

## Synchronization, Desynchronization, and Intermediate Regime of Breathing Solitons and Soliton Molecules in a Laser Cavity

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We report on the experimental and numerical observations of synchronization and desynchronization of bound states of multiple breathing solitons (breathing soliton molecules) in an ultrafast fiber laser. In the desynchronization regime, although the breather molecules as wholes are not synchronized to the cavity, the individual breathers within a molecule are synchronized to each other with a delay (lag synchronization). An intermediate regime between the synchronization and desynchronization phases is also observed, featuring self-modulation of the synchronized state. This regime may also occur in other systems displaying synchronization. Breathing soliton molecules in a laser cavity open new avenues for the study of nonlinear synchronization dynamics.

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First recognized in 1665 by Huygens in a letter to his father [1], synchronization, i.e., the ability of coupled oscillators to lock to a common frequency, is a general and ubiquitous feature of nature, occurring for biological clocks, chemical reactions, mechanical or electrical oscillators, and lasers to mention a few well-known examples [2]. Apart from being of significant interest in fundamental science, synchronization phenomena also find a wide range of practical applications. For instance, in the field of optics, the synchronization of multiple microresonators can break the power limitation of a single microresonator [3–5]. Breathing solitons, manifesting themselves as localized temporal or spatial structures that exhibit periodic oscillatory behavior, are fundamental modes of many nonlinear physical systems and relate to a wide range of important nonlinear dynamics. In optics, initially studied in single-pass fiber systems [6,7], the breathing soliton concept has been extended to passive Kerr cavities and microresonators [8–11] as well as to ultrafast fiber lasers [12–24]. The studies in [9,14] have shown that cavity-based optical systems can support breather oscillations that are subharmonically entrained to the cavity roundtrip time. This subharmonic entrainment of breathers, which is a generalized form of synchronization wherein a harmonic of the breathing frequency  $f_b$  synchronizes with the cavity frequency  $f_r$ , results from the competition between the two intrinsic frequencies to the system  $f_r$  and  $f_b$ . In [21], we have reported frequency locking at Farey fractions of a breather fiber laser by demonstrating for the first time that

the winding numbers  $f_b/f_r$  show the hierarchy of a Farey tree and the structure of a devil's staircase, in accordance with the predictions from the theory of nonlinear systems with two competing frequencies [25].

These breather synchronization studies pertain to the dynamics of single breathers. In like manner to their stationary counterparts [26–31], multiple interacting breathing solitons in dissipative systems can also organize themselves into moleculelike bound states [10,13,22]. It remains elusive whether breather molecular complexes also display synchronization. Furthermore, while single breather oscillations in a laser represent a convenient nonlinear dynamical system to study two-frequency interactions [21], multi-breather complexes add new degrees of freedom into the system—the breathing frequencies of the elementary constituents—thereby opening the possibility to study the dynamics of nonlinear systems with three or more interacting frequencies, which is an important topic in nonlinear science [32–35].

In this Letter, we present the results of a further study showing that breather molecules can synchronize to a laser cavity, featuring a breathing frequency that equals a subharmonic of the cavity fundamental frequency. In the desynchronized phase, while the breather molecule when whole is not synchronized to the cavity, lag synchronization among the constituent breathers is observed. An intermediate regime between the synchronized and desynchronized phases is also observed, featuring a subharmonic breathing frequency with nonsubharmonic sidebands.

