OPTIMAL TECHNIQUES FOR ON-LINE CONTROL SYSTEMS WITH AN APPLICATION TO THE CONTROL OF A POWER-STATION BOILER

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SUMMARY.

The thesis describes the development of techniques for the optimal control of industrial processes by an on-line digital computer. The process model is taken to be a set of linear differential equations describing the behaviour of the process for small displacements about a particular operating point. The design of the control system is based on the minimisation of a performance index which is a quadratic function of the displacements of the system variables. The minimisation is subject to amplitude constraints on the control inputs.

An investigation is carried out into the theoretical and practical value of existing techniques for optimal on-line control of systems with constrained inputs. One result of the investigation is a new theorem on the equivalence of single-stage and N-stage optimal control policies. Because of practical limitations of existing methods, a new technique is developed in the thesis. The theoretical value of this technique is considered to be not less than that of any comparable technique. In each case the optimal control policy is derived using the concepts of dynamic programming. The proposed technique involves the exact minimisation of a single-stage performance index at each successive sampling instant, and uses a computational procedure based on the geometrical properties of the index to solve the resulting quadratic

SUMMARY (contd)

programming problem. The method of solution of the control problem is such that most of the difficulties associated with the practical implementation of other methods are removed.

A study by simulation is included of the application of these techniques to the dynamic control of a power-station boiler under changing load conditions. Results are given of tests carried out using both digital and hybrid computers. The practical limitations of existing methods are demonstrated, and it is shown that the proposed technique yields system responses which in certain cases show a significant improvement over responses achieved using other methods.

CONTENTS.

List of symbols.

A,B	=	system matrices
$\underline{a}$, $\underline{b}$, $\underline{M}$	=	control input bound vectors
C,V	=	performance indices
C(k+1)	=	$\underline{x}'(k+1)\ Q\ \underline{x}(k+1)$
$\underline{d}$	=	discrete system disturbance vector corresponding to $\underline{n}$
E	=	expected value
F	=	matrix of feedback coefficients
$\underline{f}$, $\underline{g}$, $\underline{h}$	=	vector functions
G	=	discrete system matrix corresponding to B
H	=	additional performance index matrix associated with $\underline{u}$
I	=	unit matrix
i, j, k	=	integers
L	=	$I - G(G'QG)^{-1}G'Q$
M	=	mass of liquid in drum
m	=	rank of Q
N	=	number of stages/quantity related to area of throttle valve opening
n	=	order of system/variable related to steam flow
$\underline{n}$	=	disturbance vector
P,R,S,W,Z	=	recurrence relation matrices for N-stage control
Q	=	performance index matrix
$\underline{q}$	=	vector of elements of principal diagonal of Q
r	=	number of control inputs to system

List of Symbols (contd)

$\underline{r}$	=	reference-input vector
s	=	number of zeros on principal diagonal of Q
T	=	sampling period
T_s	=	steam temperature
$\underline{u}$	=	control input vector
$\underline{x}$	=	state vector
$\underline{y}$	=	an n-vector
$\underline{\lambda}$	=	vector of elements of principal diagonal of H
ρ	=	rank of G
ρ_B	=	saturated vapour density corresponding to drum pressure
ϕ	=	state transition matrix corresponding to A

1) INTRODUCTION.

1.1) Modern control theory.

Conventional automatic control theory has made great advances over the past thirty years and industrial applications of control technology have contributed much to engineering progress. The limitations of conventional theory have however become apparent and over the past few years there has been a growing interest in a more sophisticated approach to control system design.

Many complex processes have been controlled by a number of individual control loops, rather than by an overall control scheme. Interactions between the process variables can and do cause interactions between the control loops, leading to a reduced performance of the system. In addition the criteria used to judge the performance of individual control loops have not been consistent. There is a case for the design of integrated control schemes using performance criteria which are related to the actual requirements of the process. The term "modern control theory" is used when the approach to system design is of this kind.

The first result of this approach is that a much better knowledge of the process dynamics is required. Secondly, techniques must be available for the analysis of the resulting mathematical model, and for the subsequent

1) contd.

1.1) contd.

synthesis of a system to control the process. It is doubtful if much significant progress would have been made in this second direction without the help of high-speed digital computers. The digital computer has stimulated thought into problems whose solution it was previously considered impracticable to attempt. The modern control problem often falls into this class.

Let us assume that the dynamics of a process are completely determined. The control problem may then be expressed as the development of a control law so as to minimise (or maximise) an index of overall system performance. The index will be a function of the process variables and possibly also of the control inputs to the process. This minimisation will be subject to constraints determined by physical limitations.

An n^{th} order system can usually be described at time t by means of a set of quantities $x_1(t), x_2(t), \dots x_n(t)$. These quantities are often called the state variables of the system, and together constitute the state vector $\underline{x}(t)$. If it can be assumed that the time-derivative of the state vector, $\frac{d\underline{x}}{dt}$, depends only on the current state of the system and not on past states, then the process can be described in vector notation by

1) contd.

