fierz-Pauli procedure, since, for higher spins the auxiliary fields could be introduced many ways, and although these all lead to the same free field theory, they in general lead to inequivalent interaction theories. Some progress has been made towards systematizing the Fierz-Pauli procedure, but still a large amount of arbitrariness remains. Despite the arbitrariness, the Dirac-Fierz-Pauli theory (DFP theory) could be used consistently with interactions, and is still very popular - especially as applied to the spin $\frac{3}{2}$ and spin 2 fields, originally studied by Fierz and Pauli.

In 1941 Rarita and Schwinger[4] put forward an alternative formulation of the DFP theory for half odd integer spin fields, in which the field variables are spin-tensors rather than spinors. Such a formulation has recently been given by Nack[5] for integer spin fields. The advantage of the RS formalism is that when the Lagrangian is constructed from the spin tensors, the auxiliary fields of the Fierz and Pauli type are already present and arise naturally when interactions are introduced. The RS formalism has therefore become very popular and received great usage. The difference between the DFP and RS formulations is little more than notation and both will be included under the heading "DFP theory". The arbitrariness of the DFP theory referred to above is reflected

1) contd.

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in the arbitrariness of the Lagrangian's from which the RS equations can be obtained. It has been shown by many authors [6],[7] that the most general Lagrangian leading to the RS equations depend on a certain number of arbitrary parameters. To what extent such parameters arise in physical results of the theory does not seem clear at the moment.

An effort has been made to organize the DFP theory by Fronsdal[8], who introduces a spin projection operator to do the work of the DFP subsidiary conditions - that is, to project out of an arbitrary field variable satisfying the Klein Gordon equation (Dirac first order equation) for integer spin (half odd integer spin) fields, that part corresponding to the spin required. The equations obtained using the spin-projection operator turn out to be non-local, involving inverse powers of momentum. Chang[9] has developed a systematic procedure for eliminating the non-localities by introduction of DFP type auxiliary fields. Quantization of the Fronsdal theory is however very difficult, especially in the interaction case.

The first quantization of a DFP theory was performed by Gupta[10] in 1954, who quantized the spin $\frac{3}{2}$ theory in the free field case. A consistent interaction quantization of the spin $\frac{3}{2}$ theory was not proposed until 1969, by Gupta and Repko[11]. The mathematics involved in this last work is so complicated that it is unlikely to survive as a higher spin theory. No one has yet put forward a consistent quantum field theory for an interacting system involving a field of spin greater than one. Doubts as to whether such theories exist have been expressed by a number of authors, notably Johnson

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and Sudarshan[12]. It was shown by Johnson and Sudarshan that the usual local quantum field theory of higher half odd integer spin fields suffers from an inconsistency when interactions are introduced. The problem arises out of the subsidiary conditions - or "constraints" - to which the theory is subject to give unique mass and spin. First it is shown that a consistent interaction quantization is possible only if such subsidiary conditions are present in the theory. Then it is shown for particular spin $\frac{3}{2}$ theories that the assumption of the usual Fermi-Dirac statistics leads to physical results which conflict with relativity when interaction with the electromagnetic field is considered. At present it is not clear how to get round the difficulties found by Johnson and Sudarshan for interacting fields.

The DFP spin-two theory has received a great deal of attention and again arbitrary parameters occur in the Lagrangian.[13],[14]. The free field quantization has been discussed, but work on the interaction case is lacking.

1.2) Theories Based on a First Order Field Equation.

Many authors considered the original DFP formalism as unsatisfactory because of the troublesome subsidiary conditions In 1945 Bhabha[15] therefore proposed an alternative approach to higher spin theories, which took as the basic wave equation one of the form

$$(L_\mu \partial^\mu + i\chi)\psi = 0 \qquad (1.2.1)$$

No other subsidiary conditions were imposed. Such an equation had been used to great effect by Kemmer[16] to describe scalar and vector mesons, and of course the Dirac electron equation

1) contd.

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has the matrix form (1.2.1). In his original formulation Bhabha only allowed a restricted possiblity for the L_μ, namely such that

$$I_{\mu\nu} \propto [L_\mu, L_\nu] \qquad (1.2.2)$$

where $I_{\mu\nu}$ are the infinitesimal generators of the representation of the Lorentz group according to which the wave function ψ transforms. The choice (1.2.2) was made simply because it was found to apply in the above theories of Dirac and Kemmer. As a consequence of (1.2.2) Bhabha found that (1.2.1) described multi-mass particles, and also that the DFP higher spin theories could not be put in the form (1.2.1).

Rejecting the idea of multi-mass fields Harish-Chandra[17] abandoned the assumption (1.2.2) and looked for wave equations of the type (1.2.1) describing particles with unique mass. Harish-Chandra gave conditions on the L_μ for unique mass and quantized his free field theory, giving the conditions on L_o for definiteness of charge and energy. The work of Bhabha and Harish-Chandra has been developed over the years, but mainly with a view to obtaining equations with a given mass, spin spectrum.

Bhabha obtained general representations for the L_μ (independent of (1.2.2)) such that (1.2.1) is relativistically covariant, and proposed many particular theories, including one which suggested a high energy spin $\frac{3}{2}$ state for the proton. This last theory was quantized by LeCouteur[18] using an indefinite metric, in an unsuccessful attempt to compare the theory with experiment.

It was early realized that L_o was the important

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matrix for discussing mass-spin spectra, charge and energy, definiteness and quantization. Also, if L_o is non-diagonalizable then, as shown by Wild[19], (1.2.1) does contain implicit subsidiary conditions. If L_o is diagonalizable, then free field quantization, with the usual spin-statistics connection, is only possible for fields with spin less than $\frac{3}{2}$. By regarding (1.2.1) as a general relativistic field equation, interpreting ψ as a field variable, it is possible to get good quantizable free field theories for higher spin than one, provided L_o is taken to be non-diagonalizable. An example is the DFP spin $\frac{3}{2}$ theory, which can be put in the form (1.2.1) with a non-diagonalizable L_o; also (1.2.2) is not valid[20].

The general quantization of a theory based on (1.2.1) was published in 1952 by Udgaonkar[21]. This was only a "quantization in principle" because the usual spin-statistics connection depends on certain positive definite properties of the field energy and charge (before quantization), and these properties were assumed to hold (see Section 2.5). One must first find quantizable theories based on (1.2.1) which satisfy the required definite properties.

A complete analysis of field theories described by (1.2.1) was given in 1948 by Gelfand and Yaglom[22]. This thesis is based on their work and, with some modifications, the notation and terminology used are theirs. Gelfand and Yaglom found the most general form for the L_μ, in a particularly useful representation, such that (1.2.1) is invariant under proper Lorentz transformations and space reflections, and is derivable from a real invariant Lagrangian. They found the mass and spin spectra for such a theory and studied particular theories up to

1) contd.

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the spin $\frac{3}{2}$ DFP theory. Much of the work is detailed in [24] and will be reviewed in Chapter 2 of this thesis. As Gelfand and Yaglom point out, any field equation can be reduced to a system of first order equations and so (1.2.1) in its most general context includes all relativistic field theories, provided enough freedom is allowed for the L_μ. Thus in general L_o may be singular to allow for "redundant components" of ψ which are introduced simply to reduce the order of certain field equations to first order. These redundant components do not correspond to physically independent states of the field and must be eliminated in the quantization process, so that only the independent field states go over into the particle picture. It was stated by Bhabha[23] that (1.2.1) was the most general wave equation possible on the quantum mechanical grounds that ψ, being a wave function, must satisfy a first order differential equation. If ψ is regarded as a field variable then this does not apply. A field variable can satisfy an equation of any order, and (1.2.1) is merely a convenient presentation of a set of field equations which may be equivalent to a system of higher order equations.

Gelfands work has been continued by Shelepin in a series of papers[25],[26],[27],[28] which study the algebra of the L_μ matrices. Shelepin has considered the irreducibility of (1.2.1), the algebraic relations between the L_μ for various physical requirements, the relation between $I_{\mu\nu}$ and L_μ and other aspects of the relativistic covariance of (1.2.1). The question of quantization and definteness properties of energy and charge were not considered. Recently Shelepin and Lizin[29] have extended Shelepin's work to calculating cross-sections for

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particles with arbitrary spin. Use is made of work by Fedorov[30] which is relevant to the work of this thesis. Fedorov constructs projection operators onto mass-spin states by use of the minimal polynomials of the operator $p_\mu L^\mu$ and the helicity operator for a given theory. These projection operators can be used to obtain expressions for transition probabilities between states in terms of the trace of certain matrices. Also, compact trace conditions can be obtained for the definiteness of charge and energy in the theory. In Section 3.4 of this thesis we obtain similar conditions to Fedorov's, but which are stronger and more useful for practical work.

The analysis of theories based on (1.2.1) is very difficult in practice, even for modest spins. As we will see, the conditions for Lorentz covariance, quantization, and the determination of mass-spin spectra, are all expressible in terms of mathematical conditions on the L_o matrix. These conditions are generally difficult to satisfy, and the main problem tackled in this thesis is that of the practical determination of matrix representations of the L_o matrix corresponding to good quantizable theories. The major difficulty lies in ensuring the necessary positive definite requirements for the charge and energy in the theory. Previously these field quantities have been explicitly calculated for a particular theory and their definitness exhibited directly. In Chapter 3 a fairly systematic procedure is discussed for finding theories of a certain type (each spin state occurs at most once) which satisfy the required definite properties. Examples of this procedure are given in Chapter 4. In this procedure, use is

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made of certain graphical diagrams depicting the representation of the Lorentz Group according to which ψ in (1.2.1) transforms under Lorentz transformations. Similar diagrams were used in a limited way by LeCouteur[25],[36] in work on theories for which L_o is diagonalizable. No such restriction is placed on L_o in the theories considered in this thesis.

1.3) More Recent Theories.

In 1956 Foldy[37] initiated a new approach to the study of higher spin wave equations. Foldy's object was to construct wave equations without troublesome subsidiary conditions. For this he abandoned manifest covariance and used the relativistic Schrodinger equation to provide a representation space for irreducible representations of the Inhomogeneous Lorentz Group corresponding to particles of mass m and spin s. The wave function used by Foldy has complicated (momentum dependent) transformation properties however, and this makes the covariant introduction of interactions difficult. To avoid this difficulty, Weaver, Hammer and Good[38] constructed a relativistic Schrodinger equation whose solutions transformed according to the representation $\mathcal{A} \sim \mathcal{D}(s,0) \oplus \mathcal{D}(0,s)$ of the Homogeneous Lorentz Group. Such wave functions describe particles of unique spin s, and have simple transformation properties, which suggest ways in which interactions may be covariantly introduced. The basic idea was to postulate the required Hamiltonian in the rest frame and use a generalized Foldy-Wouthuysen transformation to derive the Hamiltonian in an arbitrary coordinate system.

The Weaver, Hammer, Good theory was studied in detail

1) contd.

1.3) contd.

by Mathews[39],[40], who analysed the general relativistic equation of the form

$$i \frac{\partial \psi}{\partial t} = H \psi$$

which is suitable for describing particles of unique spin, without further subsidiary conditions, and such that ψ transforms according to $\mathcal{D}(s,0) \oplus \mathcal{D}(0,s)$. Mathews found that the Hamiltonian H was not uniquely defined by the requirements of covariance, but he was able to ensure uniqueness by imposing the extra condition that the Hamiltonian should have a unique finite limit on the rest frame. Mathews was able to quantize the theory in a consistent manner for the case of half odd integral spin fields, but he found that in the integral spin case the wave function was not causal. This was remedied by Nelson and Good[43] in 1968.

Weinberg[41] has also studied theories based on the representation $\mathcal{D}(s,0) \oplus \mathcal{D}(0,s)$, from a somewhat different viewpoint, not making explicit use of field equations. He starts from the single particle states, defined in terms of creation and annihilation operators. Working back from the known transformation properties of these states one obtains relativistic fields representing the corresponding particles and transforming in a simple way. When the $\mathcal{D}(s,0) \oplus \mathcal{D}(0,s)$ representation is used the fields obey the Klein Gordon equation, but no other equation - since this representation yields a unique spin, no subsidiary equations are necessary to eliminate unwanted spin states. Whilst the Lagrangian

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aspect of field theory is not explicit in Weinberg's work, it is not completely avoided, as he claims. For, as shown by Shay[42], the Weinberg equations can be incorporated in a Lagrangian approach. The equivalence of the Weinberg and Weaver, Hammer, Good quantized theories has been shown by Nelson and Good[43].

The technique of Weinberg and others, of starting with the single particle states and working back to obtain the covariant fields describing these states, poses the question of the number of such covariant formulations, for a given mass and spin state. There are of course an infinite number of such formulations, corresponding to the infinite number of representations of the Lorentz group yielding a given spin state. Pursey[44] has given a complete account of this approach to covariant wave equations, showing clearly the rôle played by subsidiary conditions in eliminating superfluous spin states. Tung[45] has considered those formulations which do not contain subsidiary conditions, and he finds it unlikely that these can be used to describe particles of spin greater than one. This is consistent with the result of Wild[19] that good quantizable field theories based as a first order equation for spin greater than one must contain implicit subsidiary conditions.

The above work of Foldy, Weaver, Hammer, Good and Weinberg may yet lead to quantizable higher spin field theories satisfying the required definiteness properties for charge and energy, and we suspect that these will be equivalent to theories obtained by the approach of Section 1.2. In fact a number of authors have shown the equivalence of the various

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1.3) contd.

covariant formulations of the spin $\frac{3}{2}$ theory originally proposed by Fierz and Pauli[3], and it is likely that this carries over to arbitrary spins. In any case, it is thought that the class of field theories based on a first order differential equation derivable from a Lagrangian is sufficiently broad to justify a detailed study of such theories and their quantization. In Chapter 2 we briefly review the theory of such a field equation.

C H A P T E R 2.

RELATIVISTIC QUANTUM FIELD THEORIES DERIVABLE FROM A LAGRANGIAN LINEAR IN THE FIELD DERIVATIVES.

2) Relativistic Quantum Field Theories Derivable from a Lagrangian Linear in the Field Derivatives.

2.1) Introduction.

Let $\psi(x_1,x_2,x_3,t)$ be the field variable describing the state of the field at the space-time point (x_1,x_2,x_3,t). To allow for an internal degree of freedom such as spin, ψ will in general be a vector in some finite dimensional complex vector space. We do not consider the various possible theories involving infinite dimensional representations of the Homogeneous Lorentz group.

The space-time evolution of the field is described by a field equation, which in accordance with the principle of local action is taken to be a linear partial differential equation in ψ. Since any equation containing higher order derivatives can be reduced to one with only first order derivatives by introducing new field variables and modifying the definition of ψ, we need only consider a system of first order equations. However, if we do reduce a higher order equation to a first order equation, the new field variable ψ will contain extra components which do not represent independent degrees of freedom of the field - such components may be called "redundant". In obtaining physical quantities or in quantization the redundant components have to be eliminated so that only field components representing truly independent degrees of freedom of the field are used.

For example consider the field equation for the scalar field:-

$$\frac{\partial^2\psi}{\partial x_0^2} - \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_3^2} + \chi^2\psi = 0 \qquad (2.1.1)$$

In this case ψ has only one independent component. By introducing the new variable:-

2) contd.

2.1) contd.

$$\phi_\mu = \frac{1}{\chi}\,\partial_\mu\psi \tag{2.1.2}$$

the above equation can be written as the first order system

$$\left.\begin{aligned}\partial^\mu\phi_\mu &= \chi\psi\\ \partial_\mu\psi &= \chi\phi_\mu\end{aligned}\right\} \tag{2.1.3}$$

The new field variable is a five component object

$$\Psi = \begin{pmatrix}\psi\\ \phi_\mu\end{pmatrix} \tag{2.1.4}$$

The ϕ_μ are the redundant components. Using a first order theory we would obtain the field theory for the scalar meson in the form of the equations (2.1.3). Only ψ must play an essential rôle in the construction of physical quantities.

So we choose as our field equation:-

$$L_0\frac{\partial\psi}{\partial x_0} + L_1\frac{\partial\psi}{\partial x_1} + L_2\frac{\partial\psi}{\partial x_2} + L_3\frac{\partial\psi}{\partial x_3} + i\chi\psi = 0$$

or

$$(L_\mu\partial^\mu + i\chi)\psi = 0 \qquad (\mu = 0,1,2,3) \tag{2.1.5}$$

L_μ are constant matrices and χ is a real constant.

We require a relativistic Lagrangian field theory which can be quantized and so we have basically three aspects to consider:-

1) <u>Relativistic Covariance of the Theory</u>.

We demand manifest covariance of the theory under the orthochronous Lorentz group, $\mathcal{L}$. ψ is therefore taken as a vector in some representation space of a reducible or irreducible representation $\mathcal{R}$ of $\mathcal{L}_p$, the proper Lorentz group

2) contd.

2.1) contd.

1) contd.

$$\mathcal{R} = \sum_i \oplus \mathcal{D}(k_i, \ell_i)$$

Thus ψ transforms according to

$$\psi'(x') = T\psi(x)$$

where T is a transformation matrix independent of x.

2) Lagrangian Origin of the Theory.

The Lagrangian formalism is convenient because it provides a systematic procedure for constructing conserved physical quantities of the field such as charge, energy, etc. There are two stages in establishing a Lagrangian formalism for a relativistic field theory:-

i) A non-degenerate invariant bilinear form:-

$$(\psi_1, \psi_2) = \psi_2^\dagger \Lambda \psi_1$$

must exist. That is:-

$(\psi, \psi) = 0$ if and only if $\psi = 0$

and

$$(\psi_1', \psi_2') = (\psi_1, \psi_2)$$

where

$$\psi_i' = T\ \psi_i$$

ii) Using the bilinear form of (i) it must be possible to construct a real, invariant Lagrangian form which the field equation (2.1.5) can b e derived.

3) Quantization.

The usual quantization procedure is used, in which the relativistic fields are expanded in eigenfunctions of linear operators representing certain observables, and the coefficients of the expansion are expressed in terms

2) contd.

2.1) contd.

3) contd.

of creation and annihilation operators of the appropriate particle states for these observables. These annihilation and creation operators satisfy the usual commutation or anti-commutation relations. As is well known, if we do not allow such possibilities as indefinite metrics or para-statistics our theory must have the following spin-statistics connection:-

Integral spin: The relativistic fields must have positive definite total energy so that commutation relations (Bose-statistics) for the particle operators can be used to quantize the field and give a positive definite charge (As Pauli[45] has shown, an integral spin field cannot be consistently quantized by anti-commutation rules).

Half-odd integral spin: The relativistic fields must have definite charge, so that anti-commutation relations (Fermi-statistics) can be used to quantize the field and make the energy positive definite (Pauli showed that an half odd integral spin field must have indefinite energy).

Much work has been done on first order field theories. Gelfand and Yaglom[22] have given a complete treatment of the relativistic Lagrangian field theory, which also summarizes most of the earlier work of Bhabha, Harish Chandra, etc.

An explicit matrix representation is chosen for the Lorentz transformation of ψ and the condition for covariance of (2.1.5) gives matrix representations for the L_μ in terms of certain arbitrary complex numbers. The most general non-degenerate invariant hermitian form, in the representation

2) contd.

2.1) contd.

space $\mathcal{R}$ of the Lorentz group, is obtained. There are several, and each one leads to a different Lagrangian field theory in $\mathcal{R}$. The condition for Lagrangian origin of the equation (2.1.5) imposes further conditions on the elements of the L_μ depending on the bilinear form used. The field mass and spin states described by (2.1.5) are found by studying the matrix L_o. L_o can be partitioned into block diagonal form corresponding to the invariant subspaces of $\mathcal{R}$ under the subgroup of space rotations. To each representation of weight s of the rotation subgroup contained in $\mathcal{R}$, there corresponds one of the blocks of L_o, called the s-block. The non-zero eigenvalues of the s-block give the masses of the spin s states.

Gelfand and Yaglom considered a number of simple field theories with the correct definitness properties for quantization, but the only higher spin theory they considered was the DFP spin $\frac{3}{2}$ theory. The general problem of higher spin quantization and definitness properties was not studied by these authors.

The "quantization in principle" of the first order field theory has been carried out by Udgaonkar[21]. However, this quantization was performed for quite general fields, under the assumption that the appropriate definiteness properties hold. So far, no one seems to have analysed the general first order theory with a view to finding which representations lead to good quantizable field theories, and studying the general properties of such theories. Thus we might ask for the permissible bilinear forms and the mass and spin spectra for a particular representation $\mathcal{R}$ such that it yields a quantizable theory.

2) contd.

2.1) contd.

Further, there seems to be a large amount of arbitrariness in manifestly covariant theories with a given mass-spin spectra and we would like to know the significance of this, and the possibilities of eliminating it - if necessary. So far these aspects of relativistic field theories have been approached by largely ad hoc methods. Some particular representation space $\mathscr{R}$ is tried and found to yield a good theory, usually by trial and error. No general methods are available for finding "good" theories (i.e. quantizable field theories) or eliminating bad ones. The purpose of the work of this thesis is to examine in detail the general first order theory of Gelfand and Yaglom; to try to organize and extend it and develop general criteria for good theories. In Sections 2.2, 2.3, 2.4 we review the basic results of the first order relativistic field theory and its "quantization in principle".

2.2) The Relativistic Covariance of the First Order Field Equation.

We first consider proper transformations of the Lorentz group:-

$$x_\mu' = a_\mu{}^\nu x_\nu. \qquad (2.2.1)$$

where

$$|a_\mu{}^\nu| = 1$$

and

$$a_o{}^o > 0.$$

We therefore need only consider infinitesimal transformations:-

$$x_\mu' = (\delta_\mu{}^\nu + \epsilon_\mu{}^\nu)x_\nu \qquad (2.2.2)$$

where the $\epsilon_{\mu\nu}$ are real infinitesimal parameters and

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$$

This transformation of space-time coordinates induces the

2) contd.

2.2) contd.

following transformation of ψ in the representation space of $\mathcal{R}$:-

$$\psi'(x') = (I + \tfrac{1}{2}\epsilon^{\mu\nu} I_{\mu\nu})\psi(x) \qquad (2.2.3)$$

where the $I_{\mu\nu}$ are infinitesimal generators of the representation $\mathcal{R}$, and

$$I_{\mu\nu} = - I_{\nu\mu} \qquad (2.2.4)$$

The necessary and sufficient condition that the field equation (2.1.5) is form invariant under the transformation:-

$$\psi'(x') = T_a \psi(x) \qquad x' = ax \qquad (2.2.5)$$

is

$$\sum_{\mu} T_a^{-1} L_\mu T_a a^\mu{}_\nu = L_\nu \qquad \mu,\nu = 0,1,2,3 \qquad (2.2.6)$$

In the case of infinitesimal proper Lorentz transformations (2.2.2) and (2.2.3); substitution in (2.2.6) gives:-

$$[L_\mu, I_{\nu\rho}] = g_{\mu\nu} L_\rho - g_{\mu\rho} L_\nu \qquad (2.2.7)$$

where
$$g_{\mu\nu} = 0 \qquad \mu \neq \nu$$
$$= +1 \qquad \mu = \nu = 1,2,3$$
$$= -1 \qquad \mu = \nu = 0$$

and $\mu,\nu,\rho = 0,1,2,3$.

The $I_{\nu\rho}$ satisfy the usual Lie algebra of generators of $\mathcal{L}_p$

$$[I_{\mu\nu}, I_{\rho\nu}] = -g_{\mu\rho} I_{\nu\sigma} + g_{\mu\sigma} I_{\nu\rho} + g_{\nu\rho} I_{\mu\sigma} - g_{\nu\sigma} I_{\mu\rho} . \qquad (2.2.8)$$

To find all the L_μ satisfying (2.2.7) we choose a particular matrix representation of the $I_{\nu\rho}$ and substitute

2) contd.

2.2) contd.

in (2.2.7), solving the resulting equations in the matrix elements of L_μ. This procedure can be simplified by noting that (2.2.7) can be reduced to a smaller set of relations. For any proper Lorentz transformation can be factored into space rotations and a "boost", or pure Lorentz transformation along some space axis, say I_{o3}. Splitting the relations (2.2.7) corresponding to space and time indices we get

$$\begin{aligned} L_i &= [L_o, I_{io}] \\ [L_o, I_{ij}] &= 0 \\ [L_k, I_{io}] &= g_{ki}L_o = \delta_{ki}L_o \\ [L_k, I_{ij}] &= g_{ki}L_j - g_{kj}L_i = \delta_{ki}L_j - \delta_{kj}L_i \end{aligned} \tag{2.2.9}$$

Using the Lie algebra relations (2.2.8) and the Jacobi identity for the generators $I_{\mu\nu}$, it is found that all of the relations (2.2.9) are contained in the equations:-

$$L_i = [L_o, I_{io}] \tag{2.2.10}$$

$$[L_o, I_{12}] = [L_o, I_{13}] = [L_o, I_{23}] = 0 \tag{2.2.11}$$

$$[I_{3o}, [I_{3o}, L_o]] = L_o \tag{2.2.12}$$

The last two equations completely determine L_o for an equation (2.1.5) covariant under a representation of $\mathcal{L}_p$, and then the L_i can be found from (2.2.10). In fact, so far as all physical quantities of interest are concerned, and for the testing for quantizability of the theory, it turns out that only L_o is important.

To obtain a matrix representation for L_o we need to use some representation of the $I_{\mu\nu}$. Most convenient from our point of view is the so called "canonical representation" The irreducible representation $\mathcal{D}(k,\ell)$ of $\mathcal{L}_p$ generates representation $\mathcal{D}(s)$ of the rotation group, where $s = k+\ell,\ k+\ell-1, \ldots |k-\ell|$

2) contd.

2.2) contd.

are the weights of the representations. Let $R^{(k,\ell)}$ denote the representation space of $\mathcal{D}(k,\ell)$. $R^{(k,\ell)}$ can be completely reduced to a direct sum of subspaces $R_s^{(k,\ell)}$, invariant under space rotations, forming representation spaces for $\mathcal{D}(s)$. As is well known, the eigenvectors of the operator

$$H_3 = iI_{12} \tag{2.2.13}$$

supply an orthonormal basis

$$\left\{ \xi_{sm}^{(k,\ell)} \right\} \qquad m = s, s-1, \ldots (s-1), -s. \tag{2.2.14}$$

for the representation space $R_s^{(k,\ell)}$, which is (2s+1)-dimensional. $\xi_{sm}^{(k,\ell)}$ is an eigenvector of H_3 corresponding to eigenvalue m. The basis (2.2.14) is called the canonical basis of $R_s^{(k,\ell)}$. We select a canonical basis in each subspace $R_s^{(k,\ell)}$, then the vectors:-

$$\left\{ \xi_{sm}^{(k,\ell)} \right\} \qquad \begin{matrix} s = k+\ell,\ k+\ell-1,\ \ldots, |k-\ell| \\ m = s, s-1,\ \ldots -(s-1), -s \end{matrix} \tag{2.2.15}$$

form a basis for the irreducible representation $\mathcal{D}(k,\ell)$. In this basis the matrices representing space rotations are in completely reduced (block-diagonal) form.

We will follow Gelfand and Yaglom's notation and specify the canonical representation by the pair of numbers (ℓ_0, ℓ_1), both integers or both half odd integers for finite representations of $\mathcal{L}_p$, where

$$\ell_0 = k-\ell, \qquad \ell_1 = k+\ell+1 \tag{2.2.16}$$

In terms of these numbers ℓ_0, ℓ_1, the index s specifying the spin representations contained in the representation $\mathcal{D}(k,\ell) \equiv (\ell_0, \ell_1)$ takes the values

2) contd.

2.2) contd.

$$s = |\ell_0|,\ |\ell_0| + 1,\ \ldots,\ell_1 - 1 \qquad (2.2.17)$$

A general irreducible representation of $\mathcal{L}_p$ will be denoted by τ or τ_i.

In general we consider reducible representations of $\mathcal{L}_p$ and in this case we take $\mathcal{R}$ to be a direct sum of representations of $\mathcal{L}_p$. A canonical basis $\{\xi^{\tau}_{sm}\}$ $s = |\ell_0|,\ldots,\ell_1-1$ $m = s,s-1,\ldots -(s-1),-s$ is chosen in each irreducible subspace corresponding to the representation $\tau_i \sim (\ell_0^{(i)},\ell_1^{(i)})$, and then $\{\xi^{\tau_i}_{sm}\}$, over all s, m, τ_i, provides a basis for the reducible representation $\mathcal{R}$.

Gelfand and Yaglom have given explicit representations of the $I_{\mu\nu}$ in the canonical basis. For convenience, we introduce new operators:-

$$\left.\begin{aligned} H_+ &= iI_{23}-I_{13} \quad ; \quad H_- = iI_{23}+I_{13} \quad , \quad H_3 = iI_{12} \\ F_+ &= iI_{10}-I_{20} \quad ; \quad F_- = iI_{10}+I_{20} \quad , \quad F_3 = iI_{30} \end{aligned}\right\} \qquad (2.2.18)$$

Inspection of (2.2.11) and (2.2.12) shows that we only need the matrices for $H_{\pm}$, H_3 and F_3; in fact these relations become

$$[L_0,\ H_+] = [L_0,H_-] = [L_0,H_3] = 0 \qquad (2.2.19)$$

$$[[F_3,L_0],\ F_3] = L_0 \qquad (2.2.20)$$

In the canonical representation the operators $H_{\pm}$, H_3, F_3 are given by:-

$$\left.\begin{aligned} H_3\ \xi^{\tau}_{sm} &= m\ \xi^{\tau}_{sm} \\ H_-\ \xi^{\tau}_{sm} &= \sqrt{(s+m)(s-m+1)}\ \xi^{\tau}_{sm-1} \\ H_+\ \xi^{\tau}_{sm} &= \sqrt{(s+m+1)(s-m)}\ \xi^{\tau}_{sm+1} \end{aligned}\right\} \qquad (2.2.21)$$

$$F_3\xi^{\tau}_{sm} = C_s\sqrt{s^2-m^2}\ \xi^{\tau}_{s-1,m} - A_{sm}\xi^{\tau}_{sm} - C_{s+1}\sqrt{(s+1)^2-m^2}\ \xi^{\tau}_{s+1,m} \qquad (2.2.22)$$

2) contd.