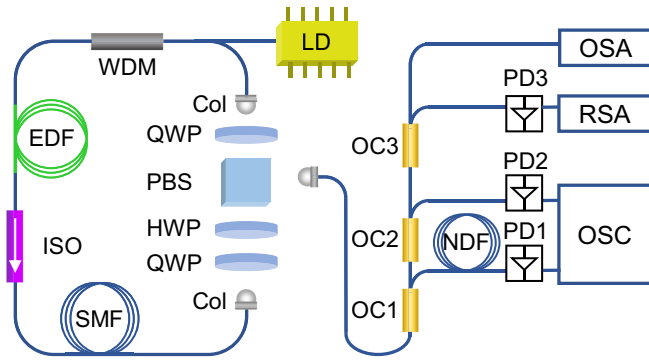


FIG. 1. The laser setup. Laser diode (LD), wavelength-division multiplexer (WDM), erbium-doped fiber (EDF), isolator (ISO), single-mode fiber (SMF), collimator (Col), quarter-wave plate (QWP), half-wave plate (HWP), polarization beam splitter (PBS), optical coupler (OC), normally dispersive fiber (NDF), photodiode (PD), oscilloscope (OSC), radio spectrum analyzer (RSA), optical spectrum analyzer (OSA).

Direct synchronization-desynchronization transitions without such a regime occur when the intracavity loss is increased. The transitions among these different phases are realized by tuning a single control parameter (the pump current). These experimental findings are confirmed by numerical simulations of a lumped laser model.

The system that we studied is a standard mode-locked fiber laser (Fig. 1). The gain medium is a 1.24 m-long erbium-doped fiber. Other fibers in the cavity are single-mode fibers. The group-velocity dispersion (GVD) parameters of the two fiber types are 61.2 and  $-18 \text{ ps}^2/\text{km}$ , respectively, resulting in a slightly anomalous net cavity dispersion ( $-0.0078 \text{ ps}^2$ ). The laser has a repetition frequency of  $f_r = 33.39 \text{ MHz}$ . Mode locking is realized through an effective saturable absorber by the nonlinear polarization evolution (NPE) effect [36]. The transfer function of NPE is controlled via three wave plates working together with a polarization beam splitter which also serves as the laser output port. The pump strength is the key parameter of the system: its tuning under given settings of the wave plates may enable switching from stationary soliton to breathing soliton states [13–24,37–39] or to soliton molecules (SMs) with various dynamics [28,40–42]. Contrary to our previous works [15,21] and to reduce as much as possible the intracavity loss, we have not implemented any genetic algorithm in the present laser. Indeed, the frequency locking dynamics of breather molecular complexes have been found to be highly affected by cavity damping (see Fig. S1 in the Supplemental Material [43]).

Figure 2 illustrates how the radiofrequency (rf) spectrum of the laser emission (measured by a radio spectrum analyser connected to photodiode 3 in Fig. 1) is influenced by the pump current (see also the measurements across a frequency range up to  $f_r$  provided in Fig. S2 of the Supplemental Material [43]). At low currents (up to

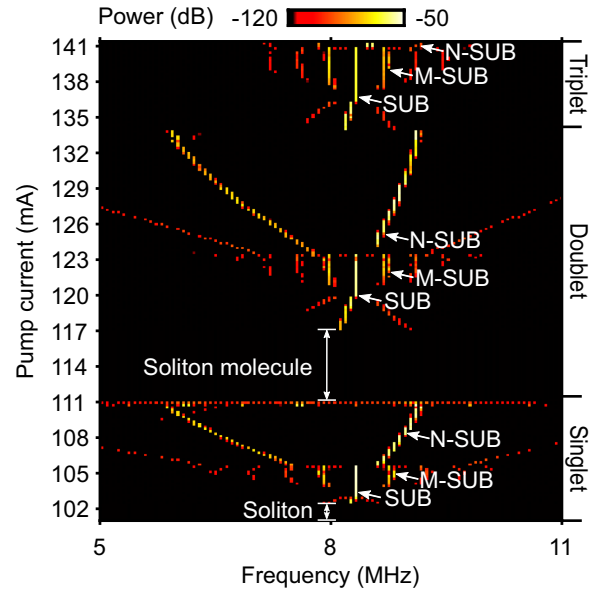


FIG. 2. Map of the laser spectral intensity in the space of rf and pump current, showing phase transitions for single breathers, and diatomic and triatomic breathing SMs. Subharmonic (SUB), nonsubharmonic (N-SUB), modulated subharmonic (M-SUB).

102 mA), our laser emits a single soliton pulse per cavity roundtrip (RT) as evidenced by the single frequency component of the rf signal at  $f_r$  (Supplemental Material [43] Fig. S2). Increase of the pump current leads to the generation of a breathing soliton with a short pulsating period of 4 RTs, as revealed by the appearance of the subharmonic narrow peaks located at multiples of  $f_b = f_r/4$ : the breathing frequency  $f_b$  is locked to the cavity repetition frequency [19,21]. The transition from stationary to breathing soliton correlates to the ubiquitous dynamics known as “Hopf bifurcation.” Further pump current increase causes new equally spaced spectral lines to appear symmetrically on both sides of the subharmonic peaks. The separation between these new lines, forming a “modulated subharmonic” structure, is associated with a long pulsating period in the time domain. Pump currents above 106 mA break the frequency locking (the modulated sidebands also vanish), and the breathing frequency continuously drifts as the pump strength varies so that  $f_b$  is no longer commensurate with  $f_r$ . We term this laser regime “non-subharmonic breathing.” For currents above 111 mA, the pulsating regime disappears, being replaced by a short stage of chaoticlike behavior and then the emission of stationary diatomic SMs up to 117 mA, when pulsating pulses are generated again in the form of breathing SMs. Between 120 and 123 mA, subharmonic and modulated subharmonic behaviors are retrieved whereas nonharmonic features are apparent between 117 and 120 mA and then between 123 and 133 mA. Above 133 mA, the diatomic breathing SM switches to a triatomic molecule, and a similar evolution pattern of the rf spectrum from

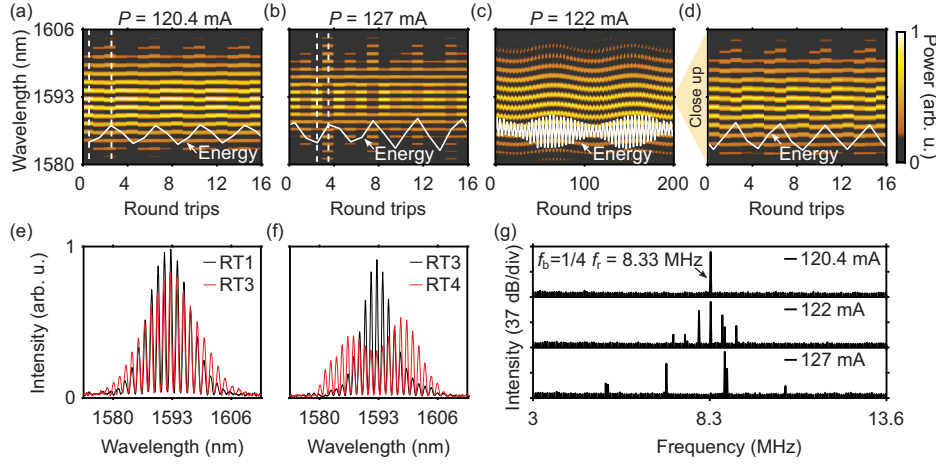


FIG. 3. Experimental observation of subharmonic, modulated subharmonic, and nonsubharmonic diatomic breathing SMs. (a)–(c) Time stretch recording of single-shot optical spectra over consecutive cavity RTs. The white curves denote the energy evolutions. (d) A magnified version of (c) showing the short period breathing. (e),(f) Single-shot spectra at the RT numbers of maximal and minimal energies within a period [indicated by dashed lines in (a) and (b), respectively]. (g) rf spectrum measurements for the three breathing regimes shown on a reduced frequency span centered on  $f_r/4$ .

subharmonic to modulated subharmonic and to nonsubharmonic breathing is again recorded.