2.2) contd.

where $$C_s = \frac{i}{s}\sqrt{\frac{(s^2-\ell_0^2)(s^2-\ell_1^2)}{4s^2-1}} \qquad (2.2.23)$$

$$A_s = \frac{i\ell_0\ell_1}{s(s+1)} \qquad (2.2.24)$$

and $\tau \sim (\ell_0,\ell_1)$; $s = |\ell_0|, |\ell_0|+1,\ldots,\ell_1-1,$

$m = s,s-1,\ldots,-(s-1),-s$

Now equations (2.2.19) tell us that L_0 commutes with all space rotations. In the canonical representations the I_{ij} are in completely reduced form and each sub-representation $\mathcal{D}^{\tau}_{(s)}$ of the rotation group is irreducible. By a natural generalization of Schur's Lemma it follows that in the canonical representation L_0 must have the form

$$L_0 = [C^{\tau\tau'}_{sm\ s'm'}] \qquad (2.2.25)$$

where

$$C^{\tau\tau'}_{sm\ s'm'} = C^{\tau\tau'}_{s}\ \delta_{ss'}\ \delta_{mm'} \qquad (2.2.26)$$

and the $C^{\tau\tau'}_{s}$ are arbitrary complex numbers depending only on the representations τ and τ' and the spin index s. This uses up all of the information conveyed by (2.2.19).

The equation (2.2.20) further restricts the $C^{\tau\tau'}_{s}$. Substituting (2.2.22) and

$$L_0\xi^{\tau}_{sm} = \sum_{\tau's'm'} C^{\tau\tau'}_{sm\ s'm'}\ \xi^{\tau'}_{s'm'} = \sum_{\tau'} C^{\tau\tau'}_{s}\ \xi^{\tau'}_{sm} \qquad (2.2.27)$$

into the following form of (2.2.0):-

$$[[F_3,L_0],\ F_3]\ \xi^{\tau}_{sm} = L_0\ \xi^{\tau}_{sm} \qquad (2.2.2)$$

gives six complicated equations, on equating coefficients of ξ^{τ}_{sm} to zero. These equations have been studied by Gelfand and Yaglom and it is found that

$$C^{\tau\tau'}_{s} = 0 \qquad (2.2.29)$$

unless either

2) contd.

2.2) contd.

i) $(\ell_0', \ell_1') = (\ell_0+1, \ell_1)$

in which case

$$\left.\begin{aligned} C_s^{\tau\tau'} &= C^{\tau\tau'}\sqrt{(s+\ell_0+1)(s-\ell_0)} \\ C_s^{\tau'\tau} &= C^{\tau'\tau}\sqrt{(s+\ell_0+1)(s-\ell_0)} \end{aligned}\right\} \qquad (2.2.30)$$

or

ii) $(\ell_0', \ell_1') = (\ell_0, \ell_1+1)$

in which case

$$\left.\begin{aligned} C_s^{\tau\tau'} &= C^{\tau\tau'}\sqrt{(s+\ell_1+1)(s-\ell_1)} \\ C_s^{\tau'\tau} &= C^{\tau'\tau}\sqrt{(s+\ell_1+1)(s-\ell_1)} \end{aligned}\right\} \qquad (2.2.31)$$

where $C^{\tau\tau'}$, $C^{\tau'\tau}$ are arbitrary complex numbers. When two representations $\tau \sim (\ell_0, \ell_1)$ and $\tau' \sim (\ell_0', \ell_1')$ are related by (i) and (ii) they are said to be linked, or to interlock, by linkages of type (i) or type (ii) respectively. The above results show that only matrix elements of L_0 which connect linked representations in R are non-zero. This result was first obtained by Bhabha[15], and of course it is independent of the matrix representation of $I_{\mu\nu}$ used - this just determines the form of the non-zero matrix elements. In fact, from (2.1.5), since $\partial_\mu \sim \mathcal{D}(\frac{1}{2},\frac{1}{2})$ and $\psi \sim \sum_i \oplus \mathcal{D}(k_i, \ell_i)$ it follows that

$$\begin{aligned} \partial_\mu \psi &\sim \sum_i \oplus \mathcal{D}(\tfrac{1}{2},\tfrac{1}{2}) \otimes \mathcal{D}(k_i, \ell_i) \\ &\sim \sum_i \oplus \Big[\mathcal{D}(k_i-\tfrac{1}{2}, \ell_i-\tfrac{1}{2}) \oplus \mathcal{D}(k_i-\tfrac{1}{2}, \ell_i+\tfrac{1}{2}) \oplus \mathcal{D}(k_i+\tfrac{1}{2}, \ell_i-\tfrac{1}{2}) \oplus \\ &\qquad \oplus \mathcal{D}(k_i+\tfrac{1}{2}, \ell_i+\tfrac{1}{2})\Big] . \end{aligned}$$

L_μ projects this space onto the space $\sum_i \oplus \mathcal{D}(k_i, \ell_i)$ of ψ and so L_μ can only connect representations $\mathcal{D}(k_i, \ell_i)$ and $\mathcal{D}(k_j, \ell_j)$ such that

$$k_i = k_j \pm \tfrac{1}{2}, \qquad \ell_i = \ell_j \pm \tfrac{1}{2}$$

i.e. the representations are linked.

2) contd.

2.2) contd.

(2.2.30) and (2.2.31) give the most general matrix L_o, such that the equation (2.1.5) is covariant under $\mathcal{L}_p$ in the canonical representation. Although other representations of the L_μ have been obtained[15], the canonical representation seems best for study of the physical properties of the theory. Of course, as much as possible should be done in a representation independent way. Much work has been done on this by Harish-Chandra and Shelepin, the latter in seeking to extend Gelfand and Yaglom's work. This involves a study of the L_μ algebra, but so far this work has been confined to the relativistic covariance of the theory. The quantization and definiteness properties required have not been studied in this algebraic manner. The complexity of the L_μ algebra for higher spin theories seems to discourage this.

Our field equation (2.1.5) is now completely specified by the set of arbitrary complex numbers $C^{\tau\tau'}$ for all pairs of linked irreducible representations τ,τ' of $\mathcal{L}_p$ in R. If we now further require covariance of the theory under space reflections, i.e. under the orthochronous Lorentz group $\mathcal{L}$, then the $C^{\tau\tau'}$ become subject to further conditions.

Any transformation of $\mathcal{L}$ can be made up from transformations of $\mathcal{L}_p$ and the reflection

$$x_o' = x_o, \quad x_i' = -x_i \qquad i = 1,2,3 \qquad (2.2.32)$$

represented in $\mathcal{R}$ relative to the canonical basis by the matrix S say. We recall that any irreducible representation of $\mathcal{L}$, on restriction to proper Lorentz transformations completely reduces into either a single irreducible self conjugate representation of $\mathcal{L}_p$, $\tau \sim (0,\ell_1)$ or two mutually conjugate irreducible representations $\tau \sim (\ell_o,\ell_1)$ and $\tau \sim (-\ell_o,\ell_1)$ of $\mathcal{L}_p$.

2) contd.

2.2) contd.

Note that we differ slightly from Gelfand and Yaglom in putting the dot denoting space conjugation beside the representation τ and not above it. This removes the ambiguity in such symbols as $\dot{\tau}'$, which could mean the conjugate of the representation linked to τ, or the representation linked to the conjugate of τ - as is easily seen these are not in general the same, or in our notation $\tau^{\bullet\prime} \neq \tau^{\prime\bullet}$.

We have two cases to consider:-

1) Self conjugate representations of $\mathcal{L}_p$.

Let τ be a self conjugate representation of $\mathcal{L}_p$, i.e.

$$\tau \sim (0, \ell_1).$$

Let the canonical basis for τ be $\{\xi^{\tau}_{sm}\}$. Then we get two possibilities for S, differing in sign, which lead to two inequivalent representations of $\mathcal{L}$ for a given self conjugate representation τ of $\mathcal{L}_p$:-

$$\left.\begin{aligned} S\,\xi^{\tau}_{sm} &= (-1)^{[s]}\,\xi^{\tau}_{sm} \\ S\,\xi^{\tau}_{sm} &= (-1)^{[s]+1}\,\xi^{\tau}_{sm} \end{aligned}\right\} \qquad (2.2.33)$$

where [s] denotes the integral part of s.

2) Mutually Conjugate Pairs of Representations of $\mathcal{L}_p$.

τ and $\tau^{\bullet}$ are conjugate. If $\{\xi^{\tau}_{sm}\}$ and $\{\xi^{\tau^{\bullet}}_{sm}\}$ are canonical bases in each representation space then we take $\{\xi^{\tau}_{sm}\}$, $\{\xi^{\tau^{\bullet}}_{sm}\}$ as a canonical basis for the representations of $\mathcal{L}$, $\tau \oplus \tau^{\bullet}$. In terms of this basis S can always be written in the form

$$\left.\begin{aligned} S\,\xi^{\tau}_{sm} &= (-1)^{[s]}\,\xi^{\tau^{\bullet}}_{sm} \\ S\,\xi^{\tau^{\bullet}}_{sm} &= (-1)^{[s]}\,\xi^{\tau}_{sm} \end{aligned}\right\} \qquad (2.2.34)$$

2) contd.

2.2) contd.

The generaliation of the above results to the case of a reducible representation of $\mathcal{L}$ are obvious.

Now consider the space reflection

$$\psi'(x') = S\,\psi(x) \quad ; \quad x_o' = x_o \quad x_i' = -x_i \tag{2.2.35}$$

Substituting in (2.2.6) we find that

$$[S,\ L_o] = 0 \tag{2.2.36}$$

is the condition for (2.1.5) to be covariant under the orthochronous Lorentz group. This imposes further conditions on the $C^{\tau\tau'}$. For $\mathcal{R}$ to provide a representation of $\mathcal{L}$ it must consist of self-conjugate and/or mutually conjugate pairs of representations of $\mathcal{L}_p$. We only have to consider pairs τ,τ' of linked representations and so we have three possibilities:-

a) $\tau \neq \tau\cdot$ and $\tau' \neq \tau'\cdot$

Substituting (2.2.34) in (2.2.36) gives

$$C^{\tau\tau'} = C^{\tau\cdot\tau'\cdot} \tag{2.2.37}$$

b) $\tau = \tau\cdot$ and $\tau' \neq \tau'\cdot$ (or $\tau \neq \tau\cdot$, $\tau' = \tau'\cdot$)

From case i) above S has two forms, given at (2.2.33), so far as the representation τ (or τ') is concerned, while it only has one form, (2.2.34), so far as τ' (or τ) is concerned. We therefore find

$$\left.\begin{aligned} (1)\quad S\,\xi^{\tau}_{sm} &= (-1)^{[s]}\,\xi^{\tau}_{sm} \\ S\,\xi^{\tau'}_{sm} &= (-1)^{[s]}\,\xi^{\tau'\cdot}_{sm} \end{aligned}\right\} \tag{2.2.3'-}$$

then

$$C^{\tau\tau'} = C^{\tau\cdot\tau'\cdot} \tag{2.2.39}$$

or

$$\left.\begin{aligned} (2)\quad S\ \xi^{\tau}_{sm} &= (-1)^{[s]+1}\ \xi^{\tau}_{sm} \\ S\ \xi^{\tau'}_{sm} &= (-1)^{[s]}\ \ \xi^{\tau'\cdot}_{sm} \end{aligned}\right\} \tag{2.2.40}$$

then

2) contd.

2.2) contd.

b) contd.

$$C^{\tau\tau'} = - C^{\tau^{\bullet}\tau'^{\bullet}} \tag{2.2.41}$$

Analogous results hold if $\tau \neq \tau^{\bullet}$ and $\tau' = \tau'^{\bullet}$.

c) <u>$\tau = \tau^{\bullet}$ and $\tau' = \tau'^{\bullet}$</u>

In this case there are no extra conditions on the $C^{\tau\tau'}$. However, there do exist different ways of representing the operator S without affecting the matrics L_{μ}. This gives rise to different transformation possibilities for ψ under reflections S - leading to tensor and pseudo-tensor theories. Also, in this case S must act in the same way in each subspace R^{τ} and $R^{\tau'}$, otherwise $C^{\tau\tau'}$ will be zero.

(2.2.30) and (2.2.31) and the three sets of conditions above (a) → (c) completely specify the form of L_o in the canonical representation such that the equation (2.1.5) is covariant under all transformations of the orthochronous Lorentz group $\mathcal{L}$.

2.3) <u>Lagrangian Field Theory</u>.

Bhabha[23] has shown that the most general Lagrangian density from which (2.1.5) may be derived, which is linear in the field derivatives can be put in the form

$$L[\psi(x)] = \frac{1}{i} \psi^{\dagger}\Lambda(L_{\mu}\partial^{\mu}+i\chi)\psi \tag{2.3.1}$$

where Λ is a non-singular matrix. The Lagrangian L must be real if it is to be used to construct real field quantities in the usual way. L is real if the bilinear form

$$(\psi_1, \psi_2) = \psi_2{}^{\dagger}\Lambda\psi_1 \tag{2.3.2}$$

is hermitian, i.e. if

$$\Lambda^{\dagger} = \Lambda \tag{2.3.3}$$

and if

2) contd.

2.3) contd.

$$(L_\mu \psi_1, \psi_2) = (\psi_1, L_\mu \psi_2) \tag{2.3.4}$$

i.e.

$$L_\mu^\dagger \Lambda = \Lambda L_\mu \tag{2.3.5}$$

or, the L_μ must be hermitian with respect to the bilinear form (,).

The covariance of the field theory requires the invariance of the Lagrangian L, i.e.

$$L[\psi'(x')] = L[\psi(x)] \tag{2.3.6}$$

Necessary and sufficient conditions for this are:-

1) The bilinear form (ψ_1, ψ_2) is invariant:-

$$(\psi_1'(x'), \psi_2'(x')) = (\psi_1(x), \psi_2(x)) \tag{2.3.7}$$

2) the equation obtained from L, (2.1.5), is covariant.

Since we have already constructed the L_μ to satisfy (2), we have now only to find the most general invariant hermitian bilinear form (,) in the representation space $\mathcal{R}$, and ensure that the L_μ satisfy (2.3.4) for this form. Henceforth we shall consider only Lagrangian field theories which are covariant under $\mathcal{L}$, the orthochronous Lorentz group. In this case a non-degenerate invariant hermitian form always exists in $\mathcal{R}$.

Gelfand and Yaglom have shown that the most general non-degenerate hermitian form, in the finite representation space $\mathcal{R}$, which is invariant under transformations of $\mathcal{L}$ (from now on $\mathcal{R}$ is assumed to be a representation space for $\mathcal{L}$) is given in the canonical basis by:-

$$(\psi_1, \psi_2) = \sum_{\tau s m} a^{\tau\tau'} e_s x^{\tau}_{sm} \overline{y^{\tau'}_{sm}} \tag{2.3.8}$$

where, in the canonical basis $\{\xi^{\tau}_{sm}\}$

2) contd.

2.3) contd.

$$\psi_1 = (x^{\tau}_{sm}) \; ; \; \psi_2 = (y^{\tau}_{sm})$$

and $e_s = (-1)^{[s]}$. The $a^{\tau\tau\cdot}$ satisfy

$$a^{\tau\tau\cdot} = \overline{a^{\tau\cdot\tau}} \tag{2.3.9}$$

and in fact, by a transformation to a new canonical basis it is always possible to ensure that

$$a^{\tau\tau\cdot} = \pm 1 \tag{2.3.10}$$

which will be assumed in the future.

The form given for (,) by Gelfand and Yaglom is more general than that above, since they also allow infinite dimensional representations for $\mathcal{R}$.

We now consider the conditions imposed on the L_μ by (2.3.4). Again we have only to consider L_0 because

$$(L_0\psi_1,\psi_2) = (\psi_1,L_0\psi_2) \tag{2.3.11}$$

implies

$$(L_i\psi_1,\psi_2) = (\psi_1,L_i\psi_2) \quad i = 1,2,3.$$

Further, the condition (2.3.11) is equivalent to

$$(L_0\xi^{\tau}_{sm},\xi^{\tau'}_{s'm'}) = (\xi^{\tau}_{sm},L_0\xi^{\tau'}_{s'm'}) \tag{2.3.12}$$

where the ξ^{τ}_{sm} are the canonical basis vectors and as usual τ', denotes a representation linked to τ. Substituting (2.2.27) in (2.3.12) gives after some manipulation

$$a^{\tau'\tau'\cdot}\, C^{\tau\tau'}_{s} = a^{\tau\tau\cdot}\, \overline{C^{\tau'\cdot\tau\cdot}_{s}} \tag{2.3.13}$$

This condition differs from that given by Gelfand and Yaglom ((24) Part II; Section 8; (27)), which is not quite correct. They give the above conditions directly on the $C^{\tau\tau'}$, but in fact this depends on the linkage of τ with τ'. First, note that since $(-\ell_0,\ell_1)$ is equivalent to $(\ell_0,-\ell_1)$ we can adopt

2) contd.

2.3) contd.

the convention that $\ell_1 > 0$ and only ℓ_0 can take positive and negative values. Then, from (2.2.17)

$$s < \ell_1 \quad , \quad \text{so } (s+\ell_1+1)(s-\ell_1) < 0 \tag{2.3.14}$$

$$s \geqslant \ell_0 \quad , \quad \text{so } (s+\ell_0+1)(s-\ell_0) > 0 \tag{2.3.15}$$

It then follows, on substituting (2.2.30) and (2.2.31) that for a:-

<u>type (i) linkage</u>

$$a^{\tau'\tau'\cdot} C^{\tau\tau'} = a^{\tau\tau\cdot} \overline{C^{\tau'\cdot\tau\cdot}} \tag{2.3.16}$$

<u>type (ii) linkage</u>

$$a^{\tau'\tau'\cdot} C^{\tau\tau'} = -a^{\tau\tau\cdot} \overline{C^{\tau'\cdot\tau\cdot}} \tag{2.3.17}$$

The hermitian form (,) can always be transformed so that $a^{\tau\tau\cdot} = \pm 1$. The different possibilities then allowed for (,) impose different conditions on the $C^{\tau\tau'}$ of L_0 and we get different possible theories in a given representation space $\mathcal{R}$. That is, the hermitian form can be chosen in a number of ways consistent with invariance and hermitivity, each leading to a different field theory.

We now have the most general matrix L_0 such that (2.1.5) is invariant under $\mathcal{L}$ and is derivable from a real invariant Lagrangian density. We summarize the complete set of the conditions on the coupling coefficients $C^{\tau\tau'}$ which determine L_0:-

In the canonical basis L_0 has the form

$$L_0 = [\, C^{\tau\tau'}_{sm\, s'm'} \,]$$

Covariance under $\mathcal{L}_p$ leads to

$$C^{\tau\tau'}_{sms'm'} = C^{\tau\tau'}_{s}\, \delta_{ss'}\delta_{mm'}$$

The $C^{\tau\tau'}_{s}$ are zero except for linking representations:-

2) contd.

2.3) contd.

Type (i) $(\ell_0', \ell_1') = (\ell_0+1, \ell_1)$

$$C_s^{\tau\tau'} = C^{\tau\tau'} \sqrt{(s+\ell_0+1)(s-\ell_0)}$$

$$C_s^{\tau'\tau} = C^{\tau'\tau} \sqrt{(s+\ell_0+1(s-\ell_0)}$$

Type (ii) $(\ell_0', \ell_1') = (\ell_0, \ell_1+1)$

$$C_s^{\tau\tau'} = C^{\tau\tau'} \sqrt{(s+\ell_1+1)(s-\ell_1)}$$

$$C_s^{\tau'\tau} = C^{\tau'\tau} \sqrt{(s+\ell_1+1)(s-\ell_1)}$$

where the $C^{\tau\tau'}$, $C^{\tau'\tau}$ are arbitrary complex numbers. Covariance under $\mathcal{L}$ imposes the further conditions

a) $\tau \neq \tau\cdot$ and $\tau' \neq \tau'\cdot$

$$C^{\tau\tau'} = C^{\tau\cdot\tau'\cdot}$$

b) $\tau = \tau\cdot$ and $\tau' \neq \tau'\cdot$

If the reflexion operator S is represented by:-

(1) $\left.\begin{aligned} S\ \xi_{sm}^{\tau} &= (-1)^{[s]} \xi_{sm}^{\tau} \\ S\ \xi_{sm}^{\tau'} &= (-1)^{[s]} \xi_{sm}^{\tau'\cdot} \end{aligned}\right\}$ then $C^{\tau\tau'} = C^{\tau\cdot\tau'\cdot}$

(2) $\left.\begin{aligned} S\ \xi_{sm}^{\tau} &= (-1)^{[s]+1} \xi_{sm}^{\tau} \\ S\ \xi_{sm}^{\tau'} &= (-1)^{[s]} \xi_{sm}^{\tau'\cdot} \end{aligned}\right\}$ then $C^{\tau\tau'} = - C^{\tau\cdot\tau'\cdot}$

c) $\tau = \tau\cdot$ and $\tau' = \tau'\cdot$

No extra conditions on the C's, but different possibilities for reflection operator, leading to tensor and pseudo-tensor theories.

Finally, if the equation (2.1.5) is obtainable from a real invariant Lagrangian density constructed using the hermitian form (2.3.8), where the $a^{\tau\tau\cdot}$ may be chosen as $\pm$ 1,

2) contd.

2.3) contd.

then for a:-

type (i) linkage

$$a^{\tau'\tau'\cdot}\, C^{\tau\tau'} = a^{\tau\tau\cdot}\, \overline{C^{\tau'\cdot\tau\cdot}}$$

type (ii) linkage

$$a^{\tau'\tau'\cdot}\, C^{\tau\tau'} = -\, a^{\tau\tau\cdot}\, \overline{C^{\tau'\cdot\tau\cdot}}$$

Using the Lagrangian density (2.3.1) we can now construct the densities of the usual field quantities, momentum, charge, energy, angular momentum, etc. Neglecting unimportant conventional constants, the corresponding densities are:-

Momentum $\quad p_k = i(L_o\psi, \partial_k\psi)$

Energy $\quad H = i(L_o\psi, \partial_o\psi)$

Charge $\quad \rho = (L_o\psi, \psi)$

Orbital Angular Momentum $\quad \mathcal{J}_{ij} = (L_o\psi, (x_i\partial_j - x_j\partial_i)\psi)$

Spin Angular Momentum $\quad \mathcal{S}_{ij} = i(L_o\psi, I_{ij}\psi)$

Total Angular Momentum

$$M_{ij} = \mathcal{J}_{ij} + \mathcal{S}_{ij} = i(L_o\psi, (x_i\partial_j - x_j\partial_i + I_{ij})\psi)$$

All these densitites are of the form

$$\sigma = (L_o\psi, \mathcal{O}\psi)$$

where σ is the density of some particular field quantity and $\mathcal{O}$ is the operator representing the corresponding observable in classical particle quantum mechanics. Note that all the densities above are real (Hence the i in $\mathcal{S}_{ij}$, because $\mathcal{L}$ invariance of the form (,) means

2) contd.

2.3) contd.

$(\overline{\psi_1, I_{ij}\psi_2}) = (I_{ij}\psi_2, \psi_1) = -(\psi_2, I_{ij}\psi_1)$. By integrating the densities over all space:-

$$M = \int_V \sigma \, dv$$

we get the totals for all the above field quantities.

2.4) Rest Mass and Spin.

Associated with a quantum mechanical particle we have a rest mass and a spin (we shall not consider other internal symmetries). In the relativistic field theory these arise as characteristic quantities of the field which, on passing to the quantized theory, can be satisfactorily interpreted as the mass and spin of the corresponding particles. To find the mass and spin states for the theory, we expand the fields in eigenfunctions of the momentum and spin operators.

First expand ψ in eigenfunctions of ∂_μ:-

$$\psi = \sum_k \phi(k) \, e^{ik^\mu x_\mu} \qquad (2.4.1)$$

where we here consider the field to be confined in a box of finite volume so that ∂_μ has a discrete set of eigenfunctions. The k_μ are arbitrary, except that the field (2.4.1) must satisfy the field equation (2.1.5), which leads to

$$(L_\mu k^\mu + \chi)\phi(k) = 0 \quad . \qquad (2.4.2)$$

There exist non-zero solutions to this equation if and only if

2) contd.

2.4) contd.

$$|L_\mu k^\mu + \chi| = 0 \qquad (2.4.3)$$

which equation gives us the allowed values of k_μ. Now from (2.2.6) we have

$$T(L_\mu k^\mu + \chi)T^{-1} = L_\mu a^\mu{}_\nu k^\nu + \chi \qquad (2.4.4)$$

so

$$|L_\mu k^\mu + \chi| = |L_\mu (a_\nu{}^\mu k^\nu) + \chi| \qquad (2.4.5)$$

So $|L_\mu k^\mu + \chi|$ must be an invariant polynomial in k^μ and so its irreducible factors must be of the form $k^2 + m_r^2$ where $k^2 = k_\mu k^\mu$. So:-

$$|L_\mu k^\mu + \chi| = \prod_r (k^2 + m_r^2) = 0 \qquad (2.4.6)$$

We identify the k_μ with the four-momentum of the "plane wave" represented by

$$u = e^{ik^\mu x_\mu}, \qquad (2.4.7)$$

which on quantization will become identified with the four-momentum of the corresponding particle. (2.4.6) then says that $\pm m_r$ are the possible rest masses associated with the plane wave (2.4.7) and associated particles. Note that since any irreducible representations of $\mathcal{L}$ is even dimensional, the L_μ is even dimensional and so the polynomial $|L_\mu k^\mu + \chi|$ will be of even degree in k^μ, as we have written it in (2.4.6).

The m_r are related to the non-zero eigenvalues of L_o. We find, by considerations in the rest frame that (2.4.2) has a non-zero solution if and only if

2) contd.

2.4) contd.

$$k^2 = - m_r^2 = - \left(\frac{\chi}{\mu_r} \right)^2 \qquad (2.4.8)$$

where $\pm \mu_r$ are the non-zero eigenvalues of L_o. The zero eigenvalues of L_o have no physical significance; they correspond to the equations defining the redundant components.

Thus the possible values of the rest mass associated with the field are given in terms of the non-zero eigenvalues $\pm \mu_r$ of L_o by

$$\pm m_r = \pm \frac{\chi}{\mu_r} \qquad (2.4.9)$$

For a reasonable physical interpretation we want the m_r to be real, so the μ_r must be real. As shown by Gelfand and Yaglom, complex eigenvalues of L_o correspond to states of zero charge and energy density. Note that associated with every state of rest mass $m_r = \chi/\mu_r$ there will be one with rest mass $-m_r = -\chi/\mu_r$. If we take $\chi/\mu_r > 0$ then by convention this corresponds to the "particle" while $-\chi/\mu_r$ corresponds to the "anti-particle" of the theory. So finally (2.1.5) has plane wave solutions of the form

$$e^{i(\underline{k}.\underline{x}+\omega_{\underline{k}r}t)} \quad ; \quad e^{i(\underline{k}.\underline{x}-\omega_{\underline{k}r}t)} \qquad (2.4.10)$$

where

$$\omega_{\underline{k}r} = + \sqrt{\underline{k}^2 + \left(\frac{\chi}{\mu_r} \right)^2} \qquad (2.4.11)$$

$$\omega_{\underline{k}r} = k_o, \quad \underline{k} = (k_1, k_2, k_3)$$

The field expansion (2.4.1) can therefore be written

$$\psi = \sum_{\underline{k}r} \left(\phi_+(\underline{k},r) e^{i(\underline{k}.\underline{x}+\omega_{\underline{k}r}t)} + \phi_-(\underline{k},r) e^{i(\underline{k}.\underline{x}-\omega_{\underline{k}r}t)} \right) \qquad (2.4.12)$$

where $\phi_+(\underline{k},r)$ and $\phi_-(\underline{k},r)$ satisfy

2) contd.

2.4) contd.

$$(\underline{L}.\underline{k} - \omega_{\underline{k}r} L_o + \chi)\phi_+(\underline{k},r) = 0 \qquad (2.4.13)$$

$$(\underline{L}.\underline{k} + \omega_{\underline{k}r} L_o + \chi)\phi_-(\underline{k},r) = 0 \qquad (2.4.14)$$

The solution spaces of (2.4.13) and (2.4.14) are subspaces of the representation space $\mathcal{R}$. We now require bases for these subspaces in order to expand $\phi_+(\underline{k},r)$ and $\phi_-(\underline{k},r)$ and for this we need an operator which commutes with $L_\mu k^\mu + \chi$. Such an operator is the covariant spin operator

$$S^2 = \frac{\omega_\alpha \omega^\alpha}{k_\mu k^\mu} \qquad (2.4.15)$$

where ω_α is the Pauli-Lubanski covariant spin-vector

$$\omega_\alpha = -\frac{1}{2i} \epsilon_{\alpha\mu\nu\lambda} I^{\mu\nu} k^\lambda \qquad (2.4.16)$$

The eigenvalues of S^2 are $s(s+1)$ where s are the spin values carried by the states $(\underline{k}, \pm\omega_{\underline{k}r})$. To show that S^2 commutes with $L_\mu k^\mu + \chi$ it is sufficient to show that

$$[L_\rho k^\rho , \omega_\alpha] = 0 \qquad (2.4.17)$$

This last equation follows from (2.2.7), which gives

$$[L_\rho k^\rho, \omega_\alpha] = i\epsilon_{\alpha\mu\nu\lambda} L^\mu k^\nu k^\lambda. \qquad (2.4.18)$$

Since the right hand side is the contraction of an anti-symmetric tensor with a symmetric tensor it must vanish, giving (2.4.17).

So S^2 splits the subspaces of $\mathcal{R}$ corresponding to states $(\underline{k}, \pm\omega_{\underline{k}r})$ into subspaces corresponding to eigenvalues $s(s+1)$ of total spin. Thus $\mathcal{R}$ can be split into "physical" subspaces $R^s_{\pm\mu_r}$ according to non-zero eigenvalues of L_o and eigen-

2) contd.

2.4) contd.

values of spin, and a "non-physical" subspace corresponding to zero eigenvalues of L_0. Each physical subspace $R^s_{\pm\mu_r}$ carries an irreducible representation $\mathcal{D}(s)$ of the rotation group and there exist 2s+1 independent vectors $\phi_{\pm\mu_r sm}$ $m = s, s-1, \ldots (s-1), -s$ in $R^s_{\pm\mu_r}$ providing a basis for this subspace. In particular the 2s+1 eigenvectors of H_3 (see (2.1.18)) provide a basis, in which case m represents the different possible values of spin projection along the x_3 axis in the rest frame.

In the canonical representation L_0 provides a neat picture of all the mass-spin states associated with a given field theory. The spin values are given by the index s and from (2.2.26) we see that L_0 can be put in block diagonal form, grouping together all the basis vectors ξ^τ_{sm} $m = s, s-1, \ldots -(s-1), -s$ corresponding to the same values of spin s. We call these blocks "s-blocks"; they have the typical form

$$\begin{bmatrix} C^{\tau\tau'} \\ s \end{bmatrix}$$

We always suppress the m dependence in the s-blocks for brevity. Each of the "elements" in our s-blocks are in fact $(2s+1) \times (2s+1)$ scalar matrices and can therefore be treated like scalars. A non-zero eigenvalue of an s-block is thus a 2s+1 repeated eigenvalue of L_0 and specifies a state of rest mass χ/μ and spin s i.e. the non-zero eigenvalues μ of an s-block correspond to a state of rest mass χ/μ and of spin s. In this way the s-blocks of L_0 provide a list of all the states described by the field theory. On quantization these states go into particle states of mass χ/μ and spin s. L_0 in fact selects out for us certain mass-spin states from the complete range of states possible for the given representation $\mathcal{R}$.

2) contd.

2.4) contd.

Recently Capri and Shamaly[31],[32] have proposed a method for finding first order equations (2.1.5) which describe fields of unique mass and spin. The method is to start with Bhabha's[15] representation for the L_μ and use a transformation due to Wild[19] to obtain what are effectively the s-blocks of Gelfand and Yaglom, described above. The requirements of unique mass and spin are then used to impose conditions on these s-blocks. This method of Capri's is in fact incorporated in what we have said above about the s-blocks. If we want to ensure a unique mass m and spin j then we take, if possible, all other s-blocks to have only zero eigenvalues and the j-block to have a single pair of non-zero eigenvalues $\pm \frac{\chi}{m}$. Capri's method is simply a special case of this general approach.

2.5) Quantization of the Free Relativistic Field.

We quantize the theory in the usual way, by expanding the field quantities such as energy, linear and angular momentum, charge, etc, as sums of contributions for single particle states. This is done by expanding the fields in eigenfunctions of the single particle operator for the corresponding quantum mechanical observable, and interpreting the coefficients in the expansion as elements of a certain operator algebra on a Hilbert space of states. From Section 2.3 we know that any field quantity density can be expressed in the form

$$\sigma = (L_o\psi, \mathcal{O}\psi) \qquad (2.5.1)$$

where $\mathcal{O}$ is the hermitian operator representing the observable

2) contd.

2.5) contd.

in classical particle quantum mechanics. The total field quantities are obtained by integration over some box V:-

$$M = \int_V \sigma \, dv \tag{2.5.2}$$

We expand ψ in simultaneous eigenfunctions of a complete set of commuting operators, which commute with $L_\mu \partial^\mu + i\chi$ and which include $\mathcal{O}$:-

$$\psi = \sum_I a_I \phi_I \tag{2.5.3}$$

where I is an appropriate index set. Then we get

$$\sigma = \sum_I \bar{a}_I \, a_I (L_o \phi_I, \mathcal{O} \phi_I)$$

$$= \sum_I \bar{a}_I a_I \lambda_I (L_o \phi_I, \phi_I)$$

where λ_I is the eigenvalue of $\mathcal{O}$ with eigenfunction ϕ_I. So:-

$$M = \sum_I \bar{a}_I a_I \lambda_I \int_V (L_o \phi_I, \phi_I) dv$$

$$= \sum_I \bar{a}_I a_I \lambda_I \tag{2.5.4}$$

using some convenient normalization of ϕ_I or a_I. By now regarding the a_I as certain operators in a Hilbert space of states and taking $\bar{a}_I$ as a_I^+, + denoting the adjoint in this Hilbert space, we get a particle representation:-

2) contd.

2.5) contd.

$$M = \sum_I N_I \lambda_I = \sum_I a_I^+ a_I \lambda_I \qquad (2.5.5)$$

for the given field quantity. The algebra we choose for the a_I operators depends on the spin-statistics nature of the field theory one is quantizing.

Referring again to Section 2.3 we notice that the orbital and spin angular momentum operators $i(x_i\partial_j - x_j\partial_i)$ and iI_{ij} do not separately commute with $L_\mu\partial^\mu + i\chi$ and so will not lead to a simple particle interpretation as obtained above if their separate eigenfunctions are used to expand the fields. However, their sum, the total angular momentum operator does commute with $L_\mu\partial^\mu + i\chi$ and so its eigenfunctions can be used as a basis for the field expansions.

The major concern in the quantization of any field theory is in the particle form of the energy and charge, which we now study in detail. The densities of these field quantities are

$$H = i(L_o\psi, \partial_o\psi) \qquad (2.5.6)$$

$$\rho = (L_o\psi, \psi) \qquad (2.5.7)$$

apart from unimportant conventional constants. ψ can be expanded in eigenfunctions of ∂_μ as shown at (2.4.12). From (2.4.13) and (2.4.14) we derive the "orthogonality relations":-

$$\phi_+^\dagger(\underline{k},r')\Lambda L_o\phi_-(\underline{k},r) = 0 \text{ for any } r,r' \qquad (2.5.8)$$

and

$$\phi_+^\dagger(\underline{k},r')\Lambda L_o\phi_+(\underline{k},r) = \phi_-^\dagger(\underline{k},r')\Lambda L_o\phi_-(\underline{k},r) = 0 \qquad (2.5.9)$$

$$\text{if } \omega_{\underline{k}r'} \neq \omega_{\underline{k}r}$$

These relations are useful in the quantization procedure.

Substituting (2.4.12) into the densities (2.5.6) and

2) contd.

2.5) contd.

(2.5.7) integrating over the box V and absorbing any constants in $\phi_+(\underline{k},r)$ and $\phi_-(\underline{k},r)$ as necessary we can now write the total energy and total charge in the form

$$E = \sum_{\underline{k}r} \omega_{\underline{k}r} \left(\phi_-^\dagger(\underline{k}r) \Lambda L_0 \phi_-(\underline{k},r) - \phi_+^\dagger(\underline{k},r) \Lambda L_0 \phi_+(\underline{k},r) \right)$$

$$Q = \sum_{\underline{k}r} \left(\phi_+^\dagger(\underline{k},r) \Lambda L_0 \phi_+(\underline{k},r) + \phi_-^\dagger(\underline{k},r) \Lambda L_0 \phi_-(\underline{k},r) \right)$$

where we have used (2.5.8) and (2.5.9).

The quantization procedure must ensure that both of these expressions have definite form in the particle number representation (i.e. are sums of positive contributions in the case of energy and have the form Σ (N-M) in the case of charge, where N counts the particles and M counts the anti-particles). Since both of these quantities transform like the fourth components of four vectors, we can restrict ourselves to the rest frame for considerations of definiteness. A transformation out of the rest frame merely multiplies the energy and charge by a positive number.

In the rest frame (2.4.13) and (2.4.14) show that $\phi_+(0,r)$ and $\phi_-(0,r)$ are eigenvectors of L_0 corresponding to eigenvalues $+\mu_r$ and $-\mu_r$ respectively. Employing the canonical basis L_0 can be reduced to block diagonal form, the blocks being the s-blocks introduced in Section 2.4, and we see that $\phi_+(0,r)$ and $\phi_-(0,r)$ can be expanded in terms of eigenvectors of the restriction of L_0 to the s-subspaces - i.e. of the s-blocks:-

2) contd.

2.5) contd.

$$\phi_+(0,r) = \sum_{rs} a_{ors}\phi_+(0,r,s)$$

$$\phi_-(0,r) = \sum_{rs} \bar{b}_{ors}\ \phi_-(0,r,s)$$

where $\phi_+(0,r,s)$ and $\phi_-(0,r,s)$ are eigenvectors of the s-block corresponding to eigenvalues $+\mu_r$ and $-\mu_r$ respectively. The a_{ors} and $\bar{b}_{ors}$ are now to be interpreted as operators in the Hilbert space of states, and the complex conjugate goes over to the adjoint in this Hilbert space.

Since Λ does not mix up the s-blocks of L_o (see (2.3.8)) we can write

$$\phi_+^\dagger(0,r)\Lambda L_o\phi_+(0,r) = \sum_{rs} \bar{a}_{ors}a_{ors}\ \phi_+^\dagger(0,r,s)\Lambda_s(L_o)_s\phi_+(0,r,s)$$

$$\phi_-^\dagger(0,r)\Lambda L_o\phi_-(0,r) = \sum_{rs} b_{ors}\bar{b}_{ors}\ \phi_-^\dagger(0,r,s)\Lambda_s(L_o)_s\phi_-(0,r,s)$$

where $\Lambda_s,(L_o)_s$ are the s-subblocks of Λ and L_o. The total energy and charge thus become in the rest frame:-

$$E = \sum_{rs} \omega_{kr}(b_{ors}b^*_{ors}\phi_-^\dagger(0,r,s)\Lambda_s(L_o)_s\phi_-(0,r,s) - a^*_{ors}a_{ors}\phi_+^\dagger(0,r,s)\Lambda_s(L_o)_s\phi_+(0,r,s))$$

$$Q = \sum_{rs} \left(a^*_{ors}a_{ors}\phi_+^\dagger(0,r,s)\Lambda_s(L_o)_s\phi_+(0,r,s) + b_{ors}b^*_{ors}\phi_-^\dagger(0,r,s)\Lambda_s(L_o)_s\phi_-(0,r,s)\right)$$

Writing:-

$$\epsilon_+(0,r,s) = \phi_+^\dagger(0,r,s)\Lambda_s(L_o)_s\ \phi_+(0,r,s)$$

$$\epsilon_-(0,r,s) = \phi_-^\dagger(0,r,s)\Lambda_s(L_o)_s\ \phi_-(0,r,s)$$

2) contd.

2.5) contd.

$$E = \sum_{rs} \underline{\omega}_{kr}\, \epsilon_+(0,r,s)\left[b_{ors}b^+_{ors}\left(\frac{\epsilon_-(0,r,s)}{\epsilon_+(0,r,s)}\right) - a^+_{ors}a_{ors}\right] \qquad (2.5.10)$$

$$Q = \sum_{rs} \epsilon_+(0,r,s)\left[b_{ors}b^+_{ors}\left(\frac{\epsilon_-(0,r,s)}{\epsilon_+(0,r,s)}\right) + a^+_{ors}a_{ors}\right] \qquad (2.5.11)$$

So the definiteness properties of the quantized energy and charge depend ultimately on the quantities $\epsilon_+(0,r,s)$ and $\epsilon_-(0,r,s)$. The usual spin-statistics connection will be satisfied if in the case of

Integral spin fields

$\epsilon_+(0,r,s)$ have the same sign for all r,s and in each case $\epsilon_-(0,r,s)$ has the opposite sign to $\epsilon_+(0,r,s)$.

Half Odd Integral Spin Fields.

$\epsilon_+(0,r,s)$ have the same sign for all r,s and in each case $\epsilon_-(0,r,s)$ has the same sign as $\epsilon_+(0,r,s)$.

The remainder of this work is concerned with the study of s-blocks and the search for field theories for which the $\epsilon_+(0,r,s)$ and $\epsilon_-(0,r,s)$ satisfy one of the above conditions. These conditions are equivalent to the usual ones of definite total energy for relativistic fields of integral spin and definite total charge for relativistic fields of half odd integral spin. Gelfand and others have studied field theories with positive energy or definite charge. It is known[24],[19] that if L_o is diagonalizable then the only field theories satisfying the above conditions are those for spins 0, $\frac{1}{2}$ or 1. For higher spins therefore $L_o$must be non-diagonalizable if we are to quantize the theory. Theories corresponding to a non-diagonalizable L_o are those

2) contd.

2.5) contd.

containing "subsidiary conditions".

In the Gelfand-Yaglom approach, the eigenvalues of the s-blocks are explicitly obtained in terms of the arbitrary parameters $C^{\tau\tau'}$. Eigenvectors corresponding to non-zero eigenvalues are found explicitly and substituted in the field expressions for the charge or energy - the definiteness properties lead in this way to conditions on the $C^{\tau\tau'}$. The object of this thesis is to organize and simplify this lengthy and complicated procedure.

Fedorov[30] has found useful conditions on Λ and L_0 for definiteness of field energy or charge, but still these are difficult to apply directly in practice. The complexity in previous attempts to find quantizable higher spin theories lies in the fact that the required conditions on the field theory are usually expressed and investigated in terms of L_0. The work can be simplified however by decomposing L_0 into s-blocks and studying the quantities $\epsilon_+(0,r,s)$ and $\epsilon_-(0,r,s)$ defined above. In effect the s-blocks enable us to study separately the spin-s particles carried by the field and their separate contributions to charge and energy.

In order to keep the algebra within reasonable bounds we find it useful to make the assumption that the field carries no more than one particle of any given spin. Thus restricting ourselves to a certain type of field theory (also we do not allow multiple representations in $\mathcal{R}$) we obtain, using certain graphs as visual aids, conditions similar to Fedorov's, but which are simpler and easier to use in practice. We will not concern ourselves with the

2) contd.

2.5) contd.

actual quantization of a field theory, rather we look for theories which are quantizable. There is no point then in explicitly exhibiting the particle or quantized form of the theory.

CHAPTER 3.

INVESTIGATION OF QUANTIZABLE FREE FIELD THEORIES.

3) Investigation of Quantizable Free Field Theories.

3.1) Introduction.

We have seen in Chapter 2 that the s-blocks, into which L_o can be partitioned, contain much of the useful information about the field theory. The non-zero eigenvalues of the s-block tell us the masses of the states of spin s described by the theory. In Section 2.5 we saw that the important quantities in the quantization of the field theory are

$$\epsilon_+(r,s) = \psi^\dagger_{+rs} \Lambda_s(L_o)_s \psi_{+rs} \qquad = \epsilon_+(0,r,s)$$

and

$$\epsilon_-(r,s) = \psi^\dagger_{-rs} \Lambda_s(L_o)_s \psi_{-rs} \qquad = \epsilon_-(0,r,s)$$

where $\psi_{\pm rs}$ are eigenvectors of the s-block $(L_o)_s$ corresponding to eigenvalues $\pm\mu_r$ - more precisely it is the signs of these quantities that we are interested in.

First we must construct the most general forms of the s-blocks. The exact form of the elements of L_o have been given by Gelfand and Yaglom in the canonical representation, as detailed in Section 2.2. However, to construct L_o and the s-blocks from these results is a lengthy, tedious job which can be organized a great deal, and this is the object of Section 3.2. We only here consider theories which do not contain multiple representations, that is $\mathcal{A}$ is a direct sum of distinct irreducible representations of $\mathcal{L}_p$. It then proves useful to exhibit the s-blocks as simple graphs in the (ℓ_o,ℓ_1) plane. Using these graphs as visual aids provides simple proofs to some results about the s-blocks and also is advantageous in algebraic manipulation. This is done in Section 3.3.

By making certain assumptions about the minimal polynomials of the s-blocks, very simple expressions can be

3) contd.

3.1) contd.

found which give the signs of $\epsilon_+(r,s)$ and $\epsilon_-(r,s)$. Also, from considerations of the graphs of the s-blocks it is easily seen that $\epsilon_+(r,s)$ and $\epsilon_-(r,s)$ have opposite signs in the case of integral spin and the same signs in the case of half odd integral spin, as required for quantization and the spin-statistics connection. The question of quantization and the definiteness properties of the field theories is considered in Section 3.4.

3.2) Construction of the s-blocks.

We give here an algorithm for the construction of the s-blocks, which takes the maximum advantage of their symmetries. Further details of the justification of the algorithm are given in Appendix A.

Useful for the understanding of the algorithm is a graphical interpretation of the problem of the structure of L_0, which can also give a deeper insight into the structure and properties of the s-blocks. The basic idea is that any reducible representation of $\mathcal{L}_p$ can be displayed as a particularly simple type of graph in the ℓ_0,ℓ_1 plane; and all the conditions of $\mathcal{L}_p$ and space reflection covariance and Lagrangian origin can be represented in terms of this graph. To help with this and to reduce the number of theories we have to consider we will make two "simplicity" assumptions. These have no physical basis, but they do make the mathematical formalism simpler and restrict the number of possible theories one has to investigate for a given maximum spin. The large amount of arbitrariness in such theories in general is well known.

3) contd.

3.2) contd.

Assumption 1.

No irreducible representation of $\mathcal{L}_p$ occurs more than once in $\mathcal{R}$.

Many people have looked at theories involving multiple representations[31],[32],[33]. In particular Bhabha[33] has shown that a quantizable field theory carrying a spin $\frac{1}{2}$ and spin $\frac{3}{2}$ state can be obtained using the representation

$$2(-\tfrac{1}{2},\tfrac{3}{2}) \oplus 2(\tfrac{1}{2},\tfrac{3}{2}) \oplus (-\tfrac{1}{2},\tfrac{5}{2}) \oplus (\tfrac{1}{2},\tfrac{5}{2}).$$

The question arises of the possibility of multi-state theories that are quantizable but do not have repeated representations. As we see later, such theories do exist.

Assumption 2.

In a theory designed for maximum spin j, $\mathcal{R}$ shall not contain any representation carrying a spin greater than j.

This assumption is not so necessary as the first, and it is sometimes necessary to forsake it - for example in the spin 0 theory. However, unless otherwise stated we will adhere to this assumption from now on.

It is convenient to treat the cases of integral and half odd integral spin separately. We will concentrate on the former in detail and simply point out the appropriate modifications for the latter. In the case of integral spin we plot only the points ℓ_0,ℓ_1, where ℓ_0,ℓ_1 are both integers, in the $\ell_0\ell_1$ plane and refer to the resulting lattice of points as the "Bose-plane". In the half odd integral spin case we

3) contd.

3.2) contd.

plot only the points for which ℓ_0, ℓ_1 are both half odd integers and refer to the lattice as the "Fermi-plane." Since the representations (ℓ_0, ℓ_1) and $(-\ell_0, -\ell_1)$ are equivalent, we can adopt the convention $\ell_1 > 0$, then we get all the irreducible finite representations of $\mathcal{L}_p$ giving integer spin theories by plotting $\ell_0 = 0, \pm 1, \pm 2, \ldots$; $\ell_1 = 0, 1, 2, \ldots$ and similarly for half odd integer spin theories by plotting $\ell_0 = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ $\ell_1 = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and confining ourselves to the "fan" $\ell_1 > |\ell_0|$. We now consider integral spin theories.

Integral Spin.

Any irreducible finite representation τ of $\mathcal{L}_p$ in our representation space $\mathcal{R}$ will be specified by two integers (ℓ_0, ℓ_1), giving a point in the Bose plane in the fan $\ell_1 > |\ell_0|$ - called the "Bose fan". See Fig(3.2.1)

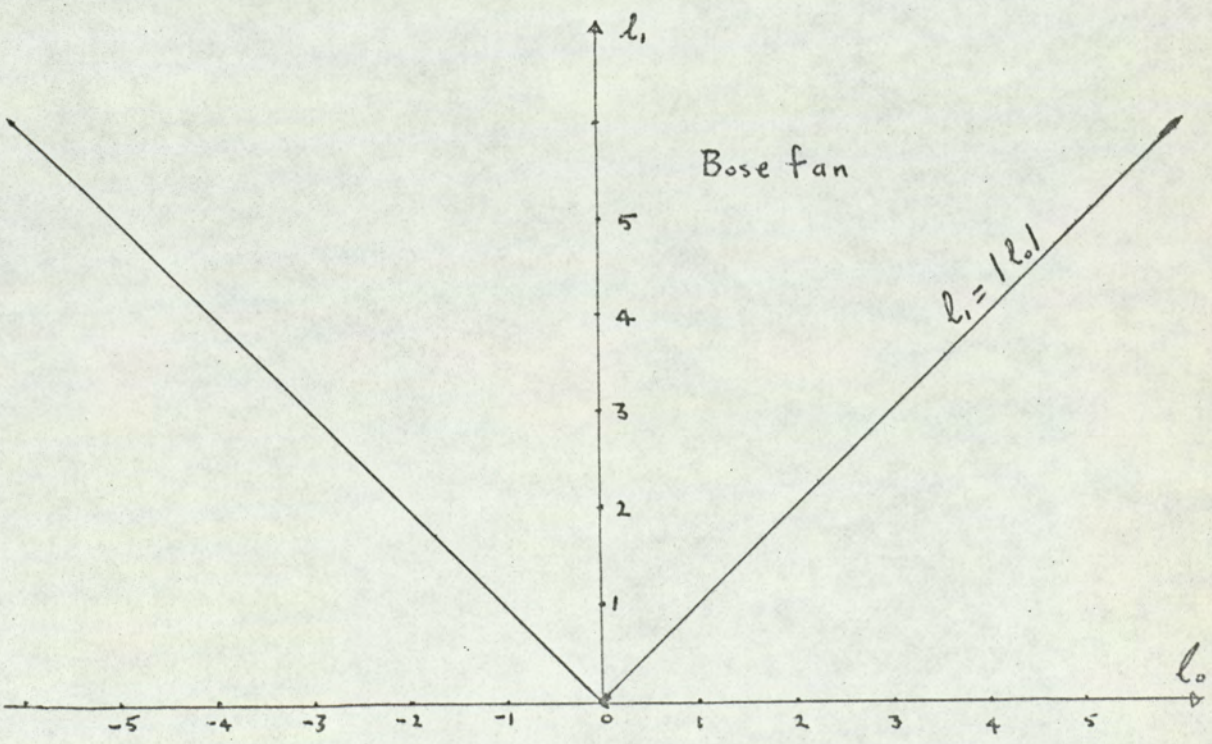


Fig.3.2.1

3) contd.

3.2) contd.

The Assumption 1 means that $\mathcal{R}$ will correspond to a subset of points or "nodes" in the Bose fan. Further, Assumption 2 means that for a theory with maximum spin j, we only consider representations $\tau \sim (\ell_0, \ell_1)$ of $\mathcal{L}_p$ such that

$$\ell_1 \leq j + 1.$$

Thus for a maximum spin j theory we have only to consider the representations τ_i in the shaded triangle and on the line $\ell_1 = j+1$ in Figure 3.2.2. $\mathcal{R}$ will be a representation

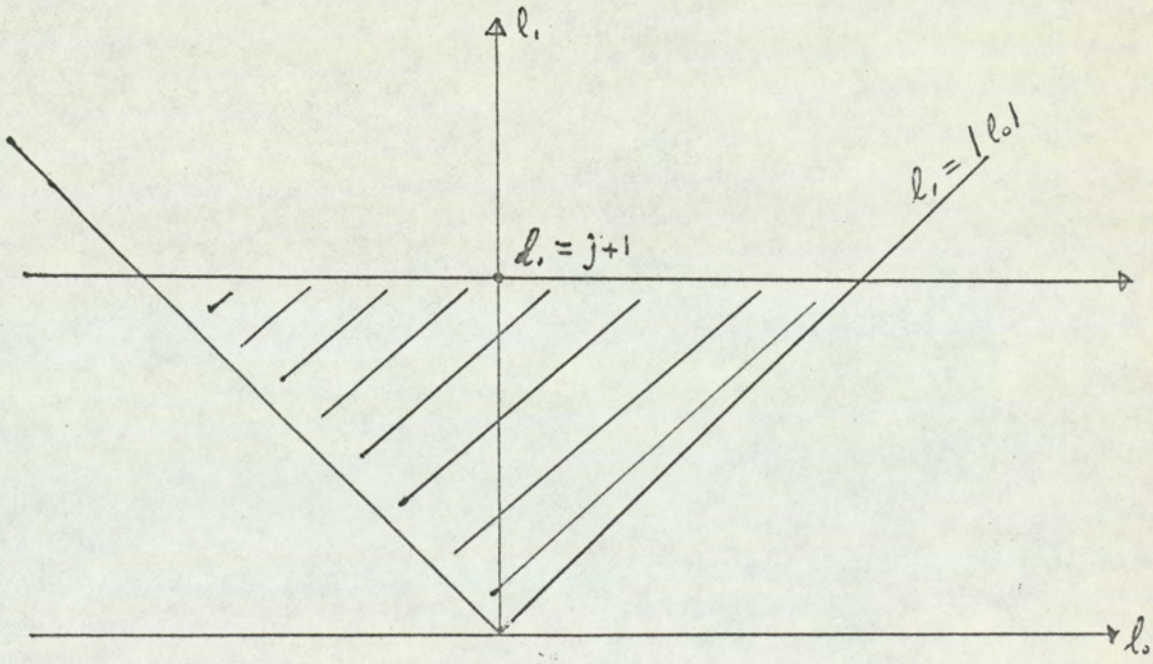


Fig.3.2.2.

space for a possible maximum spin j theory if it corresponds to a subset of points in this triangle, with at least one point on the line $\ell_1 = j+1$. It will clearly suffice to consider the most general possible such $\mathcal{R}$ - i.e. that which includes all representations in the triangle and on the $\ell_1 = j+1$ line. Any other maximum spin j representation subject to Assumptions

3) contd.

3.2) contd.

1 and 2 can then be obtained by omitting some representations, or by equating appropriate $C^{\tau\tau'}$ to zero. Unless otherwise stated we will assume $\mathcal{R}$ is the most general representation in the above sense.

We will in future denote the s-block by

$$A_s = [\, C_s^{\tau\tau'} \,] = (L_o)_s \qquad (3.2.1)$$

Its elements are scalar matrices $C_s^{\tau\tau'}[\delta_{mm'}]$ labelled by the representations τ, τ'. To construct A_s we first need the representations involved, i.e. those representations τ which carry a spin s state. From the range of spins given by (2.2.17) we see that the s-block will involve only those representations in and on the rectangle:-

$$\left.\begin{array}{ll} \ell_o = -s & \ell_o = s \\ \ell_1 = s+1 & \ell_1 = j+1 \end{array}\right\} \qquad (3.2.2)$$

which will be called the "s-rectangle". So the s-block in a maximum spin j theory will have dimension

$$(j-s+1)(2s+1) \qquad (3.2.3$$

since $(j-s+1)(2s+1)$ in the number of representations it contains. If some of the representations are not included in a particular theory then the s-block will of course be smaller.

We now adopt a useful standard numbering of the representations τ in the Bose-fan. We number the representations from left to right along rows (i.e. lines of constant ℓ_1). With this numbering, inspection of the s-rectangle in the Bose triangle shows that the s-block will contain those representations

3) contd.

3.2) contd.

$$\tau_i, \tau_{i+1}, \ldots\ldots \tau_{i+2s}$$

where $i = \ell^2 - \ell - s+1$

for $\ell = s+1, s+2, \ldots, j+1$ (3.2.4)

For a given ℓ, (3.2.4) gives the representations in the s-rectangle in the row $\ell_1 = \ell$.

We label a row or a column of the s-block by the index k of the representation τ_k to which it corresponds, in ascending values of k along rows and columns. Then A_s can be partitioned into $(2s+1)\times(2s+1)$ subblocks $B_{\ell k}(s)$ corresponding to the rows of the s-rectangle; where $B_{\ell k}(s)$ contains the rows of the s-block corresponding to the representations on the row $\ell_1 = \ell$ of the s-rectangle and the columns of the s-block corresponding to the representations on row $\ell_1 = k$ of the s-rectangle.