Our experimental observations emphasize the robustness of the phase transition phenomenon, which is independent of the number of elementary constituents of the breathing soliton structure. To enlighten the laser dynamics in the different phases, we have focused on the case of the diatomic breathing SM that is observed for pump currents in the range 117 to 134 mA. The case of the triatomic breather molecule is illustrated in Fig. S3 of the Supplemental Material [43], displaying qualitatively similar features. We have complemented the rf spectrum measurements by additional spatio-spectral measurements, which are summarized in Fig. 3. Panels (a)–(c) therein show the roundtrip-resolved optical spectra measured by the time-stretch technique [44–47] (photodiode 1 in Fig. 1) for the three breathing regimes. Periodic variations of the spectral intensity across a well-defined period of 4 cavity RTs can be observed for the subharmonic regime [Fig. 3(a)], accompanied by corresponding synchronous periodic changes of the pulse energy (white curve). By contrast, the nonsubharmonic regime [Fig. 3(b)] shows degraded periodicities in both the optical spectrum and energy. The spectra recorded at the RT numbers of maximal and minimal energy within a period [Figs. 3(e) and 3(f)] indicate a larger modulation depth for the nonsubharmonic state. In both regimes, the period of spectral fringes remains almost unchanged over cavity RTs: the two breathers within the molecule have nearly equal separation, estimated to be 5.5 ps, which does not change over cavity RTs. The modulated subharmonic regime shown in Fig. 3(d) [close-up of Fig. 3(c)] features two sets of periodicities with a long period of approximately 88 cavity RTs and a short period of 4 RTs. The details of the corresponding rf spectra in the

vicinity of  $f_r/4$  shown in Fig. 3(g) further highlight the differences among the three regimes: the subharmonic state features a single very narrow frequency component located exactly at  $f_b = f_r/4$ , while a set of equally spaced narrow sidebands on both sides of  $f_b = f_r/4$  is observed for the modulated subharmonic case. In sharp contrast to this, the nonsubharmonic regime presents much broader spectral lines, confirming the frequency unlocked operation of the laser [see also Fig. S4 in the Supplemental Material [43] for a magnification of Fig. 3(g)].

To validate our experimental findings and gain a better insight into the complex temporal dynamics of breathing SMs, we have performed numerical simulations of the laser based on a lumped model which models each part of the laser cavity separately. Pulse propagation in the optical fibers is modeled by a generalized nonlinear Schrödinger equation (NLSE), which includes the effects of GVD and self-phase modulation for all the fibers, and gain saturation and bandwidth-limited gain for the active fiber. In the scalar-field approach, this equation takes the form [21,28,30,48–51]

$$\psi_z = -\frac{i\beta_2}{2}\psi_{tt} + i\gamma|\psi|^2\psi + \frac{g}{2}\left(\psi + \frac{1}{\Omega^2}\psi_{tt}\right), \quad (1)$$

where  $\psi = \psi(z, t)$  is the slowly varying electric field moving at the group velocity along the propagation coordinate  $z$ , and  $\beta_2$  and  $\gamma$  are the GVD and Kerr nonlinearity coefficients, respectively. The dissipative terms in Eq. (1) represent linear gain as well as a parabolic approximation to the gain profile with the bandwidth  $\Omega$ . The gain is saturated according to  $g(z) = g_0/[1 + E(z)/E_{\text{sat}}]$ , where  $g_0$  is the small-signal gain,  $E(z) = \int dt|\psi|^2$  is the pulse energy, and  $E_{\text{sat}}$  is the gain saturation energy. We note