Now the only non-zero elements in the s-block are those labelled by linked representations. Two representations are linked by a type (i) linkage (see (2.2.30)) if they are adjacent in the Bose fan by a horizontal link in the lattice. Two representations are linked by a linkage of type (ii) [See (2.2.31)] if they are adjacent vertically. Thus a glance at the Bose triangle for $\mathcal{R}$ tells us the positions of the non-zero elements of A_s. From the structure of the s-rectangle we see that, with the standard numbering system, all of the $B_{k\ell}(s)$ subblocks are zero except those on, immediately above and immediately below the leading diagonal. This is because two representations can be linked only if they lie either on the same or adjacent rows. Furthermore, we see that with the

3) contd.

3.2) contd.

standard numbering the non-zero elements in the non-zero subblocks are distributed as follows:-

$B_{\ell\ell}(s)$

All elements are zero except those immediately above and below the leading diagonal of $B_{\ell\ell}(s)$.

$B_{\ell\ell+1}(s)$ and $B_{\ell+1\ell}(s)$.

These are all diagonal matrices.

The positions specified by these statements give the only places where non-zero elements of the s-block can occur. In the absence of further conditions these non-zero elements can be taken as arbitrary complex numbers. The full application of $\mathcal{L}$-covariance and Lagrangian origin restrict the form of these quantities, in the way detailed in Section 2.2. We now give a complete algorithm for the construction of any s-block in any maximum spin theory, giving the most general form after the above conditions of covariance and Lagrangian origin have been imposed. Details are given in Appendix A.

Algorithm.

First partition the s-block, as described above, into subblocks $B_{\ell k}(s)$ corresponding to the rows of the s-rectangle. The only non-zero blocks are $B_{\ell\ell}(s)$, $B_{\ell\ell+1}(s)$, $B_{\ell+1,\ell}(s)$ for $\ell = s+1, s+2, \ldots, j+1$, so

3) contd.

3.2) contd.

$$A_s = \begin{bmatrix} B_{s+1,s+1}(s) & B_{s+1,s+2}(s) & 0 & 0 & 0 \cdots & & 0 \\ B_{s+2,s+1}(s) & B_{s+2,s+2}(s) & B_{s+2,s+3}(s) & 0 & 0 \cdots & & 0 \\ 0 & B_{s+3,s+2}(s) & B_{s+3,s+3}(s) & B_{s+3,s+4}(s) & 0 \cdots & & 0 \\ 0 & & & & & & \vdots \\ \vdots & & & & & & 0 \\ & & & & & B_{j,j}(s) & B_{jj+1}(s) \\ 0 \quad \cdots\cdots & & & & \cdots\cdots & B_{j+1,j}(s) & B_{j+1\,j+1}(s) \end{bmatrix}$$

The non-zero subblocks $B_{k\ell}(s)$ are filled in as follows:-

$B_{\ell\ell}(s)$.

1) If $s = 0$, $B_{\ell\ell}(0) = 0$

2) If $s > 0$, all elements of $B_{\ell\ell}(s)$ are zero except those immediately above and below the leading diagonal.

3) Bisect the leading diagonal by the "skew diagonal" from lower left to upper right of the subblock and fill in the left upper triangle as follows:-

a) Put

$$C_s^{\tau_k \tau_{k+1}} = Z_{kk+1} \sqrt{(s+\ell_0^{(k)}+1)(s-\ell_0^{(k)})} \quad (\tau_k \sim (\ell_0^{(k)}, \ell_1^{(k)}))$$

in the position k,k+1 for $k = i(\ell), i(\ell+1), \ldots i(\ell)+s$ where $i(\ell) = \ell^2-\ell-s+1$. Z_{kk+1} represents an arbitrary complex number which is independent of s, but depends on the representations τ_k, τ_{k+1} which it links. Z_{kk+1} will be the same in each s-block which receives a contributions from

3) contd.

3.2) contd.

3) contd.

a) contd.

the representations τ_k, τ_{k+1}.

b) For k = i,i+1, ...i+s-2, in the corresponding symmetric position k+1,k put

$$C_s^{\tau_{k+1}\tau_k} = S_{kk+1}\,\overline{C_s^{\tau_k\tau_{k+1}}}$$

where $S_{kk+1} = \pm 1$. As explained in the Appendix A,

$$S_{kk+1} = S(\tau_k,\tau_{k+1}) = \frac{a^{\tau_k\tau_k^{\cdot}}}{a^{\tau_{k+1}\tau_{k+1}^{\cdot}}}.$$

(see (2.3.8)).

c) In the remaining position i+s,i+s-1 put

$$C_s^{\tau_{i+s},\tau_{i+s-1}} = \eta_{i+s}S_{i+s,i+s+1}\,\overline{C^{\tau_{i+s}\tau_{i+s+1}}}$$

where $\eta_{i+s} = \pm 1$, depending on which matrix is used to represent the reflection operator S in the representation space $\mathcal{R}$.

4) To get the remaining lower right triangle, first reflect the upper left triangle in the skew diagonal, then in the leading diagonal - but under the last reflection leave the η_{i+s} where it is.

The above set of rules for constructing the $B_{\ell\ell}(s)$ subblock is more difficult to write and read than it is to apply. More insight will be gained by studying the patterns in the final general form of the $B_{\ell\ell}(s)$ block, shown in Fig.3.2.3, where we have, for clarity, used the

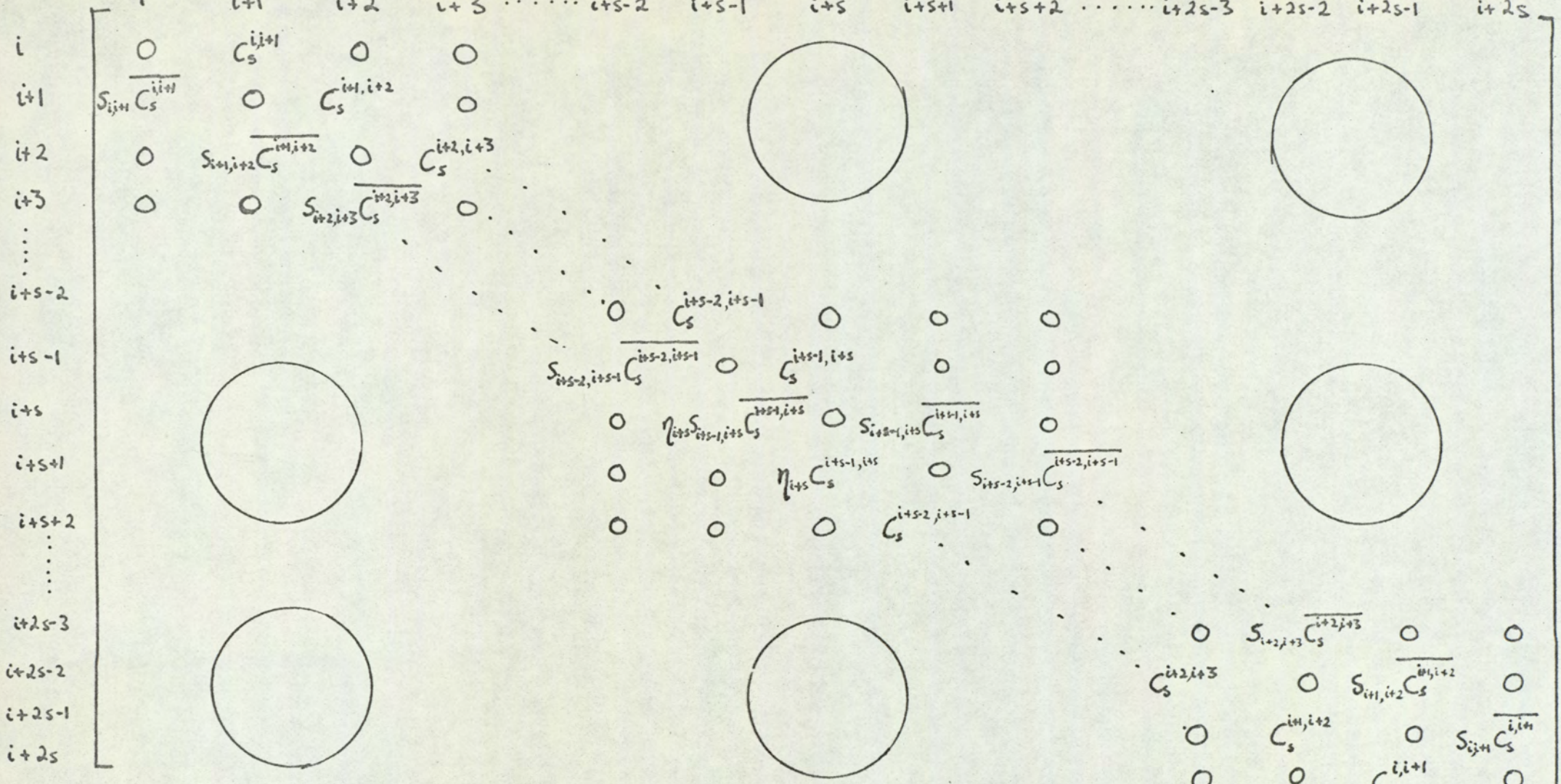


Fig: 3.2.3

3) contd.

3.2) contd.

notation $C_s^{k\ell} = C_s^{\tau_k \tau_\ell}$.

$\underline{B_{\ell\ell+1}(s) \text{ and } B_{\ell+1\ell}(s)}$.

We first construct $B_{\ell\ell+1}(s)$, to which $B_{\ell+1\ell}(s)$ is very simply related.

1) $B_{\ell\ell+1}(s)$ is diagonal. The diagonal positions are labelled by $k(\ell)$, $k(\ell+1)$, $k(\ell)=i(\ell),i(\ell)+1,\ldots i(\ell)+2s$ where $i(\ell) = \ell^2-\ell-s+1$. $B_{\ell\ell+1}(s)$ is symmetric about its skew diagonal. Fill in the positions above and on the skew diagonal with elements

$$C_s^{\tau_{k(\ell)} \tau_{k(\ell+1)}} = Z_{k(\ell)k(\ell+1)} \left| \sqrt{(s+\ell+1)(s-\ell)} \right|$$

for $k(\ell) = i(\ell), i(\ell)+1, \ldots i(\ell)+s$. To get the remaining non-zero elements just reflect in the skew diagonal. This gives a $B_{\ell\ell+1}(s)$ subblock as shown in Fig.3.2.4.

2) To get $B_{\ell+1,\ell}(s)$ from $B_{\ell,\ell+1}(s)$, simply take the complex conjugate of every element of $B_{\ell\ell+1}(s)$ and multiply the diagonal element $C_s^{k(\ell),k(\ell+1)}$ by $s_{k(\ell)k(\ell+1)}$. Thus we get a $B_{\ell+1\ \ell}(s)$ subblock as shown in Fig.3.2.5.

This completes the construction of the s-blocks in the case of integral spin. As is usual when discussing matrices, the process is much more difficult to describe than perform, and once the mechanics of the algorithm have been grasped it will be seen to provide an effective, simple way of writing down the s-blocks in any given theory. Although not essential for the construction of the s-blocks, it is

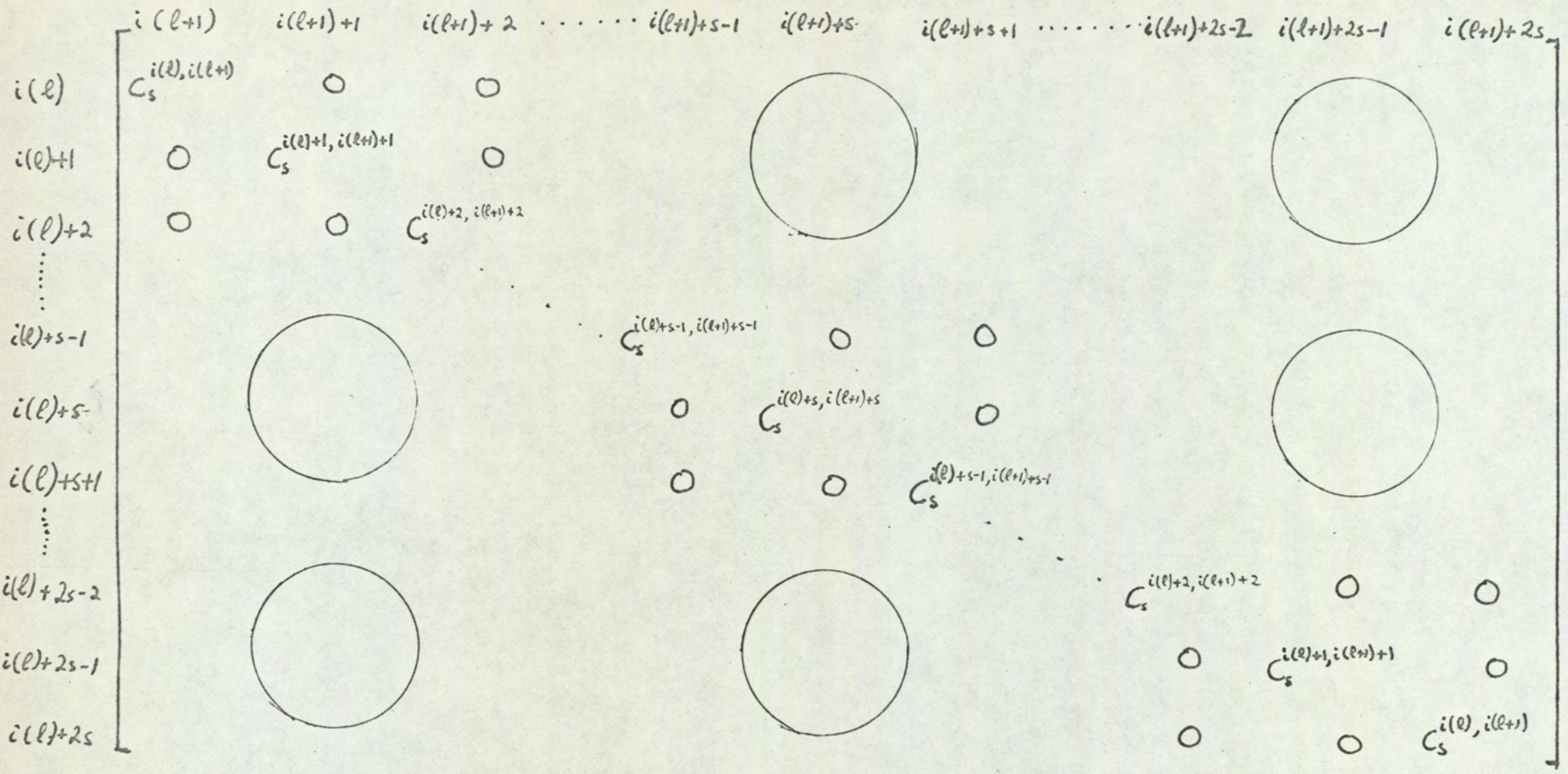

i(ℓ+1)
i(ℓ+1)+1
i(ℓ+1)+2
i(ℓ+1)+s-1
i(ℓ+1)+s
i(ℓ+1)+s+1
i(ℓ+1)+2s-2
i(ℓ+1)+2s-1
i(ℓ+1)+2s
i(ℓ)
i(ℓ)+1
i(ℓ)+2
i(ℓ)+s-1
i(ℓ)+s
i(ℓ)+s+1
i(ℓ)+2s-2
i(ℓ)+2s-1
i(ℓ)+2s
$C_s^{i(\ell),i(\ell+1)}$
$C_s^{i(\ell)+1,i(\ell+1)+1}$
$C_s^{i(\ell)+2,i(\ell+1)+2}$
$C_s^{i(\ell)+s-1,i(\ell+1)+s-1}$
$C_s^{i(\ell)+s,i(\ell+1)+s}$
$C_s^{i(\ell)+s-1,i(\ell+1)+s-1}$
$C_s^{i(\ell)+2,i(\ell+1)+2}$
$C_s^{i(\ell)+1,i(\ell+1)+1}$
$C_s^{i(\ell),i(\ell+1)}$

Fig 3.2.4

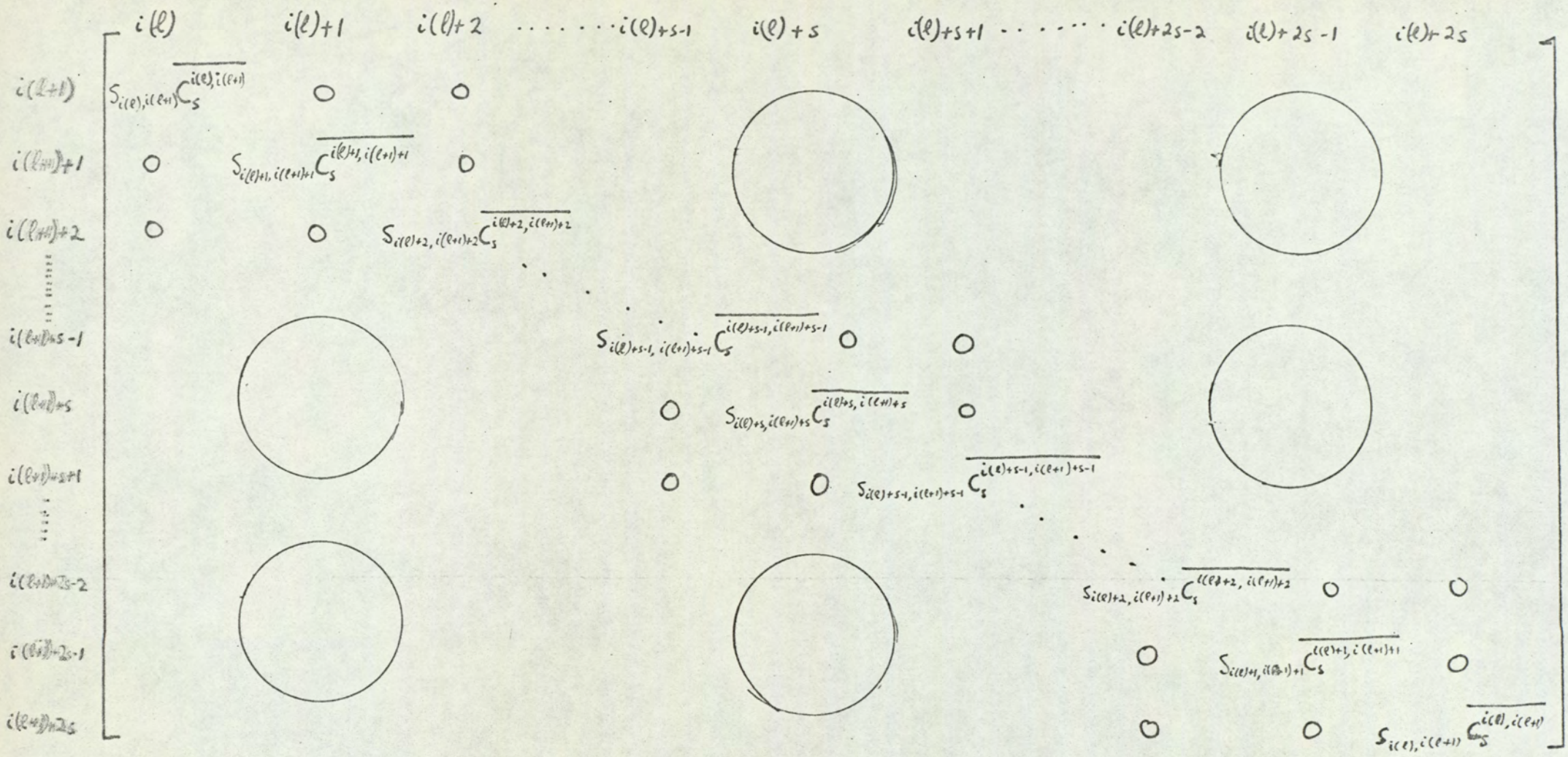

i(ℓ)
i(ℓ)+1
i(ℓ)+2
i(ℓ)+s-1
i(ℓ)+s
i(ℓ)+s+1
i(ℓ)+2s-2
i(ℓ)+2s-1
i(ℓ)+2s
i(ℓ+1)
i(ℓ+1)+1
i(ℓ+1)+2
i(ℓ+1)+s-1
i(ℓ+1)+s
i(ℓ+1)+s+1
i(ℓ+1)+2s-2
i(ℓ+1)+2s-1
i(ℓ+1)+2s
$S_{i(\ell),i(\ell+1)}\overline{C_s^{i(\ell),i(\ell+1)}}$
$S_{i(\ell)+1,i(\ell+1)+1}\overline{C_s^{i(\ell)+1,i(\ell+1)+1}}$
$S_{i(\ell)+2,i(\ell+1)+2}\overline{C_s^{i(\ell)+2,i(\ell+1)+2}}$
$S_{i(\ell)+s-1,i(\ell+1)+s-1}\overline{C_s^{i(\ell)+s-1,i(\ell+1)+s-1}}$
$S_{i(\ell)+s,i(\ell+1)+s}\overline{C_s^{i(\ell)+s,i(\ell+1)+s}}$
$S_{i(\ell)+s-1,i(\ell+1)+s-1}\overline{C_s^{i(\ell)+s-1,i(\ell+1)+s-1}}$
$S_{i(\ell)+2,i(\ell+1)+2}\overline{C_s^{i(\ell)+2,i(\ell+1)+2}}$
$S_{i(\ell)+1,i(\ell+1)+1}\overline{C_s^{i(\ell)+1,i(\ell+1)+1}}$
$S_{i(\ell),i(\ell+1)}\overline{C_s^{i(\ell),i(\ell+1)}}$

Fig 3.2.5

3) contd.

3.2) contd.

useful to bear in mind the graphical representations of the s-blocks, as described in Appendix A.

As an example of the above algorithm, we will construct the s-blocks for the maximum spin 2 theory:-

Maximum Spin 2 s-blocks.

The appropriate Bose triangle is shown in Fig. 3.2.6.

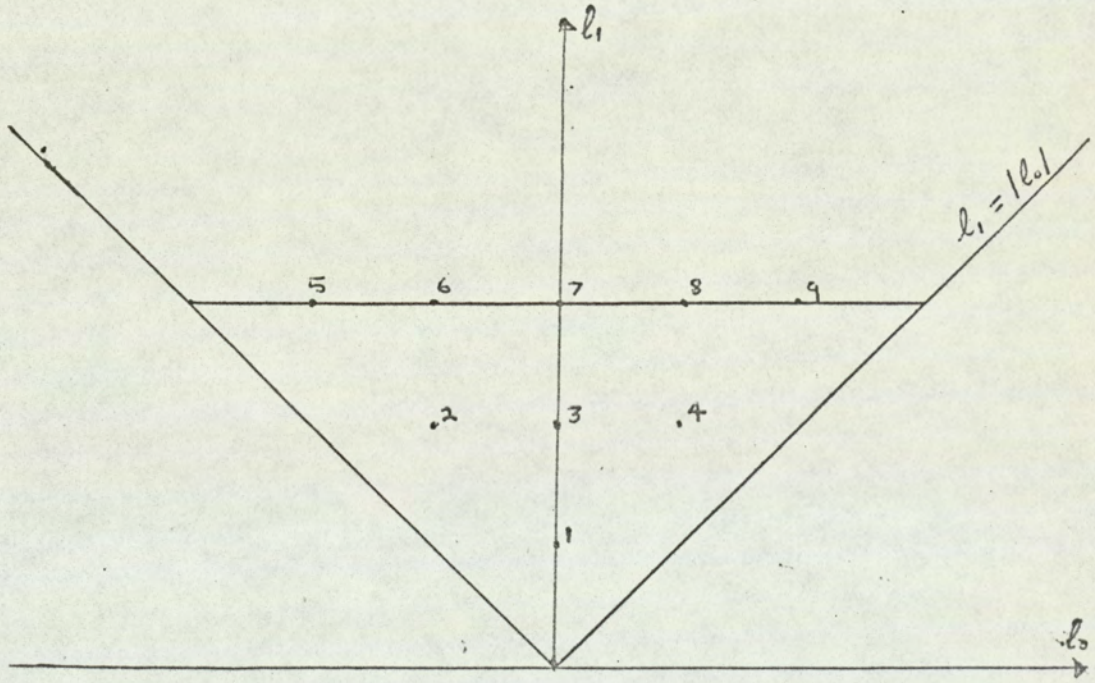


Fig.3.2.6

0-Block.

Contains the representations 1,3,7

The $B_{\ell\ell}(o)$ blocks, corresponding to the rows, are all zero.

The $B_{\ell\ell+1}(o)$ blocks are all 1 × 1, we get:-

3) contd.

3.2) contd.

$$A_0 = \begin{array}{c} \\ 1 \\ 3 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 7 \end{array} \\ \begin{bmatrix} 0 & \sqrt{(\ell_1^{(1)}+1)(-\ell_1^{(1)})}\, z_{13} & 0 \\ s_{13}\sqrt{(\ell_1^{(1)}+1)(-\ell_1^{(1)})}\, \bar{z}_{13} & 0 & \sqrt{(\ell_1^{(3)}+1)(-\ell_1^{(3)})}\, z_3 \\ 0 & s_{37}\sqrt{(\ell_1^{(3)}+1)(-\ell_1^{(3)})}\, \bar{z}_{37} & 0 \end{bmatrix} \end{array}$$

$\rightarrow$

$$\begin{array}{c} \\ 1 \\ 3 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 7 \end{array} \\ \begin{bmatrix} 0 & \sqrt{2}\,\bar{z}_{13} & 0 \\ \sqrt{2}s_{13}\bar{z}_{13} & 0 & \sqrt{6}\, z_{37} \\ 0 & s_{37}\sqrt{6}\,\bar{z}_{37} & 0 \end{bmatrix} \end{array}$$

1-block.

The 1-rectangle contains the representations 2,3,4,6,7,8 in two rows. The B-blocks are 3×3. From the algorithm and graph we obtain the matrices shown in Fig.3.2.7.

2-block.

The 2-rectangle consists of the representations 5,6,7,8,9, i.e. the $\ell_1 = 3$ row. Thus all the $B_{\ell,\ell+1}(2)$ $B_{\ell+1,\ell}(2)$ blocks are zero and the 2-block is just a single 5×5 $B_{\ell\ell}(2)$ block. Using the algorithm and graph we obtain the matrices in Fig.3.2.8.

An arbitrary complex number can be absorbed into the z's and when this is done we can finally write the s-blocks:-

	2	3	4	6	7	8
2	0	$\sqrt{(\ell_0^{(2)}+2)(1-\ell_0^{(2)})}\, z_{23}$	0	$\sqrt{(\ell_1^{(2)}+2)(1-\ell_1^{(2)})}\, z_{26}$	0	0
3	$\eta_2 s_{23}\sqrt{(\ell_0^{(3)}+2)(1-\ell_0^{(3)}}$	0	$s_{23}\sqrt{(\ell_0^{(2)}+2)(1-\ell_0^{(2)})}\, \bar{z}_{23}$	0	$\sqrt{(\ell_1^{(3)}+2)(1-\ell_1^{(3)})}\, z_{37}$	0
4	0	$\eta_2\sqrt{(\ell_0^{(2)}+2)(1-\ell_0^{(2)})}\, z_{23}$	0	0	0	$\sqrt{(\ell_1^{(2)}+2)(1-\ell_1^{(2)})}\, z_{26}$
6	$s_{26}\sqrt{(\ell_1^{(2)}+2)(1-\ell_1^{(2)})}\, \bar{z}_{26}$	0	0	0	$\sqrt{(\ell_0^{(6)}+2)(1-\ell_0^{(6)}}\, z_{67}$	0
7	0	$s_{37}\sqrt{(\ell_1^{(3)}+2)(1-\ell_1^{(3)})}\, \bar{z}_{37}$	0	$\eta_6 s_{67}\sqrt{(\ell_0^{(6)}+2)(1-\ell_0^{(6)})}\, \bar{z}_{67}$	0	$s_{67}\sqrt{(\ell_0^{(6)}+2)(1-\ell_0^{(6)})}\, \bar{z}_{67}$
8	0	0	$s_{26}\sqrt{(\ell_1^{(2)}+2)(1-\ell_1^{(2)})}\, \bar{z}_{26}$	0	$\eta_6\sqrt{(\ell_0^{(6)}+2)(1-\ell_0^{(6)})}\, z_{67}$	0

$\longrightarrow$

	2	3	4	6	7	8
2	0	$\sqrt{2}\, z_{23}$	0	$2\, z_{26}$	0	0
3	$\eta_2 s_{23}\sqrt{2}\, \bar{z}_{23}$	0	$s_{23}\sqrt{2}\, \bar{z}_{23}$	0	$2\, z_{37}$	0
4	0	$\eta_2\sqrt{2}\, z_{23}$	0	0	0	$2\, z_{26}$
6	$2 s_{26}\, \bar{z}_{26}$	0	0	0	$\sqrt{2}\, z_{67}$	0
7	0	$2 s_{37}\, \bar{z}_{37}$	0	$\eta_6 s_{67}\sqrt{2}\, \bar{z}_{67}$	0	$s_{67}\sqrt{2}\, \bar{z}_{67}$
8	0	0	$2 s_{26}\, \bar{z}_{26}$	0	$\eta_6\sqrt{2}\, z_{67}$	0

Fig 3.2.7

	5	6	7	8	9
5	0	$\sqrt{(3+\ell_0^{(5)})(2-\ell_0^{(5)})}\, z_{56}$	0	0	0
6	$s_{56}\sqrt{(3+\ell_0^{(5)})(2-\ell_0^{(5)})}\, \bar{z}_{56}$	0	$\sqrt{(3+\ell_0^{(6)})(2-\ell_0^{(6)})}\, z_{67}$	0	0
7	0	$\eta_6 s_{67}\sqrt{(3+\ell_0^{(6)})(2-\ell_0^{(6)})}\, \bar{z}_{67}$	0	$s_{67}\sqrt{(3+\ell_0^{(7)})(2-\ell_0^{(7)})}\, \bar{z}_{67}$	0
8	0	0	$\eta_6\sqrt{(3+\ell_0^{(7)})(2-\ell_0^{(7)})}\, z_{67}$	0	$s_{56}\sqrt{(3+\ell_0^{(5)})(2-\ell_0^{(5)})}\, \bar{z}_{56}$
9	0	0	0	$\sqrt{(3+\ell_0^{(6)})(2-\ell_0^{(5)})}\, z_{56}$	0

$\longrightarrow$

	5	6	7	8	9
5	0	$2 z_{56}$	0	0	0
6	$2 s_{56} \bar{z}_{56}$	0	$\sqrt{6}\, z_{67}$	0	0
7	0	$\eta_6 s_{67}\sqrt{6}\, \bar{z}_{67}$	0	$s_{67}\sqrt{6}\, \bar{z}_{67}$	0
8	0	0	$\eta_6\sqrt{6}\, z_{67}$		$2 s_{56} \bar{z}_{56}$
9	0	0	0	$2 z_{56}$	0

Fig 3.2.8

3) contd.