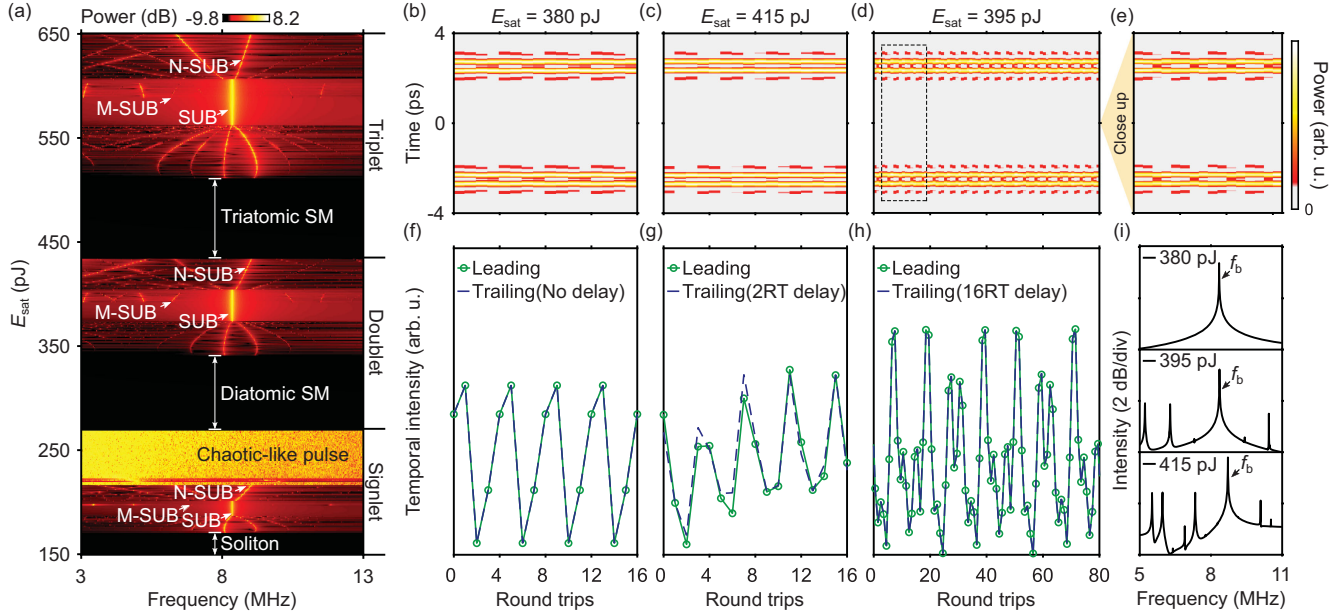


FIG. 4. Numerical modeling results. (a) Transitions among different laser states in the space of rf and gain saturation energy. (b)–(d) Temporal evolutions of the intensity over consecutive cavity RTs for subharmonic, nonsubharmonic, and modulated subharmonic diatomic breathing SMs, respectively. Close-up views of the evolutions for the modulated subharmonic breathing SM are shown in panel (e). (f)–(h) Evolutions of the peak intensities of the leading and trailing breathers for the three breather molecule states. The trailing pulse is delayed by a certain number of RTs in the nonsubharmonic and modulated subharmonic states to evidence the synchronization of the two breathers. (i) rf spectra of the three breather molecule states.

that  $E_{\text{sat}}$  can be used to change the pulse energy in the numerical simulations, thus playing a similar role to the pump current in the experiment. To match the numerical and experimental pulse energies to the same scale, however, rate equations governing the laser gain dynamics should be included into the numerical model. The discrete effects of the effective nonlinear saturation involved in the NPE mode-locking technique are modeled by an instantaneous and monotonous nonlinear transfer function for the field amplitude:  $T = \sqrt{1 - q_0 - q_m/[1 + P(t)/P_s]}$ , where  $q_0$  is the unsaturated loss due to the absorber,  $q_m$  is the saturable loss (modulation depth),  $P(z, t) = |\psi(z, t)|^2$  is the instantaneous pulse power, and  $P_s$  is the saturation power. The numerical model is solved with a standard split-step propagation algorithm and uses similar parameters to the experimental values (Supplemental Material [43], Table 1).