3.2) contd.

0-block

$$\begin{bmatrix} 0 & z_{13} & 0 \\ s_{13}\bar{z}_{13} & 0 & \sqrt{3}\, z_{37} \\ 0 & s_{37}\sqrt{3}\, z_{37} & 0 \end{bmatrix}$$

1-block

$$\begin{bmatrix} 0 & z_{23} & 0 & z_{26} & 0 & 0 \\ \eta_2 s_{23}\bar{z}_{23} & 0 & s_{23}\bar{z}_{23} & 0 & \sqrt{2}\, z_{37} & 0 \\ 0 & \eta_2 z_{23} & 0 & 0 & 0 & z_{26} \\ s_{26}\bar{z}_{26} & 0 & 0 & 0 & z_{67} & 0 \\ 0 & s_{37}\sqrt{2}\ \bar{z}_{37} & 0 & \eta_6 s_{67}\bar{z}_{67} & 0 & s_{67}\bar{z}_{67} \\ 0 & 0 & s_{26}\bar{z}_{26} & 0 & \eta_6 z_{67} & 0 \end{bmatrix}$$

2-block

$$\begin{bmatrix} 0 & z_{56} & 0 & 0 & 0 \\ s_{56}\bar{z}_{56} & 0 & \sqrt{3}z_{67} & 0 & 0 \\ 0 & \eta_6\sqrt{3}s_{67}\bar{z}_{67} & 0 & \sqrt{3}s_{67}\bar{z}_{67} & 0 \\ 0 & 0 & \eta_6\sqrt{3}z_{67} & 0 & s_{56}\bar{z}_{56} \\ 0 & 0 & 0 & z_{56} & 0 \end{bmatrix}$$

Half Odd-Integral Spin.

The main difference between integral spin and half odd-integral spin theories lies in the fact that in the latter case there can be no self conjugate representations, that is, no representations lying on the ℓ_1 axis. The $B_{k\ell}$ sub-blocks are therefore always even dimensional. The $B_{\ell\ell}$ sub-blocks have non-zero elements in the central positions on the skew diagonal; there elements are

3) contd.

3.2) contd.

$$C^{\tau}_{i(\ell)+\frac{2s-1}{2},\,i(\ell)+\frac{2s+1}{2}} \quad \text{and} \quad s_{i(\ell)+\frac{2s-1}{2},\,i(\ell)+\frac{2s+1}{2}} \quad \overline{C^{\tau}_{i(\ell)+\frac{2s-1}{2},\,i(\ell)+\frac{2s+1}{2}}}$$

As regards Lagrangian origin and $\mathcal{L}_p$ covariance, the same conditions apply as for integral spin theories. For space reflection covariance the only difference is that the cases (c) $\tau=\tau^{\cdot}$ and $\tau'^{\cdot}=\tau'$, or (b) $\tau=\tau^{\cdot}$ and $\tau'\neq\tau'^{\cdot}$ can never occur in half odd integral spin theories. Thus, there is no η_ℓ factor in the $B_{\ell\ell}(s)$ sub-block. It follows that we can use the above algorithm for half-odd integral spin also, except that part (c) is omitted in the construction of the $B_{\ell\ell}(s)$ sub-block, and any instruction elsewhere about η_{i+s} is ignored, whilst the elements above are placed on the skew diagonal. The only difference in the construction of the $B_{\ell\ell+1}(s)$ and $B_{\ell+1\ell}(s)$ sub-blocks is that in part 1) we need only fill in positions <u>above</u> the skew diagonal and not <u>on</u> it. In an even-dimensional matrix the skew diagonal does not contain an element of the leading diagonal.

Having set up the s-blocks in any given theory, we are next interested in their eigenvalues and eigenvectors, and therefore their properties in general. In the next section we see how some simple graph theoretic ideas can provide information about the s-blocks and help us in constructing their characteristic polynomials.

3.3) <u>Properties of the s-blocks</u>.

We are interested in the real non-zero eigenvalues of the s-blocks, and their corresponding eigenvectors. Also, it has been shown by Gelfand and Yaglom[23],[24], and differently

3) contd.

3.3) contd.

by Wild[19], that for spin > 1, good quantizable theories are only possible if L_0 is non-diagonalizable. Thus in any given theory for maximum spin >1, at least one of the s-blocks must be non-diagonalizable and must therefore have repeated eigenvalues.

In general, s-blocks depend on three quantities η_k, $s_{k\ell}$, $z_{k\ell}$, which have the following significance:-

$z_{k\ell}$ = An arbitrary complex number appearing whenever the representations τ_k, τ_ℓ are linked in any s-block. It is possible, by a "permissable" transformation of the basis in the $\mathcal{R}$-space to convert some of these complex numbers to real positive numbers, but this makes little difference, in practice, to the work involved.

η_k = can take the value +1 or -1, depending on which operator is used to represent space reflection in $\mathcal{R}$. It only occurs in integral spin theories.

$s_{k\ell}$: again, can take the values ± 1; in fact:-

$$s_{k\ell} = \frac{a^{\tau_k \tau_k^*}}{a^{\tau_\ell \tau_\ell^*}}$$

$a^{\tau\tau^*}$ being the arbitrary coefficients in the invariant bilinear form (2.3.8), which as stated at (2.3.10) can always be chosen such that $a^{\tau\tau^*} = \pm 1$, and so $s_{k\ell} = \pm 1$. A choice of a set of $s_{k\ell}$ is thus equivalent to a choice of a certain invariant hermitian form for the theory.

3) contd.

3.3) contd.

The conditions imposed on the eigenvalues and eigenvectors of the s-blocks will impose restrictions on the η, s, z and their final forms and arbitrariness delineate the possible theories for a given maximum spin.

If we consider just the eigenvalue problem for an s-block of general form, we immediately meet a complication, even for quite modest s. We cannot solve, in a closed form, an equation of higher than fourth degree; and in practice even a cubic is unpleasant to handle. Thus to get general conditions on the η, s, z we have to resort to use of conditions on the coefficients of the characteristic equations of the s-blocks, which are notoriously complicated; and even then we cannot find general forms for the eigenvectors. This practical obstacle will occur in all higher spin field theories, unless some sort of simplicity assumption is made to reduce the s-blocks, or to simplify the characteristic equations. Otherwise particular values must be substituted for the η, s, z and numerical techniques used. However, we can still make useful and interesting observations on the characteristic equation of the s-blocks, and for this it proves useful to use some ideas from graph theory. Later we will consider reducing the generality of the theories to make the mathematics easier. For a brief review of our needs in graph theory see Appendix B.

We regard the s-rectangle as a graph representing the s-block matrix; that is, non-zero elements can only occur in those positions corresponding to connected nodes of the graph. However, since the s-block is not in general symmetric, we have to split each link in the Bose

3) contd.

3.3) contd.

or Fermi-triangle into two oppositely directed branches, so that we have a directed graph. The branches directed from node i to node j will represent $C_s^{\tau_i\tau_j}$ and that directed from j to i represents $C_s^{\tau_j\tau_i}$. Thus the s-block will be represented by a graph of the type shown in Figure 3.3.1 - provided we have a completely general theory, using all possible representations. If some of the representations are missing then the s-block will be represented by a subgraph of that in the Figure.

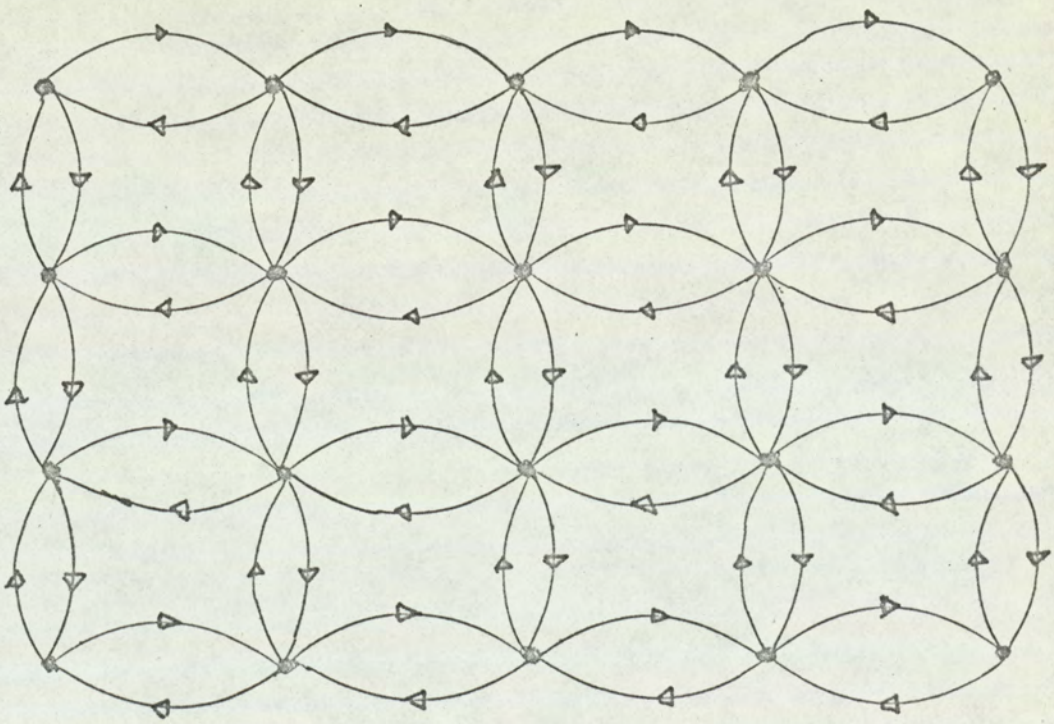


Figure 3.3.1

Let us consider such graphs in general - that is, graphs which are topologically equivalent to a square grid or lattice. We are interested in the eigenvalues of matrices represented by such graphs. First we must say a word or two about the graphical interpretation of a determinant.

Let ζ be any graph with n nodes and $\mathcal{M}(\zeta)$ any

3) contd.

3.3) contd.

n ×n matrix which can only have non-zero elements in the position i,j if the nodes i and j are connected by a branch in ζ. Thus each branch $i \to j$ of ζ represents a possible non-zero element m_{ij} of $\mathcal{M}(\zeta)$. Any term in the determinant $|\mathcal{M}(\zeta)|$ must be of the form

$$\pm m_{1i_1} m_{2i_2} \cdots\cdots m_{ni_n}$$

where $i_1\ i_2 \ldots i_n$ is some permutation:-

$$P = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

of the numbers 1 to n. Since any permutation can be written as a product of disjoint cycles it follows that the non-zero terms of $|\mathcal{M}(\zeta)|$ must correspond to sets of disjoint cycles, or loops of the graph $\zeta^{(34)}$. In the general case, this graphical interpretation of a determinant would be of little use because of the difficulty of searching the graph to find sets of disjoint loops. However, in the case of our simple graph it is quite useful.

Now let G be any lattice graph of the type shown in Figure 3.3.1, with n nodes. Let M(G) be any n × n matrix which can only have non-zero elements in the i,j position if nodes i and j are connected by a branch on the graph G. Then:-

Theorem 3.3.1

If G contains an even number of nodes, n, the characteristic equation of M(G) is of the form

$$|M(G) - \lambda I| = P_1(\lambda^2) = 0$$

3) contd.

3.3) contd.

where P_1 is a polynomial of degree n/2.

If G contains an odd number of nodes, the characteristic equation is of the form

$$|M(G) - \lambda I| = \lambda P_2(\lambda^2) = 0$$

where P_2 is a polynomial of degree (n-1)/2.

Proof.

Let m_{ij} be the i,j element of M(G).

The matrix $M(G) - \lambda I$ is represented by a lattice graph of the type of G, except that each node has a self loop, which corresponds to the non-zero diagonal element $-\lambda$, as shown in Fig.3.3.2. Apart from these self

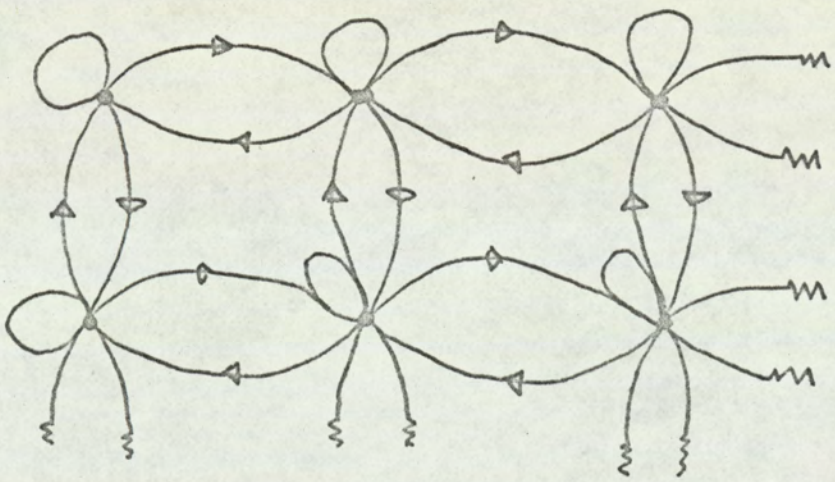


Fig.3.3.2

loops, all loops of such a graph clearly have an even number of branches. The self loops are odd and of length 1. Any non-zero term of the expansion of $|M(G) - \lambda I|$ must correspond to a set of disjoint loops of the graph representing $M(G) - \lambda I$. Hence any term of $|M(G) - \lambda I|$ must be of the form:-

$$\pm m_{i_1 j_1} m_{i_2 j_2} \cdots m_{i_p j_p} (-\lambda)^r$$

where p must be even, since the m-part of this term must

3) contd.

3.3) contd.

correspond to a union of disjoint loops with an even number of branches. The $(-\lambda)^r$ part represents the self loops of r nodes. Now r+p = n, the order of $M(G) - \lambda I$, and the number of nodes in the graph. Since p is even it follows that r is even or odd with n, from which the theorem follows.

Theorem 3.3.2.

Theorem 3.3.1 applies for any subgraph G' of G and also for any union of disjoint graphs of the type of G.

Proof.

Any subgraph of G is obtained by deleting some of the branches of G, which is equivalent to putting appropriate m_{ij} equal to zero in M(G). This would not change the arguments of the proof of theorem 3.3.1.

If G" is a union of disjoint graphs of type G, then by an appropriate numbering of nodes M(G") can always be put into fully reduced form. Blocks on the diagonal of M(G") are then $M(G_i)$ where G_i are the disjoint graphs comprising G". The theorem (3.3.1) applies to each $M(G_i)$, and since the characteristic polynomial of a block diagonal matrix is the product of the characteristic polynomials of the blocks, the second part of the theorem follows.

The proof of theorem 3.3.1 suggests a method of writing down the characteristic polynomial of M(G) directly from the graph G, using it as a visual aid. This method can be made systematic and has a definite advantage over the expansion of the determinant $|M(G) - \lambda I|$. Of course, for large n, either way of finding the characteristic polynomial becomes prohibitive.

The graphical method consists of writing down

3) contd.

3.3) contd.

all the contributions to the coefficient of $(-\lambda)^r$ for each of the necessary values of r (taking account of theorem 3.3.1) by inspection of the graph. Each possible combination of r nodes is inspected (this corresponds to the term containing $(-\lambda)^r$ in the expansion of $|M(G) - \lambda I|$) and the graph is searched for sets of disjoint loops besides the r self loops of these nodes. If, for some particular combination of r nodes, there is no such set of loops having a total of n-r nodes, then the contribution of that particular combination of nodes to the term in $(-\lambda)^r$ is zero. Otherwise the factors corresponding to the sets of disjoint loops are written down as contributions to the coefficient of $(-\lambda)^r$, with a sign which is determined by the parity of the permutation of the node numbers from natural order represented by the set of loops. There is an easy way to find this sign; it is simply $(-1)^{\ell}$ where ℓ is the number of disjoint loops comprising the set of loops. This follows from the nature of our graph, which is clearly such that every loop has an even number of nodes and so corresponds to a permutation cycle of even degree - such a cycle always has parity -1 from the theory of permutations.

As an example of this graphical technique of writing down the characteristic polynomial, we will derive the characteristic polynomial for the graph of Figure 3.3.3, whose matrix can be represented by

3) contd.

3.3) contd.

$$A = \begin{bmatrix} 0 & a_{12} & 0 & a_{14} & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 & a_{25} & 0 \\ 0 & a_{32} & 0 & 0 & 0 & a_{36} \\ a_{41} & 0 & 0 & 0 & a_{45} & 0 \\ 0 & a_{52} & 0 & a_{54} & 0 & a_{56} \\ 0 & 0 & a_{63} & 0 & a_{65} & 0 \end{bmatrix} \qquad (3.3.1)$$

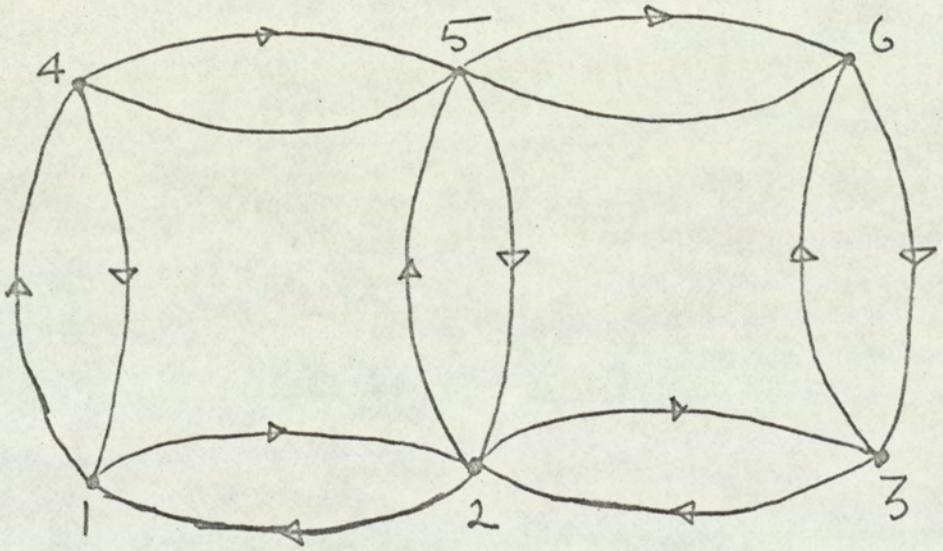


Fig. 3.3.3

Note that A has the general form of the 1-block given at the end of section 3.2 for the general maximum spin two theory. The graph in the figure is that of the 1-rectangle in the Bose triangle for a maximum spin two theory. Thus, this example will be useful in section 4.2, where the spin 2 theory is studied.

We know from Theorem 3.3.1 that the characteristic polynomial will be of the form

$$\Delta(-\lambda) = (-\lambda)^6 + C_4(-\lambda)^4 + C_2(-\lambda)^2 + C_0 \quad .$$

3) contd.

3.3) contd.

We have now to calculate the C_i.

C_2 and C_4

There are 15 possible pairs of nodes and 15 possible sets of four nodes. These two sets of node sets are complements of each other with respect to the set {1,2,3,4,5,6} and are therefore conveniently tabulated together. The node sets, together with their contributions to the coefficients C_2, C_4, obtained by inspection of the graph, are listed in Table I.

TABLE I.

C_4		C_2	
Nodes	Terms	Nodes	Terms
3456	$-a_{12}a_{21}$	12	$a_{45}a_{54}a_{36}a_{63}$
2456	0	13	0
2356	$-a_{14}a_{41}$	14	$a_{25}a_{52}a_{36}a_{63}+a_{23}a_{32}a_{56}a_{65}-a_{23}a_{36}a_{65}a_{52}-a_{25}a_{56}a_{63}a_{32}$
2346	0	15	0
2345	0	16	$a_{45}a_{54}a_{23}a_{32}$
1456	$-a_{23}a_{32}$	23	$a_{14}a_{41}a_{56}a_{65}$
1356	0	24	0
1346	$-a_{25}a_{52}$	25	$a_{14}a_{41}a_{36}a_{63}$
1345	0	26	0
1256	0	34	$a_{12}a_{21}a_{56}a_{65}$
1246	0	35	0
1245	$-a_{36}a_{63}$	36	$a_{14}a_{41}a_{25}a_{52}+a_{12}a_{21}a_{45}a_{54}-a_{25}a_{54}a_{41}a_{12}-a_{21}a_{14}a_{45}a_{52}$
1236	$-a_{45}a_{54}$	45	$a_{12}a_{21}a_{36}a_{63}$
1235	0	46	0
1234	$-a_{56}a_{65}$	56	$a_{14}a_{41}a_{23}a_{32}$

C_0

This is the sum of all terms corresponding to sets of disjoint loops containing a total of six branches. By inspection of the graph we find:-

3) contd.

3.3) contd.

$$C_o = + a_{14}a_{41}a_{23}a_{36}a_{65}a_{52}$$
$$+ a_{14}a_{41}a_{25}a_{56}a_{63}a_{32}$$
$$+ a_{36}a_{63}a_{25}a_{54}a_{41}a_{12}$$
$$+ a_{36}a_{63}a_{21}a_{14}a_{45}a_{52}$$
$$- a_{14}a_{41}a_{25}a_{52}a_{36}a_{63}$$
$$- a_{14}a_{41}a_{56}a_{65}a_{23}a_{32}$$
$$- a_{36}a_{63}a_{45}a_{54}a_{12}a_{21}$$
$$- a_{12}a_{23}a_{36}a_{65}a_{54}a_{41}$$
$$- a_{14}a_{45}a_{56}a_{63}a_{32}a_{21}$$

Collecting together the above results we get for the characteristic equation of **A**:-

$$\Delta(\lambda) = \lambda^6-(a_{12}a_{21}+a_{23}a_{32}+a_{14}a_{41}+a_{25}a_{52}+a_{36}a_{63}+a_{45}a_{54}+a_{56}a_{65})\lambda^4$$
$$+(a_{45}a_{54}a_{36}a_{63}+a_{25}a_{52}a_{36}a_{63}+a_{23}a_{32}a_{56}a_{65}+a_{45}a_{54}a_{23}a_{32}$$
$$+ a_{14}a_{41}a_{56}a_{65}+a_{14}a_{41}a_{36}a_{63}+a_{12}a_{21}a_{56}a_{65}+a_{14}a_{41}a_{25}a_{52}$$
$$+ a_{12}a_{21}a_{45}a_{54}+a_{12}a_{21}a_{36}a_{63}+a_{14}a_{41}a_{23}a_{32}-a_{23}a_{36}a_{65}a_{52}$$
$$- a_{25}a_{56}a_{63}a_{32}-a_{25}a_{54}a_{41}a_{12}-a_{21}a_{14}a_{45}a_{52})\lambda^2$$
$$-(a_{14}a_{41}a_{25}a_{52}a_{36}a_{63}+a_{14}a_{41}a_{56}a_{65}a_{32}a_{23}+a_{36}a_{63}a_{45}a_{54}a_{12}a_{21}$$
$$+a_{12}a_{23}a_{36}a_{65}a_{54}a_{41}+a_{14}a_{45}a_{56}a_{63}a_{32}a_{21}-a_{14}a_{41}a_{23}a_{36}a_{65}a_{52}$$
$$-a_{14}a_{41}a_{25}a_{56}a_{63}a_{32}-a_{36}a_{63}a_{25}a_{54}a_{41}a_{12}-a_{36}a_{63}a_{21}a_{14}a_{45}a_{52})$$
$$= 0 \qquad (3.3.2)$$

Notice that the coefficient of $(-\lambda)^4$ is simply the sum of the terms corresponding to all the loops of length two in the graph. This is no coincidence, and in general:-

Theorem 3.3.3

The coefficient of $(-\lambda)^{n-2}$ in the characteristic

3) contd.

3.3) contd.

polynomial of a lattice type graph with n nodes is equal to the sum of the terms corresponding to all the loops of length two in the graph.

Proof.

Clearly only loops of length two can contribute to $(-\lambda)^{n-2}$. Also, every loop contributes at most once, since it corresponds to one possible selection of n-2 nodes of the graph. Further each loop contributes at least once, since corresponding to the pair of nodes of the loop there is a complementary set of n-2 nodes giving a factor $(-\lambda)^{n-2}$. This proves the theorem.

Theorem 3.3.3 will prove useful in finding the mass states in a particular simplified type of theory to which we will later restrict ourselves.

We now return to the study of the s-blocks. Applying theorems (3.3.1) and (3.3.2) in this case we get

Theorem (3.3.4).

The characteristic polynomial of an s-block containing an even number of representations is of the form

$$\Delta(\lambda) = P_1(\lambda^2) \qquad (3.3.3)$$

and that of an s-block containing an odd number of representations is of the form

$$\Delta(\lambda) = (-\lambda)\, P_2(\lambda^2) \qquad (3.3.4)$$

where P_1 and P_2 are polynomials of appropriate degree. This applies for the s-blocks of a theory based on any reducible representation of $\mathcal{L}_p$.

Theorem (3.3.4) is a stronger result than that

3) contd.

3.3) contd.

obtained in Section 2.4. It was first shown by Bhabha that the characteristic polynomial $|L_o - \lambda I|$ of L_o is a polynomial in $(-\lambda)^2$. (3.3.3) and (3.3.4) show that the field states of a given spin occur in pairs with "masses" of opposite sign, as discussed in Section 2.4.

We can further show for the s-blocks that:-

Theorem 3.3.5

The characteristic and minimum polynomials of the s-blocks have real coefficients.

Proof

Let A_s be any s-block. We have, from the hermiticity of L_o with respect to the bilinear form (,), (see (2.3.5), (2.3.2)):-

$$L_o^\dagger \Lambda = \Lambda L_o$$

where Λ is the matrix of the bilinear form (2.3.8). From (2.3.8) we see that Λ does not mix up the s-subspaces and so if Λ_s is the restriction of Λ to the s-subspace, we have

$$A_s^\dagger \Lambda_s = \Lambda_s A_s \qquad (3.3.5)$$

$$(A_s^\dagger - xI)\Lambda_s = \Lambda_s (A_s - xI)$$

where x is an arbitrary complex number. Since (,) is non-degenerate, we have

$$|\Lambda_s| \neq 0$$

and so

$$|A_s^\dagger - xI| = |A_s - xI|$$

or

$$\overline{|A_s - \bar{x}I|} = |A_s - xI|$$

$$= \Delta(x)$$

$$= \sum_{r=o}^{n} C_r x^r$$

3) contd.

3.3) contd.

$\Delta(x)$ being the characteristic polynomial of the $n \times n$ A_s, and C_r are its coefficients. So

$$\overline{\sum_{r=0}^{n} C_r \bar{x}^r} = \sum_{r=0}^{n} C_r x^r$$

or

$$\sum_{r=0}^{n} \bar{C}_r x^r = \sum_{r=0}^{n} C_r x^r$$

whence, since x is arbitrary,

$$\bar{C}_r = C_r$$

as required.

Now let the minimum polynomial of A_s be

$$m(x) = \sum_{r=0}^{m} D_r x^r$$

So

$$m(A_s) = \sum_{r=0}^{m} D_r A_s^r = 0.$$

Then since

$$A_s^\dagger = \Lambda_s A_s \Lambda_s^{-1}$$

we also have

$$m(A_s^\dagger) = \Lambda_s m(A_s) \Lambda_s^{-1} = 0$$

Hence

$$m(A_s^\dagger)^\dagger = \sum_{r=0}^{m} \bar{D}_r A_s^r = 0$$

By the uniqueness of the minimum polynomial

3) contd.

3.3) contd.

$$\overline{D}_r = D_r$$

as required.

Theorem 3.3.5 provides a rough check on the rather complicated expressions for the coefficients of the characteristic and minimal equations in practice.

Earlier in this section we remarked on the difficulties of carrying through any general analysis for the eigenvalues and eigenvectors of the s-blocks, because of their size and the complexity of the characteristic equation. It seems we must therefore make some simplifying assumption which will permit more convenient discussion of the s-blocks; of the many ways we could do this, the following is the most reasonable:-

Assumption.

No two particle-anti-particle pairs of states in the theory will have the same spin.