The rf spectrum can be obtained by Fourier transformation of the pulse energy in the simulations. The rf spectral intensity as a function of the gain saturation energy plotted in Fig. 4(a) shows good agreement with the experimental results (Fig. 2) and confirms the existence of an intermediate modulated subharmonic breathing state between the subharmonic and nonsubharmonic phases. One may see that compared to the experiments, the simulations tend to overestimate the range of existence of the chaoticlike regime occurring between single and multipulse laser emissions. Such a discrepancy can be ascribed to the extremely large parameter space used in the

simulations. By increasing the linear intracavity loss in the model or, equivalently, by changing the polarization state of the laser in the experiments, direct synchronization-desynchronization transitions (without the intermediate state) can also be observed in our breather laser (Supplemental Material [43] Fig. S5). Such direct transitions are saddle-node bifurcations [2]. The experiments (Fig. 2) and simulations [Fig. 4(a)] also reveal another interesting phenomenon: the higher the number of elementary constituents, the more robust the subharmonic breather structures against pump power (gain saturation energy) variations. Details are given in Fig. S6 of the Supplemental Material [43].

The roundtrip evolutions of the temporal intensity profiles of subharmonic, nonsubharmonic, and modulated subharmonic diatomic breathing SMs are shown in panels (b)–(d) of Fig. 4, respectively. The corresponding spectral evolutions are given in the Supplemental Material [43] (Fig. S7). The evolutions of the peak intensities of the leading and trailing pulses in the subharmonic breathing SM [Fig. 4(f)] reveal that the two breathers are synchronized to each other. While nonsubharmonic breathing SMs are quasiperiodic, there is a constant delay (of 2 cavity RTs) between the two breathers [Fig. 4(c)], and by adding this delay to the trailing pulse, the evolutions of the two pulses become synchronous [Fig. 4(g)]. This suggests that the dynamics of the two pulses pertains to lag synchronization, which has been extensively studied in chaotic systems

[2,52–55]. Modulated subharmonic breathing SMs also show lag synchronization [Fig. 4(h)]. Furthermore, we have also observed lag synchronization in the dynamics of triatomic breathing SMs. The evolutions of the temporal intensity profiles [Figs. 4(b)–4(e)] highlight a general feature of the three breathing SM states: each breather within the molecule is split into multiple subpulses, which arises from higher-order solitonlike evolution inside the anomalous dispersion segment of the laser cavity (see Figs. S8 and S9 in the Supplemental Material [43]).

In conclusion, we have observed subharmonic and non-subharmonic breathing soliton structures in an ultrafast fiber laser in both experiments and numerical simulations. We have also unveiled the existence of an intermediate state—modulated subharmonic breathing—between the two phases. These findings could stimulate parallel research on the synchronization of other coupled nonlinear systems where such a state has not been found yet. Furthermore, our results show that transitions among the three states are not restricted to a single-breather laser emission regime, but alike transitions also occur when the laser operates in a diatomic or triatomic breather SM generation regime, which further substantiates the universal nature of synchronization-desynchronization phenomena and opens the possibility to study the dynamics of nonlinear systems with three or more interacting frequencies. Mode-locked fiber lasers are very general physical platforms wherein the fiber propagation dynamics are governed by a generalized NLSE. This equation arises in many physical settings including, but not limited to, Bose-Einstein condensates, surface gravity waves, and superconductivity. Therefore, it is reasonable to assume that similar transitions may also arise in these physical systems when a feedback mechanism is included. Following the long history of analogies established between nonlinear fiber lasers and the various states of matter with the establishment of the concepts of molecules, crystals, rains, and gases of optical solitons among the others [56], one can also imagine to qualitatively relate the synchronization and desynchronization of breathing solitons and breather molecular complexes reported in this Letter with commensurate-incommensurate phase transitions [57], which are ubiquitous phenomena in many areas of condensed-matter physics and beyond.

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