This leads to a considerable mathematical simplification, since it means that any s-block can only have two non-zero eigenvalues, $\pm m_s$, say. Thus the characteristic polynomial of an $n \times n$ s-block must now be of the form

$$\Delta(\lambda) = \lambda^{n-2}(\lambda^2 - m_s^2) \qquad (3.3.6)$$

where m_s^2 is positive.[47] m_s^2 is very easy to obtain graphically, since from Theorem 3.3.3 it is simply the sum of the terms corresponding to all the loops of length two on the graph representing the s-block. In general the coefficients C_i of the characteristic polynomial of an s-block will be complicated functions of η_k, $s_{k\ell}$, $z_{k\ell}$. The above assumption

3) contd.

3.3) contd.

requires that we impose the condition

$$C_i = 0 \qquad i < n-2 \qquad (3.3.7)$$

which will further restrict the arbitrary quantities η, s, z. Further, for a real mass we must have

$$m_s^2 \geq 0 \qquad (3.3.8)$$

which gives a further restriction on the arbitrary parameters of our theory. Finally for a consistent quantizable theory we require certain conditions, detailed in Section 2.5, to be satisfied by the quantities

$$\epsilon_+(r,s) = \psi^\dagger_{+rs} \Lambda_s A_s \psi_{+rs} \qquad (3.3.9)$$

$$\epsilon_-(r,s) = \psi^\dagger_{-rs} \Lambda_s A_s \psi_{-rs} \quad , \qquad (3.3.10)$$

where $\psi_{\pm rs}$ are eigenvectors of the s-block A_s corresponding to non-zero eigenvalues $\pm \mu_r$. These conditions are studied in the next section.

3.4) Quantizable Field Theories and Conditions on the s-blocks.

In this section A will represent a general $n \times n$ s-block with, following the assumption of Section 3.3, eigenvalues $\pm$ m (each once) and zero (in general, repeated). Thus the characteristic polynomial of A will be

$$\Delta(\lambda) = \lambda^{n-2}(\lambda^2 - m^2) = 0 \qquad (3.4.1)$$

Since the minimum polynomial of a matrix contains the same irreducible factors as the characteristic polynomial, the minimum polynomial of A will be of the form

$$m(\lambda) = \lambda^q(\lambda^2 - m^2) \qquad (3.4.2)$$

3) contd.

3.4) contd.

where $1 \leq q \leq n-2$. So A satisfies the minimum equation

$$A^q(A^2 - m^2) = 0 \qquad (3.4.3)$$

A is diagonalizable if and only if $q = 1$.

If $\psi_{\pm m}$ represent eigenvectors of A corresponding to eigenvalues $\pm m$ respectively, then from (3.3.9) and (3.3.10) we are interested in the signs of the quantities

$$\epsilon_+(m) = \psi_{+m}^{\dagger} \Lambda A \psi_{+m} \qquad (3.4.4)$$

$$\epsilon_-(m) = \psi_{-m}^{\dagger} \Lambda A \psi_{-m} \qquad (3.4.5)$$

where Λ is always to be taken as the restriction to the s-subspace, i.e. Λ_s.

Suppose A satisfies the minimum polynomial (3.4.3). Then A also satisfies

$$A^p (A^2 - m^2) = 0 \qquad p \geq q. \qquad (3.4.6)$$

Now consider the matrices

$$P_{\pm m}(A) = A^p(A \pm m). \qquad (3.4.7)$$

Neither of these is zero, because each lacks an irreducible factor of the minimum polynomial. Also, $P_m(A)$ satisfies

$$(A - m)P_m(A) = 0 \qquad (3.4.8)$$

and so the columns of $P_m(A)$ are either zero or eigenvectors of A corresponding to eigenvalue m. There is only one such independent vector and so $P_m(A)$ has exactly one independent column, that is:-

$$\text{Rank}(P_m(A)) = 1 \qquad (3.4.9)$$

An eigenvector of A corresponding to eigenvalue $+m$ is thus

3) contd.

3.4) contd.

$$\psi_m = P_m(A)\,\psi \tag{3.4.10}$$

where ψ is an arbitrary vector.

Similarly $P_{-m}(A)$ satisfies

$$(A+m)P_{-m}(A) = 0, \tag{3.4.11}$$

has $\text{Rank}(P_{-m}(A)) = 1$ and supplies an eigenvector

$$\psi_{-m} = P_{-m}(A)\,\psi \tag{3.4.12}$$

ψ arbitrary,

of A corresponding to eigenvalue -m.

We can therefore write, from (3.4.4)

$$\begin{aligned}\text{sign}(\epsilon_+(m)) &= \text{sign}(\psi^\dagger P_m^\dagger(A)\Lambda A P_m(A)\psi)\\ &= \text{sign}(\psi^\dagger \Lambda A P_m(A)P_m(A)\psi)\\ &\qquad \text{using (3.3.5)}\\ &= \text{sign}(mP_m(m)\,\psi^\dagger \Lambda P_m(A)\psi)\\ &\qquad \text{from (3.4.8)}, \; P_m(A)P_m(A) = P_m(m)P_m(A),\\ &= \text{sign}(\psi^\dagger \Lambda P_m(A)\psi) \qquad (3.4.13)\\ &\qquad \text{since } m > 0.\end{aligned}$$

Similarly, from (3.4.5), (3.3.5), (3.4.11) we get:-

$$\text{sign}(\epsilon_-(m)) = (-1)^p\ \text{sign}(\psi^\dagger \Lambda P_{-m}(A)\psi) \tag{3.4.14}$$

Now consider $\psi^\dagger \Lambda P_m(A)\psi$. Since ψ is arbitrary this is positive or negative as $\Lambda P_m(A)$ is positive or negative semi-definite. But $\text{Rank}(\Lambda P_m(A)) = \text{Rank}(P_m(A)) = 1$, since Λ is non-singular. Hence $\Lambda P_m(A)$ has one non-zero eigenvalue, which will be positive or negative as $\Lambda P_m(A)$ is positive or negative semi-definite, as $\Lambda P_m(A)$ is hermitian. This eigenvalue is given by

$$\rho_m = \text{Trace}\,(\Lambda P_m(A)) = \text{Tr}(\Lambda P_m(A)) \tag{3.4.15}$$

since the trace of a matrix is the sum of its eigenvalues.

3) contd.

3.4) contd.

So finally

$$\text{sign}(\psi^{\dagger}\Lambda P_m(A)\psi) = \text{sign}(\text{Tr}(\Lambda P_m(A)) \tag{3.4.16}$$

Similarly, we find

$$\text{sign}(\psi^{\dagger}\Lambda P_{-m}(A)\psi) = \text{sign}(\text{Tr}(\Lambda P_{-m}(A))) \tag{3.4.17}$$

So finally we get

$$\text{sign}(\epsilon_+(m)) = \text{sign}(\text{Tr}(\Lambda P_m(A))) \tag{3.4.18}$$

$$\text{sign}(\epsilon_-(m)) = (-1)^p \text{ sign}(\text{Tr}(\Lambda P_{-m}(A))) \quad p \geq q \tag{3.4.19}$$

Thus our problem of definiteness of charge or energy has been reduced to one of finding the signs of the quantities

$$\rho_m = \text{Tr}(\Lambda P_m(A)) \tag{3.4.20}$$

$$\rho_{-m} = \text{Tr}(\Lambda P_{-m}(A)) \tag{3.4.21}$$

for our particular type of theory, specified by the assumption of Section 3.3. Now to calculate ρ_m and ρ_{-m} we need to find quantities like

$$\text{Trace}(\Lambda A^{\ell})$$

where ℓ is some natural number. Now

$$\text{Tr}(\Lambda A^{\ell}) = \sum_{i,j=1}^{n} (\Lambda)_{ij}(A^{\ell})_{ji} \tag{3.4.22}$$

and $(\Lambda)_{ij} = 0$ unless row i and column j correspond to two mutually conjugate representations on the s-rectangle, in which case

$$(\Lambda)_{ij} = (-1)^{[s]}\, a^{\tau\tau\cdot} \tag{3.4.23}$$

where row i corresponds to the representation τ and column j to the representation $\tau\cdot$. It follows therefore from (3.4.22)

3) contd.

3.4) contd.

that the coefficient of $(-1)^{[s]}\, a^{\tau\tau\cdot}$ in $\mathrm{Tr}(\Lambda A^{\ell})$ is $(A^{\ell})_{\tau\cdot\tau}$, in obvious notation, which is given by the sum of the terms corresponding to all paths of length ℓ from the representation $\tau\cdot$ to its conjugate representation τ on the graph of the s-rectangle. In particular, it is clear from the corresponding graphs in the Bore and Fermi-planes that:-

For Integral spin theories there are no paths of odd length between a representation and its conjugate.

For half odd integral spin theories there are no paths of even length between a representation and its conjugate.

It therefore follows that:-

1) For Integral spin s-blocks:-

$$\mathrm{Tr}(\Lambda A^{\ell}) = 0 \qquad \text{if } \ell \text{ odd} \tag{3.4.24}$$

2) For Half odd integral spin s-blocks

$$\mathrm{Tr}(\Lambda A^{\ell}) = 0 \qquad \text{if } \ell \text{ even.} \tag{3.4.25}$$

Now consider

$$\Lambda P_m(A) = \Lambda A^p (A + m)$$

we have

$$\mathrm{Tr}(\Lambda P_m(A)) = \mathrm{Tr}(\Lambda A^{p+1}) + m\mathrm{Tr}(\Lambda A^p)$$

Then, for integral spin s-blocks the above results give:-

$$\left.\begin{aligned} \mathrm{Tr}(\Lambda P_m(A)) &= \mathrm{Tr}(\Lambda A^{p+1}) \quad \text{if p odd} \\ &= m\mathrm{Tr}(\Lambda A^p) \quad \text{if p even} \end{aligned}\right\} \tag{3.4.26}$$

And for half odd integral spin s-blocks we have

$$\left.\begin{aligned} \mathrm{Tr}(\Lambda P_m(A)) &= m\mathrm{Tr}(\Lambda A^p) \quad \text{if p is odd} \\ &= \mathrm{Tr}(\Lambda A^{p+1}) \quad \text{if p is even} \end{aligned}\right\} \tag{3.4.27}$$

Similarly for $\Lambda P_{-m}(A)$, we have

3) contd.

3.4) contd.

$$\mathrm{Tr}(\Lambda P_{-m}(A)) = \mathrm{Tr}(\Lambda A^{p+1}) - m\mathrm{Tr}(\Lambda A^{p})$$

so for integral spin s-blocks:-

$$\left.\begin{aligned}\mathrm{Tr}(\Lambda P_{-m}(A)) &= \mathrm{Tr}(\Lambda A^{p+1}) \text{ if } p \text{ is odd} \\ &= m\mathrm{Tr}(\Lambda A^{p}) \text{ if } p \text{ is even}\end{aligned}\right\} \qquad (3.4.28)$$

And for half odd integral spin s-blocks:-

$$\left.\begin{aligned}\mathrm{Tr}(\Lambda P_{-m}(A)) &= -m\mathrm{Tr}(\Lambda A^{p}) \quad \text{if } p \text{ is odd} \\ &= \mathrm{Tr}(\Lambda A^{p+1}) \quad \text{if } p \text{ is even}\end{aligned}\right\} \qquad (3.4.29)$$

Using these results in (3.4.18) and (3.4.19) we can simplify the conditions for quantizable theories. We summarize the results here for integral and half odd integral spin theories separately.

Integral Spin.

If A satisfies the characteristic equation

$$A^{n-2}(A^2 - m^2) = 0$$

and the minimal equation

$$A^{q}(A^2 - m^2) = 0$$

where $1 \leqslant q \leqslant p$, then

$$\left.\begin{aligned}\mathrm{sign}(\epsilon_{+}(m)) &= \mathrm{sign}(\mathrm{Tr}(\Lambda A^{p+1})) \quad p \text{ odd} \\ &= \mathrm{sign}(\mathrm{Tr}(\Lambda A^{p})) \quad p \text{ even}\end{aligned}\right\} \qquad (3.4.30)$$

$$\left.\begin{aligned}\mathrm{sign}(\epsilon_{-}(m)) &= -\mathrm{sign}\ \mathrm{Tr}(\Lambda A^{p+1}) \quad p \text{ odd} \\ &= -\mathrm{sign}\ \mathrm{Tr}(\Lambda A^{p}) \quad p \text{ even}\end{aligned}\right\} \qquad (3.4.31)$$

We note that in this case of integral spin

$$\mathrm{sign}(\epsilon_{+}(m)) = -\ \mathrm{sign}(\epsilon_{-}(m)). \qquad (3.4.32)$$

This is part of the condition for consistent quantization stated in Section 2.5, for integral spin fields. Thus in the notation of Section 2.5, our special type of theory satisfies the condition that, for integral spin fields

3) contd.

3.4) contd.

$\epsilon_+(o,r,s)$ and $\epsilon_-(o,r,s)$ should have the opposite sign. The other part of the condition for consistent quantization is that the different $\epsilon_+(m)$ for each s-block must have the same sign. Without loss of generality we can always take this sign to be positive. Then the program for looking for quantizable integral spin theories is as follows.

Examine each s-block A_s and ensure that its characteristic polynomial is

$$\Delta(\lambda) = \lambda^{n-2}(\lambda^2-m^2)$$

Choose some integer p, $1 \leqslant p \leqslant n-2$ and ensure that A_s satisfies

$$A_s{}^p(A_s{}^2 - m_s{}^2) = 0$$

This need not be the minimal equation, which may be

$$A_s{}^q(A_s{}^2 - m_s{}^2) = 0$$

for any q such that $1 \leqslant q \leqslant p$. Then we get a quantizable theory if we ensure that

$$\left.\begin{array}{ll} \mathrm{Tr}(\Lambda_s A_s{}^{p+1}) > 0 & p \text{ odd} \\ \text{or} \quad \mathrm{Tr}(\Lambda_s A_s{}^{p}) > 0 & p \text{ even} \end{array}\right\} \qquad (3.4.33)$$

These trace conditions are final conditions on the η, s, z for a consistent quantizable relativistic free field theory, for integral spin.

Half Odd Integral Spin.

If A satisfies the characteristic equation

$$A^{n-2}(A^2 - m^2) = 0$$

and the minimal equation

$$A^q(A^2 - m^2) = 0$$

where $1 \leqslant q \leqslant p$, then

3) contd.

3.4) contd.

$$\left.\begin{aligned} \text{sign}(\epsilon_+(m)) &= \text{sign}(\text{Tr}(\Lambda A^p)) \quad p \text{ odd} \\ &= \text{sign}\,(\text{Tr}(\Lambda A^{p+1})) \quad p \text{ even} \end{aligned}\right\} \tag{3.4.34}$$

$$\left.\begin{aligned} \text{sign}(\epsilon_-(m)) &= \text{sign}(\text{Tr}(\Lambda A^p)) \quad p \text{ odd} \\ &= \text{sign}(\text{Tr}(\Lambda A^{p+1})) \quad p \text{ even} \end{aligned}\right\} \tag{3.4.35}$$

We note that in this case of half odd integral spin

$$\text{sign}(\epsilon_+(m)) = \text{sign}(\epsilon_-(m)) \tag{3.4.36}$$

This is part of the condition for consistent quantization of half odd integral spin fields, stated in Section 2.5. Again, as in the case of integral spin fields we will adopt the convention that $\epsilon_+(m) > 0$, then for consistent quantization it is necessary that the $\epsilon_+(m)$ for each s-block should be positive. The program for looking for quantizable half odd integral spin theories is thus as follows.

Examine each s-block A_s and ensure that its characteristic polynomial is

$$\Delta(\lambda) = \lambda^{n-2}(\lambda^2 - m^2)$$

Choose some integer p, $1 \leqslant p \leqslant n-2$ and ensure that A_s satisfies

$$A_s{}^p(A_s{}^2 - m_s{}^2) = 0$$

Then we get a quantizable theory if we ensure that

$$\left.\begin{aligned} \text{Tr}(\Lambda_s A_s{}^{p+1}) &> 0 \quad \text{if } p \text{ even} \\ \text{Tr}(\Lambda_s A_s{}^p) &> 0 \quad \text{if } p \text{ odd} \end{aligned}\right\} \tag{3.4.37}$$

These impose the final conditions on the η, s, z.

The above results tell us how to find good quantizable theories for higher spin. The major practical

3) contd.

3.4) contd.

difficulties arise in ensuring that the s-blocks, A_s, satisfy the required equations, and the trace conditions specified. We obtain sets of equations in the η, z, s and any solutions to these equations give good theories. While there certainly is still a lot of work involved to obtain and solve these equations, the preliminary work we have done in this section simplifies the job a great deal. In practical calculations it is useful to remember that the i,j position in A_s^{ℓ} is given by the sum of terms corresponding to all the possible paths of length ℓ from the representation corresponding to i to that corresponding to j, in the s-rectangle. Thus, the graphs of the s-rectangle provide a useful visual aid to the practical calculations required in the search for good theories. In the next Chapter we study some particular examples of"good"theories. Also, in the examples it will become clear that some labour can be saved in calculating the Traces involved.

CHAPTER 4.

EXAMPLES OF QUANTIZABLE THEORIES

4) Examples of Quantizable Theories.

4.1) Spin 0 and Spin 1 Theories.

The first thing to note is that with the assumption 2 of Section 3.2, it is not possible to have a spin 0 theory! This is simply because, in order for the spin zero theory to be put in the form of a first order equation we must introduce another representation besides $\tau_1 \sim (0,1)$. If τ_1 alone is used then the field must satisfy a second order equation - the Klein Gordon equation. This situation arises because τ_1 is the only representation with maximum spin 0. So, for the special case of spin 0 we break with our assumption of Section 3.2. Of course, the easiest way to get a first order spin zero theory is to introduce the representation $\tau_3 \sim (0,2)$. This is simply equivalent to converting the second order equation to a first order equation by the process used in (2.1.1) to (2.1.4). It is then convenient to consider the spin 0 and spin 1 theory together, the appropriate triangle appearing in Figure 4.1.1.

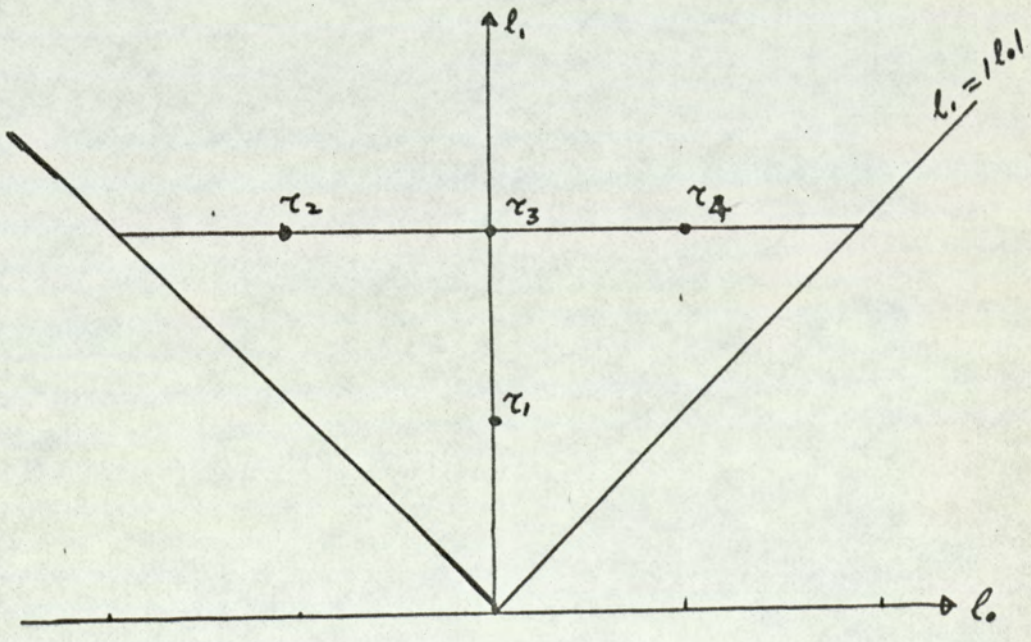


Fig.4.1.1.

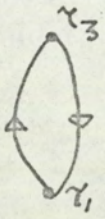
4) contd.

4.1) contd.

Using (3.2.3), the 0-block is 2 × 2 and the 1-block is 3 × 3. Applying the integral spin algorithm of Section 3.2 to the graphs of the 0-block and 1-block we have:-

<u>0-block.</u>

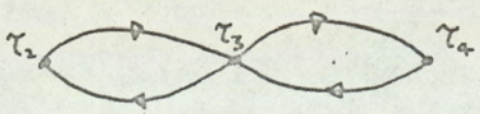
Graph



$$0\text{-block} = A_0 = \begin{bmatrix} 0 & \sqrt{2}\, z_{13} \\ s_{13}\sqrt{2}\, \bar{z}_{13} & 0 \end{bmatrix} \qquad (4.1.1)$$

<u>1-block.</u>

Graph



$$1\text{-block} = A_1 = \begin{bmatrix} 0 & \sqrt{2}\, z_{23} & 0 \\ \eta_2 s_{23}\sqrt{2}\, \bar{z}_{23} & 0 & s_{23}\sqrt{2}\, \bar{z}_{23} \\ 0 & \eta_2\sqrt{2}\, z_{23} & 0 \end{bmatrix} \qquad (4.1.2)$$

The eigenvalues of the 0-block are

$$\lambda = \pm m_0 = \pm \sqrt{2 s_{13} p_{13}} \qquad (4.1.3)$$

giving $s_{13} = +1$ for real mass. Here we are using the notation

$$p_{k\ell} = |z_{k\ell}|^2 \qquad (4.1.4)$$

The characteristic equation of the 1-block is

$$-\lambda(\lambda^2 - 4\eta_2 s_{23} p_{23}) = 0 \qquad (4.1.5)$$

giving eigenvalues

4) contd.

4.1) contd.

$$\lambda = 0$$

and $$\lambda = \pm m_1 = \pm 2\sqrt{\eta_2 s_{23} p_{23}} \qquad (4.1.6)$$

giving $\eta_2 = s_{23}$ for real mass.

Applying the results of Section 3.4, noting that both for the 0-block and 1-block there is only one choice for the minimal equation we see that for positive energy contribution we must have

0-block $$\mathrm{Tr}(\Lambda_0) > 0 \qquad (4.1.7)$$

1-block $$\mathrm{Tr}(\Lambda_1 A_1^2) > 0 \qquad (4.1.8)$$

From (4.1.7) we get

$$a^{\tau_1\tau_1} = +1 \qquad (4.1.9)$$

From (4.1.8) we get

$$\mathrm{Tr}(\Lambda_1 A_1^2) = (-1)^{[1]}\Big[a^{\tau_2\tau_2\bullet}(2s_{23}p_{23}) + a^{\tau_3\tau_3}(4\eta_2 s_{23}p_{23}) + a^{\tau_2\tau_2\bullet}(2s_{23}p_{23})\Big]$$

$$= -4(a^{\tau_2\tau_2\bullet}s_{23}p_{23} + a^{\tau_3\tau_3}\eta_2 s_{23}p_{23})$$

$$= -4a^{\tau_3\tau_3}(p_{23} + \eta_2 s_{23}p_{23})$$

$$= -8a^{\tau_3\tau_3}p_{23} \quad \text{since from (4.1.6), } \eta_2 s_{23} = +1$$

$$> 0$$

Hence (4.1.8) gives

$$a^{\tau_3\tau_3} = -1 \qquad (4.1.10)$$

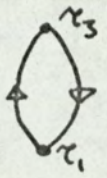
We note that we cannot have a spin 0 and spin 1 theory together here, since (4.1.9), (4.1.10) and $s_{13} = 1$ are incompatible. If we have just a spin 0 theory, then we must have, from (4.1.6)

$$p_{23} = 0$$

which reduces us to the theory based on the graph

4) contd.

4.1) contd.



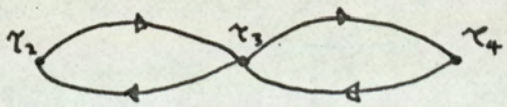
This is in fact the usual spin zero theory. We have $s_{13} = +1$, and the masses are:-

$$\pm \frac{\chi}{\sqrt{2}\, p_{13}} = \pm \frac{\chi}{\sqrt{2}|z_{13}|}$$

If we have just a spin 1 theory, then we must have, from (4.1.3)

$$p_{13} = 0$$

which reduces us to the theory based on the graph



This is the usual spin one theory studied by Gelfand, et al. We have $a^{\tau_3\tau_3} = -1$ and $\eta_{33} s_{33} = +1$, and the masses are:-

$$\pm \frac{\chi}{2\sqrt{p_{33}}} = \pm \frac{\chi}{2\,|z_{33}|}$$

4.2) <u>The Spin 2 Theory.</u>

We look for quantizable theories describing a spin 2 state and possibly also a spin 0 or spin 1 state. Initially we will keep the analysis as general as possible, but eventually it will be necessary to restrict our theories to avoid excessive algebra. What we will do will be sufficient to illustrate the procedure to be followed in any case.

4) contd.

4.2) contd.

The 0,1,2 blocks for the general spin 2 theory have been constructed in Section 3.2. They are

0-block

$$A_0 = \begin{bmatrix} 0 & z_{13} & 0 \\ s_{13}\bar{z}_{13} & 0 & \sqrt{3}\, z_{37} \\ 0 & s_{37}\sqrt{3}\,\bar{z}_{37} & 0 \end{bmatrix} \qquad (4.2.1)$$

1-block

$$A_1 = \begin{bmatrix} 0 & z_{23} & 0 & z_{26} & 0 & 0 \\ \eta_2 s_{23}\bar{z}_{23} & 0 & s_{23}\bar{z}_{23} & 0 & \sqrt{2}\, z_{37} & 0 \\ 0 & \eta_2 z_{23} & 0 & 0 & 0 & z_{26} \\ s_{26}\bar{z}_{26} & 0 & 0 & 0 & z_{67} & 0 \\ 0 & s_{37}\sqrt{2}\,\bar{z}_{37} & 0 & \eta_6 s_{67}\bar{z}_{67} & 0 & s_{67}\bar{z}_{67} \\ 0 & 0 & s_{26}\bar{z}_{26} & 0 & \eta_6 z_{67} & 0 \end{bmatrix} \qquad (4.2.2)$$

2-block

$$A_2 = \begin{bmatrix} 0 & z_{56} & 0 & 0 & 0 \\ s_{56}\bar{z}_{56} & 0 & \sqrt{3}\, z_{67} & 0 & 0 \\ 0 & \sqrt{3}\eta_6 s_{67}\bar{z}_{67} & 0 & \sqrt{3}s_{67}\bar{z}_{67} & 0 \\ 0 & 0 & \eta_6\sqrt{3}z_{67} & 0 & s_{56}\bar{z}_{56} \\ 0 & 0 & 0 & z_{56} & 0 \end{bmatrix} \qquad (4.2.3)$$

We will assume that no $z_{k\ell}$ is zero for the present. We study each block in turn, investigating the conditions imposed on the η_k, $s_{k\ell}$, $z_{k\ell}$ by the requirements detailed in Section 3.4 ensuring a quantizable theory. We have a certain amount of choice as to the minimal polynomial of the s-blocks in this case, and each choice will lead to different conditions on the η_k, $s_{k\ell}$, $z_{k\ell}$, which have to be investigated.

4) contd.

4.2) contd.

0-Block.

The characteristic equation is easily calculated in this case as

$$-\lambda[\lambda^2 - (s_{13}p_{13}+3s_{37}p_{37})] = 0 \qquad (4.2.4)$$

So the eigenvalues of the 0-block are

$$\lambda = 0$$

$$\lambda = \pm m_0 = \pm \sqrt{s_{13}p_{13}+3s_{37}p_{37}} \qquad (4.2.5)$$

There are now two possibilities:-

i) $s_{13}p_{13} + 3s_{37}p_{37} \neq 0$

In this case a spin 0 state is carried by the theory, and to make it a physical state we must have real mass, so:-

$$s_{13}p_{13} + 3s_{37}p_{37} > 0 \qquad (4.2.6)$$

All the eigenvalues are different and so the 0-block will in this case be diagonalizable, and its minimum equation will coincide with its characteristic equation

$$m(A_0) = A_0[A_0^2 - (s_{13}p_{13}+3s_{37}p_{37})] = 0 \ .$$

For the spin 0 state to make a positive energy contributions we must therefore have, from Section 3.4

$$\mathrm{Tr}(\Lambda_0 A_0^2) > 0 \qquad (4.2.7)$$

The 0-block graph is shown in Fig.4.2.1. Using

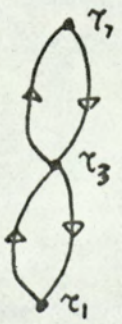


Fig.4.2.1.

4) contd.

4.2) contd.

i) contd.

this as a visual aid, as described in the proof of (3.4.24) and (3.4.25), we find

$$\mathrm{Tr}(\Lambda_o A_o^2)=(-1)^o\left[a^{\tau_1\tau_1}s_{13}p_{13}+a^{\tau_3\tau_3}(s_{13}p_{13}+3s_{37}p_{37})+a^{\tau_7\tau_7}3s_{37}p_{37}\right]$$

$$= a^{\tau_7\tau_7}(s_{13}^2 s_{37}p_{13}+s_{37}(s_{13}p_{13}+3s_{37}p_{37})+3s_{37}p_{37})$$

$$= a^{\tau_7\tau_7}s_{37}((1+s_{13})p_{13}+3(1+s_{37})p_{37})$$

$$= a^{\tau_3\tau_3}((1+s_{13})p_{13}+3(1+s_{37})p_{37}) \qquad (4.2.8)$$

We require $\mathrm{Tr}(\Lambda_o A_o^2) > 0$, and since from (4.2.6):-

$$(1+s_{13})p_{13}+3(1+s_{37})p_{37} > 0 \qquad (4.2.9)$$

this implies

$$a^{\tau_3\tau_3} > 0 \quad ; \text{ i.e. } a^{\tau_3\tau_3} = +1 \qquad (4.2.10)$$

So, for the spin 0 state to contribute positive energy we must have $a^{\tau_3\tau_3} = +1$. The masses of the spin 0 states will then be

$$\pm\frac{\chi}{m_o} = \pm\frac{\chi}{\sqrt{s_{13}p_{13}+3s_{37}p_{37}}} \qquad (4.2.11)$$

ii) $\underline{s_{13}p_{13}+3s_{37}p_{37} = 0}$

In this case all the eigenvalues of the 0-block are zero, and there are no physical spin 0 states. In this situation we have:-

$$p_{13} = -3s_{13}s_{37}p_{37}$$

giving, since all $p_{k\ell} > 0$:-

$$s_{13} = -s_{37} \text{ and } p_{13} = 3p_{37}. \qquad (4.2.12)$$

4) contd.

4.2) contd.

1-Block.

Comparing the 1-block (4.2.2) with (3.3.1) we see from (3.3.2) that the characteristic equation of the 1-block is

$$\lambda^6 + C_4\lambda^4 + C_2\lambda^2 + C_0 = 0 \qquad (4.2.13)$$

where

$$C_4 = -m_1^2 = -2(s_{26}p_{26} + s_{37}p_{37} + \eta_2 s_{23}p_{23} + \eta_6 s_{67}p_{67}) \qquad (4.2.14)$$

$$C_2 = 2\eta_6 s_{26}s_{67}p_{26}p_{67} + 4s_{26}s_{37}p_{26}p_{37} + 4\eta_2\eta_6 s_{23}s_{67}p_{23}p_{67} + p_{26}^2 + 2\eta_2 s_{23}s_{26}p_{23}p_{26} - 2\sqrt{2}\, s_{23}s_{37}(\eta_2+\eta_6)R \qquad (4.2.15)$$

$$C_0 = -2(s_{37}p_{37}p_{26}^2 + \eta_2\eta_6 s_{23}s_{26}s_{67}p_{23}p_{26}p_{67} + s_{23}s_{26}s_{67}p_{23}p_{26}p_{67} - \sqrt{2}\, s_{23}s_{26}s_{37}(\eta_2+\eta_6)p_{26}R) \qquad (4.2.16)$$

and $$R = \mathrm{Re}(z_{23}z_{37}\bar{z}_{26}\bar{z}_{67}) \qquad (4.2.17)$$

Note that all the coefficients are real, in accordance with Theorem (3.3.5).

The assumption of Section 3.3, that there is only one particle-antiparticle state with spin 1 thus leads to the conditions

$$s_{26}p_{26} + s_{37}p_{37} + \eta_2 s_{23}p_{23} + \eta_6 s_{67}p_{67} > 0 \qquad (4.2.18)$$

for real non-zero mass, and:-

$$2\eta_6 s_{23}s_{37}p_{26}p_{67} + 4s_{26}s_{37}p_{26}p_{37} + 4\eta_2\eta_6 s_{23}s_{67}p_{23}p_{67} + p_{26}^2 + 2\eta_2 s_{23}s_{26}p_{23}p_{26} - 2\sqrt{2}\, s_{23}s_{37}(\eta_2+\eta_6)R = 0 \qquad (4.1.19)$$

$$s_{37}p_{26}(p_{37}p_{26} + \eta_2\eta_6 p_{23}p_{67} + p_{23}p_{67} - \sqrt{2}\, s_{23}s_{26}(\eta_2+\eta_6)R) = 0 \qquad (4.2.20)$$

(4.2.20) shows that we must have

$$\eta_2 = \eta_6 \qquad (4.2.21)$$

because the case $\eta_2 = -\eta_6$ leads to $p_{37}p_{26}^2 = 0$, which is not permitted. Thus the conditions become

4) contd.

4.2) contd.

$$s_{26}p_{26}+s_{37}p_{37}+\eta_2(s_{23}p_{23}+s_{67}p_{67}) > 0 \qquad (4.2.22)$$

$$2\eta_2 s_{23}p_{26}(s_{37}p_{67}+s_{26}p_{23})+4s_{26}s_{37}p_{26}p_{37}+4s_{23}s_{67}p_{23}p_{67} + p_{26}^2 - 4\sqrt{2}\, s_{23}s_{27}\,\eta_2 R = 0 \qquad (4.2.23)$$

$$p_{23}p_{26}+2p_{23}p_{67}-2\sqrt{2}\,\eta_2 s_{23}s_{26}R = 0 \qquad (4.2.24)$$

These conditions ensure that the characteristic equation of the 1-block will be

$$\Delta(\lambda) = \lambda^4(\lambda^2 - m_1^2) = 0 \qquad (4.2.25)$$

m_1 real. The minimal polynomial of the 1-block must contain the same irreducible factors as $\Delta(\lambda)$ and so must be one of

$$m(\lambda) = \lambda^r(\lambda^2 - m_1^2) \qquad (4.2.26)$$

where $1 \leqslant r \leqslant 4$. Applying the results of Section 3.4 we now have two ways of ensuring that the spin 1 state has a positive energy contribution:-

i) Ensure that the minimal equation of A_1 is

$$A_1^r(A_1^2 - m_1^2) = 0$$

where $1 \leqslant r \leqslant 2$ and $m_1^2 > 0$, then take

$$\mathrm{Tr}(\Lambda_1 A_1^2) > 0$$

or

ii) Allow complete freedom to the minimal equation - all we need to demand is the characteristic equation, then take

$$\mathrm{Tr}(\Lambda_1 A_1^4) > 0$$

The case when the minimal polynomial is $\lambda^3(\lambda^2 - m_1^2)$ is included in (ii). If either (i) or (ii) hold then the spin 1 state will have a positive energy contribution.

<u>First consider (i)</u>.

It is sufficient to demand

4) contd.

4.2) contd.

$$A_1^2(A_1^2 - m_1^2) = 0 \qquad (4.2.27)$$

and $$\mathrm{Tr}(\Lambda_1 A_1^2) > 0 \qquad (4.2.28)$$

By substituting for A_1 in (4.2.27) (powers of A_1 can be calculated very effectively from the graph), we find it satisfied if the following equations are satisfied:-

$$2\sqrt{2}\eta_2 s_{23}s_{37}R - 2\eta_2 s_{23}s_{26}p_{23}p_{26} - 2s_{23}s_{67}p_{23}p_{67} - 2s_{26}s_{37}p_{26}p_{37} - p_{26}^2 - \eta_2 s_{23}s_{37}p_{26}p_{67} = 0 \qquad (4.2.29)$$

$$2\sqrt{2}s_{23}s_{37}R + s_{23}s_{37}p_{26}p_{67} - 2\eta_2 s_{23}s_{67}p_{23}p_{67} = 0 \qquad (4.2.30)$$

$$-s_{26}p_{26}[z_{26}z_{67} + \sqrt{2}z_{23}z_{37}] = 0 \qquad (4.2.31)$$

$$4\sqrt{2}\eta_2 s_{23}s_{37}R - 2s_{26}p_{26}(\eta_2 s_{23}p_{23} + 2s_{37}p_{37}) - 4s_{23}s_{67}p_{23}p_{67} = 0 \qquad (4.2.32)$$

$$-\eta_2 s_{26}p_{26}(s_{23}z_{26}\bar{z}_{23} + \sqrt{2}s_{67}z_{37}\bar{z}_{67}) = 0 \qquad (4.2.33)$$

$$2\sqrt{2}\eta_2 s_{23}s_{37}R - \eta_2 s_{23}s_{26}p_{23}p_{26} - p_{26}^2 - 2s_{23}s_{67}p_{23}p_{67} - 2\eta_2 s_{23}s_{37}p_{26}p_{67} - 2s_{26}s_{37}p_{26}p_{37} = 0 \qquad (4.2.34)$$

$$s_{23}(2\sqrt{2}s_{37}R + s_{26}p_{23}p_{26} - 2\eta_2 s_{67}p_{23}p_{67}) = 0 \qquad (4.2.35)$$

$$2s_{37}(2\sqrt{2}\eta_2 s_{23}R - 2s_{26}p_{26}p_{37} - \eta_2 s_{23}p_{26}p_{67} - 2s_{26}p_{23}p_{67}) = 0 \qquad (4.2.36)$$

These equations are easier to solve than they look.

(4.2.32) - (4.2.36) gives

$$s_{67}p_{23}p_{26} - s_{23}p_{26}p_{67} = 0$$

or $$p_{23} = s_{23}s_{67}p_{67}$$

which, since $p_{23} > 0$, means

$$p_{23} = p_{67} \qquad (4.2.37)$$

and $$s_{23} = s_{67} \qquad (4.2.38)$$

Now from (4.2.31) we have

$$z_{23}z_{37} = -\frac{1}{\sqrt{2}}\, z_{26}z_{67}$$

hence

$$R = \mathrm{Re}(\bar{z}_{23}\bar{z}_{37}z_{26}z_{67}) = -\frac{1}{\sqrt{2}}\, p_{26}p_{67} \qquad (4.2.39)$$

4) contd.

4.2) contd.

Substituting this in (4.2.30) gives on simplification

$$p_{26} = -\, 2\eta_2 s_{37} s_{67} p_{23}$$

or
$$\eta_2 s_{37} s_{67} = -\,1 \qquad (4.2.40)$$

$$p_{26} = 2p_{23} \qquad (4.2.41)$$

Now substitute the results so far in (4.2.35), we find

$$p_{23} = (2+\eta_2 s_{26} s_{67}) p_{67}$$

But from (4.2.37) $p_{23} = p_{67}$, so:-

$$\eta_2 s_{26} s_{67} = -1 \qquad (4.2.42)$$

From (4.2.29) we now get

$$p_{23} = p_{37}$$

and it is found that this also satisfies (4.2.34).

We can thus summarize (4.2.29) - (4.2.36) as:-

$$p_{26} = 2p_{23} = 2p_{67} = 2p_{37}$$

$$s_{23} = s_{67}$$

$$\eta_2 s_{23} s_{26} = -1$$

$$s_{23} z_{26} \bar{z}_{23} + \sqrt{2} s_{67} z_{37} \bar{z}_{67} = 0 = z_{26} \bar{z}_{23} + \sqrt{2} z_{37} \bar{z}_{67}$$

$$z_{26} z_{67} + \sqrt{2} z_{23} z_{37} = 0$$

If these conditions are satisfied then A_1 satisfies (4.2.27). The equations (4.2.23) and (4.2.24) will then be satisfied and we have now to satisfy (4.2.22). We find from this:-

$$s_{37}(2+\eta_2 s_{37} s_{67}) > 0$$

i.e. $$s_{37} = +1$$

We finally have

$$s_{26} = s_{37} = 1 \qquad (4.2.43)$$

$$\eta_2 = -\, s_{67} = -\, s_{23} \qquad (4.2.44)$$

$$p_{26} = 2p_{23} = 2p_{37} = 2p_{67} \qquad (4.2.45)$$

$$z_{26} \bar{z}_{23} + \sqrt{2}\, z_{37} \bar{z}_{67} = 0 \qquad (4.2.46)$$

4) contd.

4.2) contd.

$$z_{26}z_{67} + \sqrt{2}\, z_{23}z_{37} = 0 \qquad (4.2.47)$$

(4.2.43) - (4.2.47) are the final conditions that the 1-block has the characteristic equation

$$\lambda^4(\lambda^2 - m_1^2) = 0 \qquad m_1^2 > 0$$

and satisfies the equation

$$A_1^2(A_1^2 - m_1^2) = 0$$

although this need not be the minimal equation of A_1, it could in fact be $A_1(A_1^2 - m_1^2) = 0$.

We now impose the condition (4.2.28):-

$$\mathrm{Tr}(\Lambda_1 A_1^2) > 0$$

This trace can be easily written down from the graph of the 1-block, observing that the coefficient of $(\Lambda_1)_{\tau\tau^\bullet}$ is the sum of the terms corresponding to all paths of length two from the representation $\tau^\bullet$ to the representation τ. We find, using (4.2.43) -(4.2.47) to simplify the result, that

$$\mathrm{Tr}(\Lambda_1 A_1^2) = -\,4a^{\tau_7\tau_7^\bullet}p_{37} \qquad (4.2.48)$$

and so (4.2.28) gives

$$a^{\tau_7\tau_7^\bullet} = -1 \qquad (4.2.49)$$

From (4.2.43) this gives

$$a^{\tau_3\tau_3^\bullet} = -1 \qquad (4.2.50)$$

Now at (4.2.10) we found that the spin zero state gives a positive contribution to the energy if and only if $a^{\tau_3\tau_3^\bullet} = 1$. So (4.2.50) shows that in this case (i), when the minimal equation of A_1 is

$$A_1^{\,r}(A_1^2 - m_1^2) = 0 \qquad r = 1,2$$

we cannot have a spin zero and spin one state together in a quantizable theory.

4) contd.

4.2) contd.

Case (ii).

In this case we simply demand that A_1 has the characteristic equation

$$\lambda^4(\lambda^2 - m_1^2) = 0 \qquad (4.2.51)$$

and that

$$Tr(\Lambda_1 A_1^4) > 0 \qquad (4.2.52)$$

We have only to satisfy the equations (4.2.22) - (4.2.24) to ensure the first condition, while $Tr(\Lambda_1 A_1^4)$ will be a real expression in η_k, $s_{k\ell}$, $p_{k\ell}$ and $z_{k\ell}$. In this case there is a lot more freedom in these parameters and it is possible to satisfy (4.2.51) and (4.2.52) irrespective of the value of $a^{\tau_3\tau_3}$. Thus in this case we can have a spin zero and spin one state together in a quantizable theory.

Spin One State Absent.

Finally if a spin one state is not present in the theory, we must have, in addition to (4.2.23) and (4.2.24):-

$$m_1^2 = s_{26}p_{26} + s_{37}p_{37} + \eta_2 s_{23}p_{23} + \eta_6 s_{67}p_{67} = 0 \qquad (4.2.53)$$

In this case A_1 can have a minimal equation:-

$$A_1^r = 0 \qquad (4.2.54)$$

r = 1,2, ...6, each value of r giving a different formulation of the spin two theory.

2-Block.

The 2-block graph is shown in Fig.4.2.2.

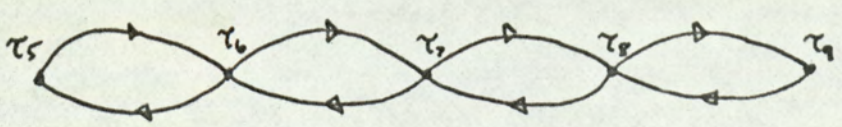


Fig.4.2.2.

4) contd.

4.2) contd.

Using the graph as a visual aid we obtain for the characteristic equation of the 2-block:-

$$-\lambda(\lambda^4 - 2(s_{56}p_{56} + 3\eta_6 s_{67}p_{67})\lambda^2 + p_{56}^2 + 6\eta_6 s_{56}s_{67}p_{56}p_{67}) = 0 \qquad (4.2.55)$$

Uniqueness of the spin 2 state means that:-

$$p_{56}^2 + 6\eta_6 s_{56}s_{67}p_{56}p_{67} = 0 \qquad (4.2.56)$$

$$s_{56}p_{56} + 3\eta_6 s_{67}p_{67} > 0 \qquad (4.2.57)$$

These give

$$p_{56} = 6p_{67} \qquad (4.2.58)$$

$$s_{56} = 1 \qquad (4.2.59)$$

$$\eta_6 s_{67} = -1 \qquad (4.2.60)$$

The possible minimal equations are

$$A_2^r(A_2^2 - m_2^2) = 0 \qquad (4.2.61)$$

where r = 1,2,3. For positive energy theories it is sufficient to demand either

i) $$A_2^2(A_2^2 - m_2^2) = 0 \qquad (4.2.62)$$

and $$\mathrm{Tr}(\Lambda_2 A_2^2) > 0 \qquad (4.2.63)$$

or

ii) $$A_2^3(A_2^2 - m_2^2) = 0 \qquad (4.2.64)$$

and $$\mathrm{Tr}(\Lambda_2 A_2^4) > 0 \qquad (4.2.65)$$

Case (i)

By direct substitution for A_2 we find that A_2 only satisfies (4.2.62) if p_{56} and p_{67} are zero, which possibility we do not allow. Sc case (i) cannot yield a spin two state if all the $z_{k\ell}$ are non-zero.

Case (ii)

We find, using (4.2.58) - (4.2.60) that

4) contd.

4.2) contd.

$$\mathrm{Tr}(\Lambda_2 A_2^4) = 6a^{\tau_7\tau_7\bullet} p_{56}p_{67} \qquad (4.2.66)$$

Hence (4.2.65) gives

$$a^{\tau_7\tau_7\bullet} = +\frac{1}{2} \qquad (4.2.67)$$

So for a quantizable spin/state we must have (4.2.67). We see that this is compatible with a quantizable spin zero state, but not with a spin one theory of the type discussed in case (i) for the 1-block. However, with a 1-block satisfying a minimal equation which is the same as its characteristic equation we can have a quantizable theory describing a spin 0, spin 1 and spin 2 state. As none of the $z_{k\ell}$ are zero, this theory includes all of representations of $\mathcal{L}_p$ in the maximum spin two Bose triangle. This may not be necessary, of course, and good theories may be obtainable with more restricted reducible representations of $\mathcal{L}_p$. A complete discussion of the spin two theory along the lines indicated in this section would be a lengthy, but not too difficult project.

4.3) Discussion.

In Sections 4.1 and 4.2 we have illustrated how the results of Chapter 3 may be applied to find particular quantizable theories. Whilst we have systematized the search for quantizable theories to some extent, it cannot be denied that the procedure still involves very depressing amounts of algebra. This seems inevitable, especially as we increase the spin. However, there seems to be much scope for general results in the theory and the methods of Chapter 3 may help in obtaining these.

Recently a paper has been published by Amar

4) contd.

4.3) contd.

and Dozzio[46] on the subject of quantizable Gel'Fand-Yaglom equations for arbitrary integral spin, and it is interesting to discuss the relation between their work and this thesis. Amar and Dozzio give a sufficient condition for a theory to be quantizable, namely that exactly one s-block shall have just two non-degenerate non-zero eigenvalues, and the remaining s-blocks shall have no non-zero eigenvalues. Their proof is an indirect one, which uses Pauli's theorem[45]. We see that the sufficienty of the condition in fact follows from (2.5.10) and (2.5.11), both in the case of integral and half-integral spin. If only one s-block has non-zero eigenvalues, and only two of these, $\pm m$, then there will be no summation in (2.5.10) and (2.5.11), which may be written

$$E = \omega_{\underline{k}r}\epsilon_+(0)\left[b_o b_o^+ \left(\frac{\epsilon_-(0)}{\epsilon_+(0)}\right) - a_o^+ a_o\right]$$

$$Q = \epsilon_+(0)\left[b_o b_o^+ \left(\frac{\epsilon_-(0)}{\epsilon_+(0)}\right) + a_o^+ a_o\right]$$

The s-blocks will all have characteristic and minimal polynomials of the forms (3.4.1) and (3.4.2), so the results of Section 3.4 apply. In particular, in the case of integral spin $\epsilon_-(0)$ and $\epsilon_+(0)$ have opposite signs, which is sufficient to ensure that E and Q have the correct quantizable forms, since they depend only on the single quantity $\epsilon_+(0)$. A similar argument follows in the case of half odd integral spin.

As Amar and Dozzio emphasize, their condition is not necessary. Indeed we have in the previous section

4) contd.

4.3) contd.

obtained quantizable theories which do not satisfy their condition, and which display mass and spin spectra. Furthermore, we have not used multiple representations of $\mathcal{L}_p$. We feel that, if possible it is better, on the grounds of simplicity, to construct quantizable theories which do not use multiple representations. However, it would be interesting to extend the graph theoretical treatment used here to this case.

Another interesting extension of our work would be to the inclusion of s-blocks having more general characteristic polynomials than (3.4.1), obtainable by dropping the assumption of Section 3.3. A simple example is given by Amar and Dozzio, corresponding to the particular type of graph shown in Fig.4.3.1 for integral spin. The form of the s-blocks and their characteristic polynomials,

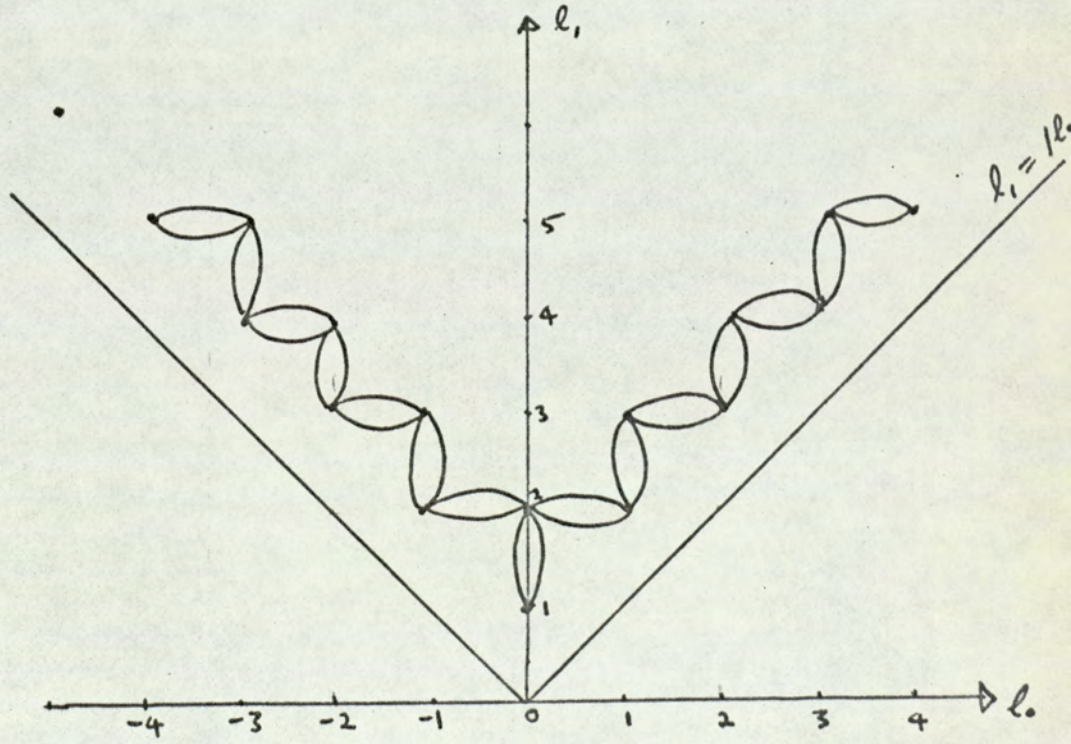


Fig.4.3.1.

4) contd.

4.3) contd.

as stated by Amar and Dozzio, are clear from a little consideration of the graph. The characteristic polynomial contains twice repeated non-zero roots. If we stick to the assumption of 3.3, that each eigenvalue has a unique eigenvector then this implies that the minimal polynomial must have repeated factors. By a result of Udgaonkar this implies that the charge and energy density will vanish. For non-zero charge and energy density it is therefore essential to allow multiple eigenvectors corresponding to the non-zero eigenvalues, that is, to abandon the assumption of 3.3. By doing this in this simple case, Amar and Dozzio have obtained a unique mass theory with indefinite energy and charge, thereby displaying the insufficiency of the unique mass condition in ensuring quantizable theories.

APPENDIX A.

CONSTRUCTION OF THE S-BLOCKS.

Appendix A. Construction of the s-blocks.

A.1 Integral Spin.

With the standard numbering of the representations in a general maximum spin j theory (see Section 3.2) the s-rectangle will contain $(j-s+1)(2s+1)$ representations, and the representations in the row $\ell_1 = \ell$ will be

$$\tau_i, \tau_{i+1}, \cdots \tau_{i+2s} \qquad (A.1.1)$$

where $i = \ell^2 - \ell - s+1$.

We take the standard ordering of the rows and columns of the s-block to be the same as that of the representations in the Bose-plane. Then, the general form of the s-block will be triple block diagonal. The non-zero subblocks on, above and below the diagonal correspond to the rows of the s-rectangle. Thus the s-block has the form:-

$$\begin{bmatrix}
B_{s+1,s+1} & B_{s+1,s+2} & 0 & 0 & \cdots & 0 \\
B_{s+2,s+1} & B_{s+2,s+2} & B_{s+2,s+3} & 0 & \cdots & \vdots \\
0 & B_{s+3,s+2} & B_{s+3,s+3} & \cdots & \cdots & \vdots \\
0 & 0 & & & & \vdots \\
\vdots & \vdots & \vdots & & B_{j,j-1} \; B_{j,j} & B_{j,j+1} \\
0 & \cdots & \cdots & 0 & B_{j+1,j} & B_{j+1,j+1}
\end{bmatrix} \qquad (A.1.2)$$

where the $(2s+1) \times (2s+1)$ subblocks $B_{\ell m}$ correspond to the representations in rows $\ell_1 = \ell$, and $\ell_1 = m$. We construct the s-block by building up these $B_{\ell m}$ subblocks.

$B_{\ell\ell}$ subblocks.

The diagonal subblocks contain elements $C_s^{\tau\tau'}$ where τ and τ' lie in the row $\ell_1 = \ell$. Such representations are linked horizontally and we have

$$C_s^{\tau\tau'} \neq 0$$

only if τ and τ' are horizontally adjacent, in which case, from (2.2.30), if τ is "to the left of τ' " on the graph

$$\left.\begin{aligned} C_s^{\tau\tau'} &= C^{\tau\tau'}\sqrt{(s+\ell_0+1)(s-\ell_0)} \\ C_s^{\tau'\tau} &= C^{\tau\tau'}\sqrt{(s+\ell_0+1)(s-\ell_0)} \end{aligned}\right\} \qquad (A.1.3)$$

With the standard numbering of rows and columns of the s-blocks, the $B_{\ell\ell}$ subblocks have zeros everywhere except immediately above and below the leading diagonal, and are of the form:-

$$\begin{bmatrix} 0 & C_s^{\tau_i\tau_{i+1}} & 0 & \cdots & 0 \\ C_s^{\tau_{i+1}\tau_i} & 0 & C_s^{\tau_{i+1}\tau_{i+2}} & \cdots & 0 \\ 0 & C_s^{\tau_{i+2}\tau_{i+1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ & & & C^{\tau_{i+2s-1}\tau_{i+2s-2}} \; 0 & C_s^{\tau_{i+2s-1}\tau_{i+2s}} \\ 0 & 0 & 0 \; \cdots \; 0 & C^{\tau_{i+2s}\tau_{i+2s-1}} & 0 \end{bmatrix} \qquad (A.1.4)$$

where $i = \ell^2 - \ell - s + 1$. The real factor

$$\rho(s,\ell_0) = \sqrt{(s+\ell_0+1)(s-\ell_0)} \qquad (A.1.5)$$

is easily calculated by looking at the abscissa (ℓ_0) of the representation $\tau \sim (\ell_0,\ell_1)$ in the Bose plane. τ is always the representation "on the left". The factor $\rho(s,\ell_0)$ is the same for four elements of the $B_{\ell\ell}$ block. To see this, we note that if $\tau \sim (\ell_0,\ell_1)$ then $\tau\cdot \sim (-\ell_0,\ell_1)$ and $\tau' \sim (\ell_0+1,\ell_1)$ so $\tau'\cdot \sim (-\ell_0-1,\ell_1)$. Thus since $\tau'\cdot$ is on the left of $\tau\cdot$, we have:-

$$\left.\begin{aligned} C_s^{\tau'\cdot\tau\cdot} &= C^{\tau'\cdot\tau\cdot}\sqrt{(s-\ell_0)(s+\ell_0+1)} \\ C_s^{\tau\cdot\tau'\cdot} &= C^{\tau\cdot\tau'\cdot}\sqrt{(s-\ell_0)(s+\ell_0+1)} \end{aligned}\right\} \qquad (A.1.6)$$

Graphically, what this says is that $\rho(s,\ell_0)$ is not only the same for elements corresponding to the two oppositely directed branches between two given representations, but also for the minor images of these branches in the ℓ_1 axis. This fact is useful in constructing the $B_{\ell\ell}$ block. It means that the factors $\rho(s,\ell_0)$ in the $B_{\ell\ell}$ block are symmetrical about the leading and skew (top right to bottom left) diagonals. Thus, in obvious notation we can write the $B_{\ell\ell}$ block as in (A.1.7).

$$\begin{bmatrix}
0 & \rho(s,\ell_0^{(i)})C^{\tau_i\tau_{i+1}} & 0 & \cdots & 0 \\
\rho(s,\ell_0^{(i)})C^{\tau_{i+1}\tau_i} & 0 & \rho(s,\ell_0^{(i+1)})C^{\tau_{i+1}\tau_{i+2}} & & \\
0 & \rho(s,\ell_0^{(i+1)})C^{\tau_{i+2}\tau_{i+1}} & 0 & & \\
\vdots & & & \ddots & \\
 & & 0 & & \rho(s,\ell_0^{(i)})C^{\tau_{i+2s-1}\tau_{i+2s}} \\
0 & \cdots & & \rho(s,\ell_0^{(i)})C^{\tau_{i+2s}\tau_{i+2s-1}} & 0
\end{bmatrix} \qquad (A$$

We now apply the conditions for Lagrangian origin and space reflection covariance. Because all the representations in a $B_{\ell\ell}$ subblock are linked horizontally by a type (i) linkage, the condition for Lagrangian origin, from (2.3.16) can be written

$$C^{\tau\tau'} = \frac{a^{\tau\tau^\bullet}}{a^{\tau'\tau'^\bullet}} \overline{C^{\tau'^\bullet\tau^\bullet}}$$

$$= s(\tau,\tau') \overline{C^{\tau'^\bullet\tau^\bullet}} \qquad \text{(A.1.8)}$$

where the $a^{\tau\tau^\bullet}$ and therefore the $s(\tau,\tau')$ can be chosen as ± 1, as explained in 2.3. The condition for space reflection covariance can take two forms, corresponding to the possibilities a), b) considered in Section 2.2. The possibility (c) considered there cannot occur in a $B_{\ell\ell}$ subblock because it corresponds to two representations τ,τ' linked vertically on the ℓ_1 axis, whereas the $B_{\ell\ell}$ subblock only contains representations which are all on the same row. It is convenient to use (A.1.8) in the conditions of Section 2.2 and then summarize the conditions on the $C^{\tau\tau'}$ as follows:-

a) $\underline{\tau \neq \tau^\bullet \text{ and } \tau' \neq \tau'^\bullet}$

$$C^{\tau\tau'} = C^{\tau^\bullet\tau'^\bullet} \qquad \text{(A.1.9)}$$

$$C^{\tau\tau'} = s(\tau,\tau') \overline{C^{\tau'\tau}} \qquad \text{(A.1.10)}$$

b) $\underline{\tau = \tau^\bullet \text{ and } \tau' \neq \tau'^\bullet}$

$$C^{\tau\tau'} = \eta\, C^{\tau^\bullet\tau'^\bullet} = \eta\, C^{\tau\tau'^\bullet} \qquad \text{(A.1.11)}$$

$$C^{\tau\tau'} = \eta\, s(\tau,\tau') \overline{C^{\tau'\tau}} \qquad \text{(A.1.12)}$$

where $\eta = \pm 1$ depending on the matrix used to represent the reflection operator S in the representation space $\mathcal{R}$. Similarly for the case $\tau \neq \tau^\bullet$ and $\tau' = \tau'^\bullet$

The above conditions are illustrated on the graphs as shown below. In these graphs we omit the $\rho(s,\ell_0)$ factors for clarity.

Case (a). $\tau \neq \tau^\bullet$ and $\tau' \neq \tau'^\bullet$

Neither of the linked representations τ,τ' lie on the ℓ_1 axis. This situation is shown in Figure A.1.1. (A.1.9) shows that a directed branch and its mirror image in the ℓ_1 axis represent the

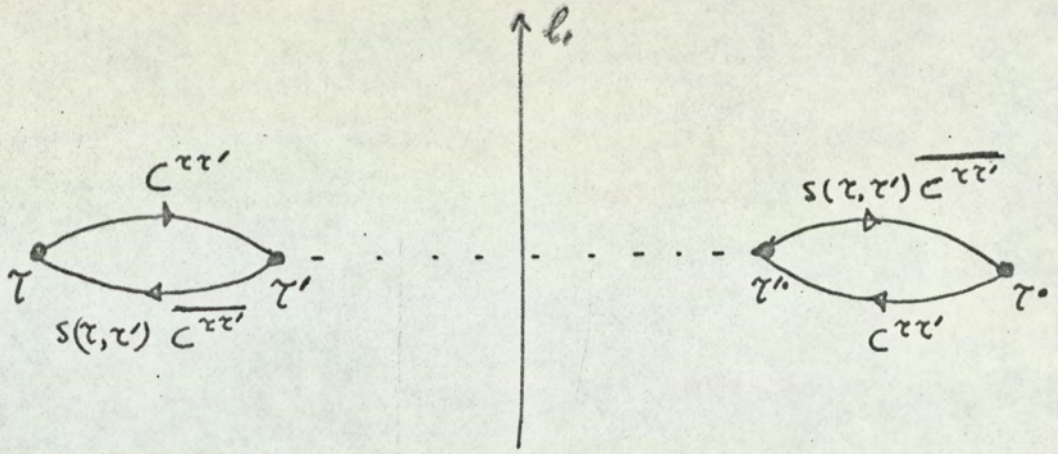


Fig.A.1.1

same complex number $C^{\tau\tau'}$. (A.1.10) then shows that a directed branch from $\tau \to \tau'$ is related to its oppositely directed counterpart by complex conjugation and multiplication by $s(\tau,\tau')$. Thus, a pair of loops such as those in Fig.A.1.1 correspond to four elements of the $B_{\ell\ell}$ subblock which all have the same $\rho(s,\ell_0)$ factor, and which depend on a single arbitrary complex number, $C^{\tau\tau'}$ say.

Case (b). $\tau = \tau\bullet$ and $\tau' \neq \tau'\bullet$

The representation τ lies on the ℓ_1 axis and is linked horizontally to τ' and $\tau'\bullet$. This situation is shown in Fig.A.1.2. (A.1.11) shows that a directed branch and its mirror image in the ℓ_1 axis represent the same complex number apart from a factor η. (A.1.12)

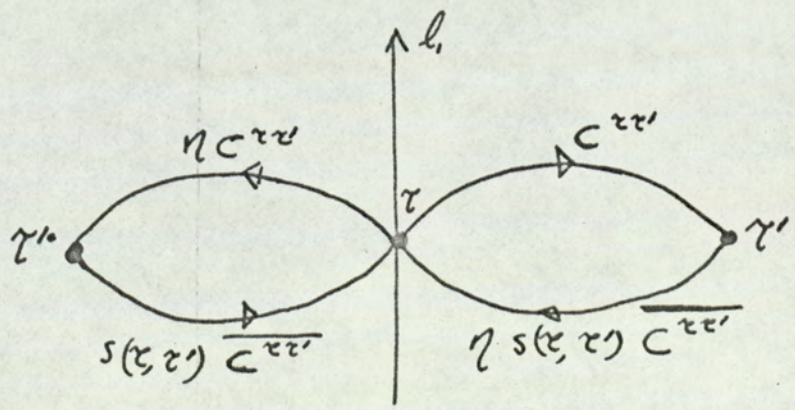


Fig.A.1.2

then shows that a directed branch from $\tau \to \tau'$ is related to its oppositely directed counterpart by complex conjugation and multiplication by $\eta\, s(\tau,\tau')$.

We see that the only difference between the cases (a) and (b) is in the η factor, which only occurs when one of the representations lies on the ℓ_1 axis. This only happens once on any row and so an η-factor only occurs once in any $B_{\ell\ell}$ block and can therefore be labelled by ℓ, thus η_ℓ.

We now see that the graph of the $B_{\ell\ell}$ subblock will be as shown in Fig.A.1.3. Again the $\rho(s,\ell_0)$ factors,

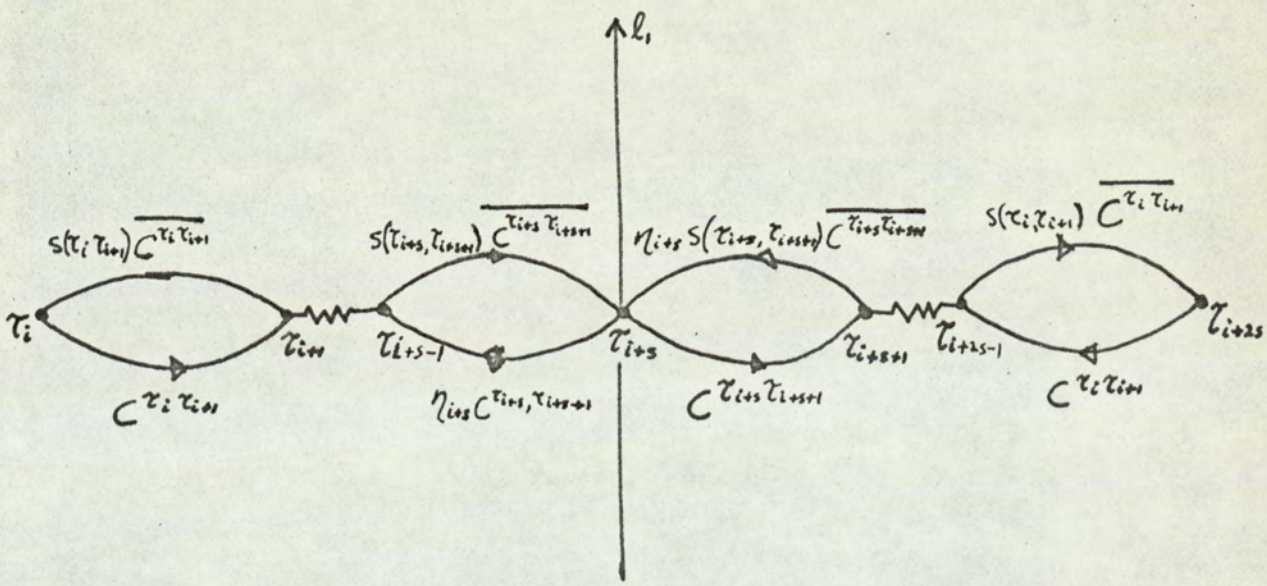


Fig.A.1.3

symmetrical about the ℓ_1 axis, have been omitted. Inspection of the graph shows how the algorithm of Section 3.2, for constructing the $B_{\ell\ell}$ subblocks comes about. The skew diagonal (diagonal from upper right to lower left) of the $B_{\ell\ell}$ subblock is a row of zeros. The part below the skew diagonal is obtained from that above by reflection in the skew diagonal, followed by reflection in the leading diagonal, but leaving the η_ℓ in position under the last reflection. In practical work with the s-blocks we introduce

the notation

$$z_{k\ell} = C^{\tau_k \tau_\ell}$$

for elements in the $B_{\ell\ell}$ subblocks.

$B_{\ell\ell+1}$ and $B_{\ell+1\ell}$ Subblocks.

These subblocks contain elements $C_s^{\tau\tau'}$ where τ and τ' lie on adjacent rows. We consider first the $B_{\ell\ell+1}$ subblocks, to which the $B_{\ell+1\ell}$ subblocks are simply related.

In the rows corresponding to the $B_{\ell\ell+1}$ block the τ,τ' are linked vertically, τ on row ℓ being linked to τ' on the row above, $\ell+1$. We have

$$C_s^{\tau\tau'} \neq 0$$

only if τ and τ' are vertically adjacent, and in this case, we have from (2.2.31)

$$C_s^{\tau\tau'} = C^{\tau\tau'} \sqrt{(s+\ell+1)(s-\ell)} \qquad \text{(A.1.13)}$$

$$C_s^{\tau\tau'} = C^{\tau'\tau} \sqrt{(s+\ell+1)(s-\ell)} \qquad \text{(A.1.14)}$$

With the standard ordering of rows and columns of the s-block, we see that the $B_{\ell\ell+1}$ subblock is diagonal and its non-zero elements are

$$C^{\tau_{k(\ell)}\tau_{k(\ell+1)}} \sqrt{(s+\ell+1)(s-\ell)} \qquad \text{(A.1.15)}$$

where $$k(\ell) = i(\ell),\ i(\ell)+1,\ \ldots\ i(\ell)+2s \qquad \text{(A.1.16)}$$

and $$i(\ell) = \ell^2-\ell-s+1 \qquad \text{(A.1.17)}$$

From (A.1.13) and (A.1.14) the quantity $\sqrt{(s+\ell+1)(s-\ell)}$ is a common factor in both the $B_{\ell\ell+1}$ and the $B_{\ell+1\ell}$ subblocks, so it can be calculated once and for all and we can concentrate on the $C^{\tau\tau'}$. Notice however that it is purely imaginary, and it is convenient later to introduce the notation

$$z_{k\ell} = i\, C^{\tau_k \tau_\ell}, \quad s_{k\ell} = s(\tau_k, \tau_\ell) \tag{A.1.18}$$

$$q(s,\ell) = |\sqrt{(s+\ell+1)(s-\ell)}| \tag{A.1.19}$$

We now consider the conditions imposed on the $C^{\tau\tau'}$ and $C^{\tau'\tau}$ by space reflection covariance and Lagrangian origin. Since all the linkages are vertical, we have for the Lagrangian origin condition, from (2.3.17):-

$$C^{\tau\tau'} = - s(\tau,\tau')\, \overline{C^{\tau'\cdot\tau\cdot}} \tag{A.1.20}$$

The conditions for space reflection covariance can take two forms, corresponding to the possibilities (a), (c) but turn out to be the same. The possibility (b) cannot arise for vertical (type (ii)) linkages. As when discussing the $B_{\ell\ell}$ subblocks, it is convenient to use (A.1.20) in these conditions (a) and (c) and then summarize the full conditions as follows:-

a) and (c) $\tau \neq \tau\cdot$ and $\tau' \neq \tau'\cdot$ or $\tau = \tau\cdot$ and $\tau' = \tau'\cdot$

$$C^{\tau\tau'} = C^{\tau\cdot\tau'\cdot} \tag{A.1.21}$$

$$C^{\tau\tau'} = - s(\tau,\tau')\, \overline{C^{\tau'\tau}} \tag{A.1.22}$$

Graphically, (A.1.21) means that a branch and its mirror image in the ℓ_1 axis correspond to the same complex number. (A.1.22) means that the $C^{\tau\tau'}$ corresponding to a directed branch are related to the $C^{\tau'\tau}$, corresponding to the oppositely directed counterpart by complex conjugation and multiplication by $-s(\tau,\tau')$. If $C^{\tau\tau'}$ is in the $B_{\ell\ell+1}$ subblock then $C^{\tau'\tau}$ is the corresponding element in the $B_{\ell+1\ell}$ subblock, and the above conditions can be summarized as follows, using the notation of (A.1.18) and (A.1.19):-

The diagonal elements of the $B_{\ell\ell+1}$ subblock are

$$q(s,\ell)\ z_{k(\ell)k(\ell+1)}$$

where $k(\ell) = i(\ell), i(\ell)+1, \ldots i(\ell)+2s$

and $i(\ell) = \ell^2 - \ell - s+1$

Then the diagonal elements of the $B_{\ell+1\ell}$ subblock are the complex conjugates of these, multiplied by

$$s_{k(\ell)k(\ell+1)} = s(\tau_{k(\ell)}, \tau_{k(\ell+1)})$$

that is:-

$$s_{k(\ell)k(\ell+1)} q(s,\ell) \overline{z_{k(\ell)k(\ell+1)}}$$

Further (from (A.1.21)), the matrix $B_{\ell\ell+1}$ is symmetrical about the skew diagonal.

The algorithm used in Section 3.2 follows directly from the above work, in the case of integral spin.

A.2 Half-Odd Integral Spin.

In the case of half odd integral spin theories we consider the lattice of points (ℓ_0, ℓ_1) for which both ℓ_0 and ℓ_1 are half-odd integral. We call this the "Fermi-plane". If we again make the restriction that a maximum spin j theory will contain no representations with $\ell_1 > j+1$, then for such a theory we are only interested in those points lying in the triangle

$$\ell_1 > \ell_0$$

$$\ell_1 \leqslant j+1$$

Again we adopt as the standard numbering of the representations in the "Fermi-fan", that from left to right along the rows (see Fig.(A.2.1)). The rows and columns of the s-blocks are numbered in the same way.

The s-block of a general maximum spin j theory will contain $(j-s+1)(2s+1)$ representations, and the representations in the row $\ell_1 = \ell$ will be

$$\tau_{(\ell)},\ \tau_{i(\ell)+1},\ \cdots\cdots\ \tau_{i(\ell)+2s}$$

where $\quad i(\ell) = \ell^2 - \ell - s + \frac{3}{4}$

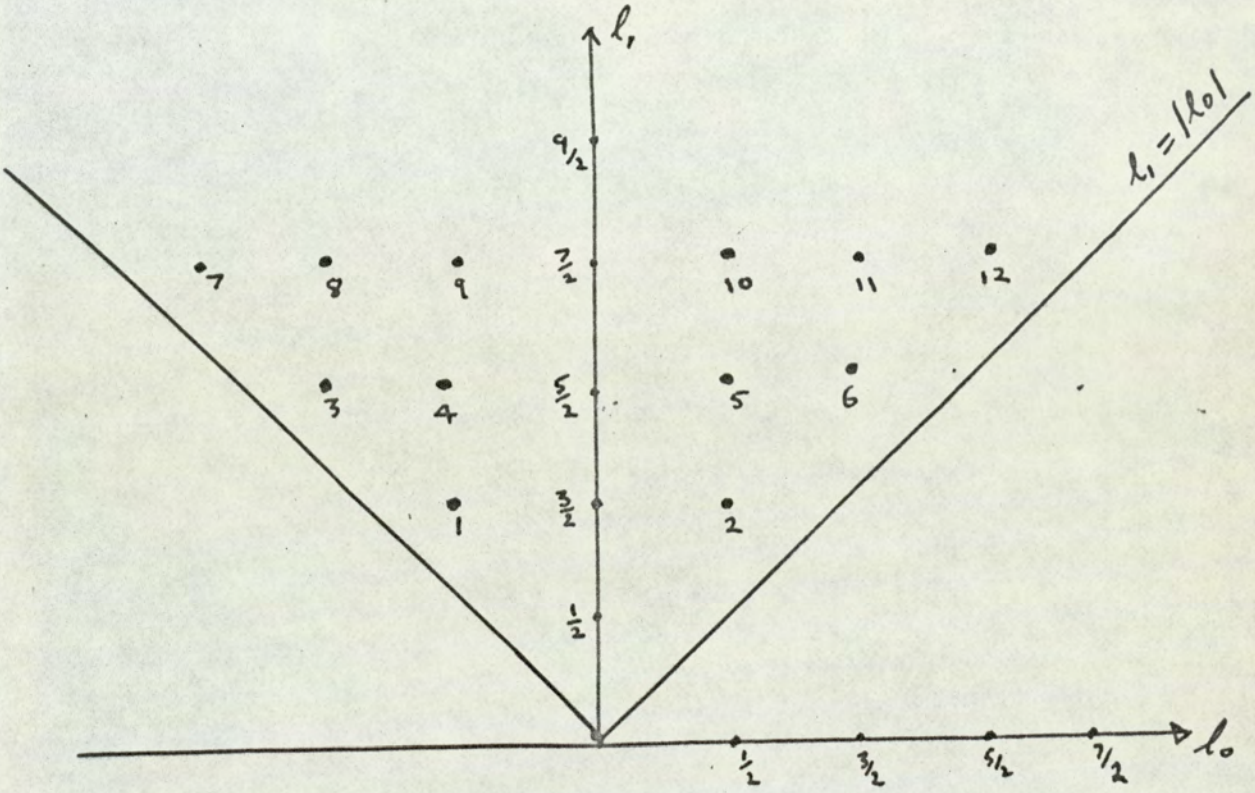


Fig.A.2.1

Again, the s-block will be triple block diagonal as in (A.1.2), and we set it up by constructing the $B_{\ell m}$ subblocks.

$B_{\ell\ell}$ subblocks.

The results (A.1.3), (A.1.6) hold in this case, so the $\rho(s,\ell_0)$ factors are symmetric about the ℓ_1 axis. The only difference between the $B_{\ell\ell}$ subblocks for half odd integral spin and those for integral spin is that in the former case the representations are linked across the ℓ_1 axis. The conditions for Lagrangian origin and space reflection covariance are unaltered, except that the possibility (b) or (c) considered in Section 2.2 cannot occur - there is thus no η-factor. The $B_{\ell m}$ subblocks are even dimensional, so the $B_{\ell\ell}$ contain non-zero elements on the skew

diagonal. From Fig.A.2.2. these are seen to be

$$C^{\tau}{}_{i(\ell)+\frac{2s-1}{2},\,i(\ell)+\frac{2s+1}{2}} \quad \text{and} \quad s_{i(\ell)+\frac{2s-1}{2},\,i(\ell)+\frac{2s+1}{2}}\,\overline{C^{\tau}{}_{i(\ell)+\frac{2s-1}{2}\;i(\ell)+\frac{2s+1}{2}}}$$

Otherwise the $B_{\ell\ell}$ subblocks can be filled in exactly as in the integral spin case.

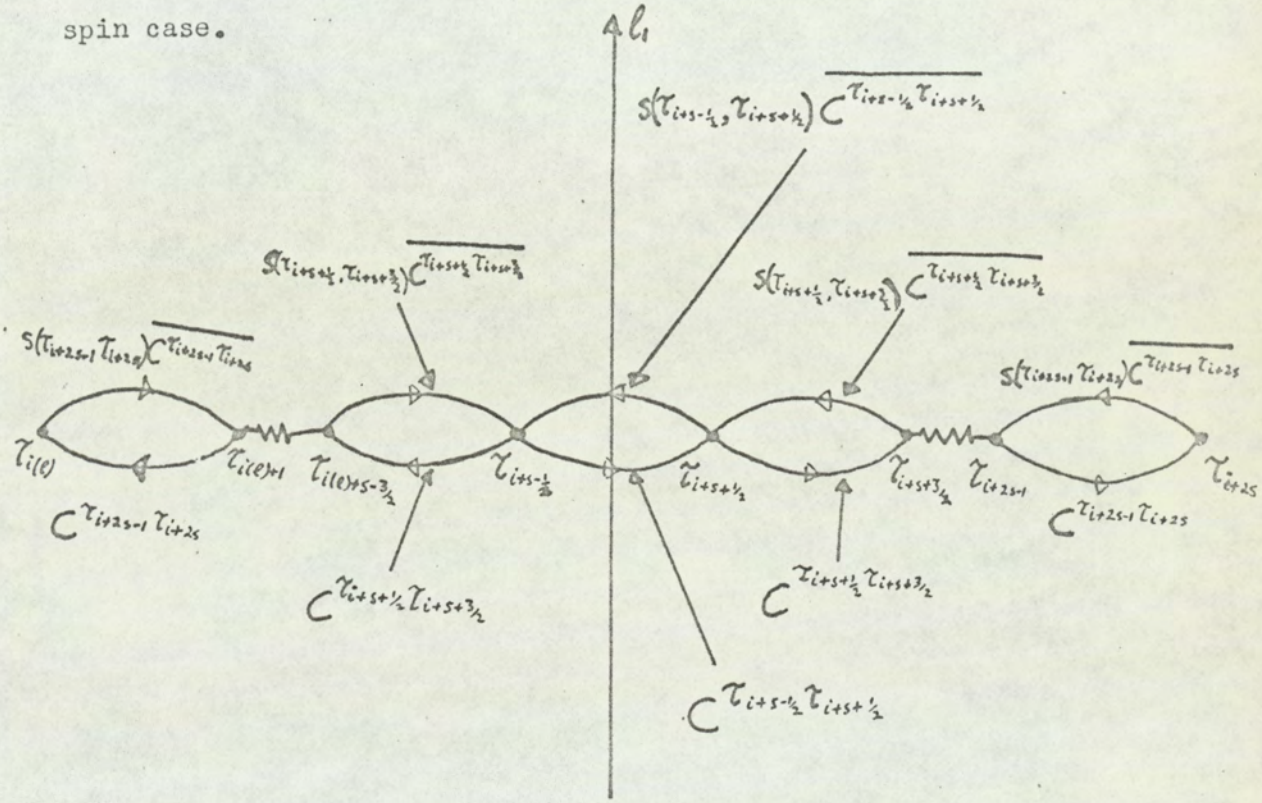


Fig.A.2.2

$B_{\ell\ell+1}$ and $B_{\ell+1\ell}$ subblocks.

These subblocks contain elements $C_s^{\tau\tau'}$ which are non-zero only if τ and τ' are linked vertically. With the standard ordering of rows and columns, the subblocks are all diagonal, and the non-zero elements in the $B_{\ell\ell+1}$ subblock are, using the notation of (A.1.18) and (A.1.19):-

$$q(s,\ell)\; z_{k(\ell)k(\ell+1)}$$

where $\quad k(\ell) = i(\ell),\ i(\ell)+1,\ \ldots\ i(\ell)+2s$

and $\quad i(\ell) = \ell^2-\ell-s+\frac{3}{4}.$

The conditions for space reflection covariance

and Lagrangian origin are the same as given at (A.1.21) and (A.1.22), except that the possibility $\tau = \tau^{\bullet}$ and/or $\tau' = \tau'^{\bullet}$ cannot arise in the half odd integral spin case. It follows then that the $B_{\ell\ell+1}$ and $B_{\ell+1\ell}$ subblocks are symmetrical about their skew diagonals, and that the $B_{\ell+1\ell}$ subblock is obtained from $B_{\ell\ell+1}$ by complex conjugation and multiplication of corresponding elements by $s_{k(\ell)k(\ell+1)}$. Thus to construct the $B_{\ell\ell+1}$ and $B_{\ell+1\ell}$ subblocks for half odd integral spin we can use the same algorithm as for the case of integral spin but ignoring the instruction about elements <u>on</u> the skew diagonal.

Notice that the essential difference between the $B_{k\ell}$ subblocks for integral and half odd integral spin is that in the former case they are always odd dimensional, and in the latter they are always even dimensional.

APPENDIX B.

ELEMENTARY GRAPH THEORY.

Appendix B. Elementary Graph Theory.

At present the terminology of graph theory does not appear to be standardized, and a particular term may be used in different ways, even by the same author. We will be using only very simple ideas from graph theory, and so we will not be too strict about our terminology, which we explain here. This may differ slightly from that of other authors, but not seriously, and in any case will be consistent with that of reference (34). Our purpose is to keep the discussion on an intuitive level and to avoid any but essential jargon.

Our graphs will be pictorially represented by sets of points, called nodes, and lines, called branches. Each branch connects exactly two nodes, although there is no limit to the number of branches which may intersect any given node. If a direction is to be associated with a branch, this is represented by an arrow on the branch, pointing from the initial node to the terminal node of the branch. A directed graph is one whose branches are all directed. We are only interested in directed graphs in this thesis. A connected graph is one in which every pair of nodes is joined by some sequence of adjacent branches (not necessarily like directed). The degree of a node is the number of branches incident at the node.

A path is a connected directed graph or sub-graph in which the terminal node of each branch is the initial node of the succeeding branch. Note that the same directed branch may be used more than once, and the same node may occur more than once in a general path. A path in which no branch occurs more than once is called simple. A path in which no node occurs more than once is called elementary. We will take the term cycle, or loop, to mean a closed elementary simple path - that is, an elementary, simple path whose initial and terminal nodes coincide. In other words, a loop

of a directed graph is a directed subgraph in which every node is of degree 2, and all the branches are like directed. A loop with a single node and branch is called a self loop of that node - its direction is not defined. The number of branches in a path is called its length.

Fig.B.1 illustrates the above ideas

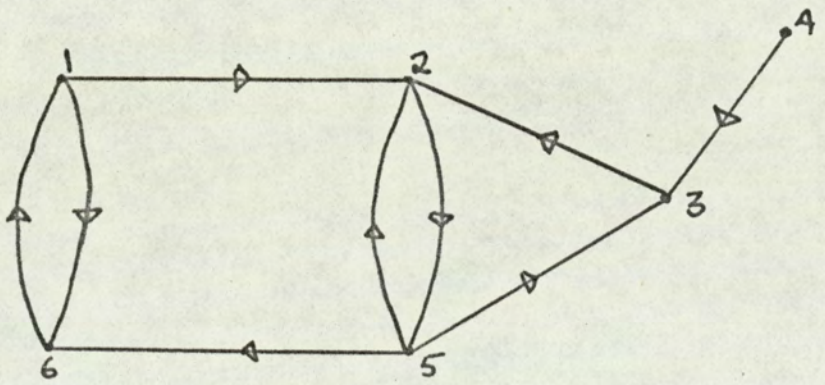


Fig.B.1

125253 is a path which is neither simple or elementary

12532 is simple but not elementary

43256 is simple and elementary

12561 is a loop

125616 is not a loop

We associate with a directed graph G a node-node matrix M(G) which has non-zero elements only in the position i,j where i is the initial node and j the terminal node of a branch of the graph G. Conversely any matrix can be associated with a graph in the same way - nodes i and j form the intial and terminal nodes of a branch of the graph if and only if the matrix has a non-zero element in the i,j position.

Basically we use two simple results:-

1) The graphical interpretation of the determinant of a matrix, see Section 3.3.

2) If A is the matrix associated with a graph G then the i,j element of A^{ℓ} is given by the sum of the terms corresponding to all paths of length ℓ from node i to node j. If there is no path from i to j then $(A^{\ell})_{ij} = 0$.

Our graphs, representing reducible representations of $\mathcal{L}_p$, have such a simple form (which we call lattice type) that the two results above can be applied to good effect.

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47) The assumption of Section 3.3 requires simply that each s-block has at most two independent eigenvectors corresponding to non-zero eigenvalues. This means that the characteristic polynomial may in fact have repeated factors. Thus, the characteristic polynomial (3.3.6) is not the most general one consistent with the assumption as it stands. However, it has been shown by Udgaonkar[31] that a necessary and sufficient condition for the charge and energy densities to be non-zero is that the minimal polynomial of L_0 shall have no repeated non-zero roots. It therefore follows from this necessary physical requirement that the minimal polynomial of an s-block must be of the form (3.4.2). It can be shown that this, together with the assumption of the uniqueness of the s-block eigenvectors, implies that the characteristic polynomial of the s-block must be of the form (3.3.6).