1.1) contd.

$$\frac{d\underline{x}}{dt} = \underline{f}\,[\underline{x}(t), \underline{u}(t), \underline{n}(t), t] \qquad \ldots (1.1)$$

where

$\underline{x}(t) = [x_1(t)\ x_2(t)\ x_3(t)\ \ldots\ x_n(t)]'$, the n-dimensional state vector;

$\underline{u}(t) = [u_1(t)\ u_2(t)\ \ldots\ u_r(t)]'$, the r-dimensional control input vector;

$\underline{n}(t) = [n_1(t)\ n_2(t)\ \ldots\ n_s(t)]'$, an s-dimensional vector whose components represent random disturbances;

and $\underline{f}$ is a known vector function $[f_1\ f_2\ \ldots\ f_n]'$

It will be assumed that all the state variables of a system are accessible for observation and measurement. Where this is not so, best estimates must be constructed from the available information.[1,32]

A typical index of system performance is

$$C = E\int_{t_0}^{t_1} F(\underline{x}(t) - \underline{r}(t), \underline{u}(t), t)\, dt \qquad \ldots (1.2)$$

where the objective function F is a non-negative scalar function; $\underline{r}(t)$ is the reference input vector to the system; t_0 is fixed

1) contd.

1.1) contd.

and t_1 may be fixed or variable; and E stands for expected value.

The optimal control policy will be obtained by the minimisation of C with respect to the control inputs, subject to a set of constraints

$$g_i(u_1, u_2, \dots u_r) \leq 0 , \; i = 1, 2, \dots \bar{r} \qquad \dots (1.3)$$

or $\underline{g}(\underline{u}) \leq 0.$

There may also be constraints on the process state variables

$$\underline{h}(\underline{x}) \leq 0 \qquad \dots (1.4)$$

Stated in this form the control problem is clearly a variational problem. Some success has been achieved by applying classical calculus of variations theory to problems in the control field[2]. However more progress has been achieved by a number of methods, all based on variational theory, which have been developed in recent years. It is not proposed to detail these methods except where a method has potential application in the field of control by on-line digital computer. An introduction to the more important optimisation techniques and a list of references is given by Leitmann[2].

1) contd.

1.2) On-line computer control.

The modern approach to control system design is based on the determination of a control policy to minimise an index of performance. The generation of the optimum policy for a complex industrial process would seem to be best accomplished by a digital computer. The computer is given all the information available on the dynamic behaviour of the process, and programmed to produce the optimum control policy as a sequence of numbers. In this context the expression "on-line" means simply that the computer is working in real time to control the process. Fig.1.1 illustrates a typical multivariable computer control system.

Using a digital computer introduces a discrete property into the system. The method of working is for the computer to sample the values of the process state variables $\underline{x}(t)$, use these to calculate the optimal control inputs $\underline{u}(t)$, and then return to sample for new values of $\underline{x}(t)$. The cycle time will be decided by reference to the characteristics of the process. After calculation the control input vector will be held constant until it is updated one cycle time (sampling period) later. $\underline{u}(t)$ will therefore be a piecewise-constant function of time (Fig.1.2)

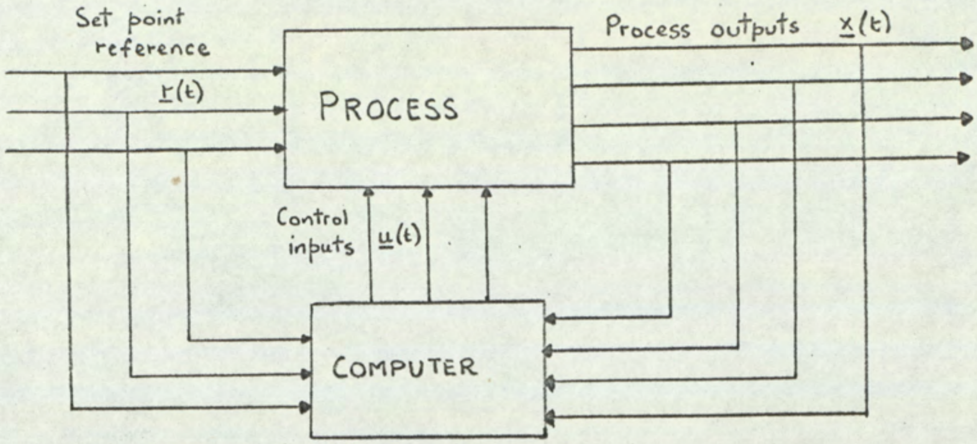


FIG. 1.1 – A COMPUTER CONTROL SYSTEM

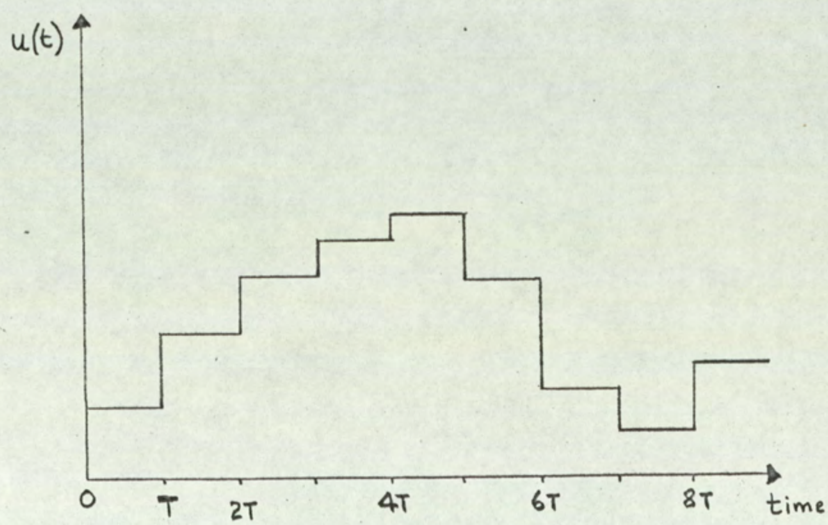


FIG. 1.2 – A PIECEWISE-CONSTANT CONTROL INPUT

1) contd.

1.2) contd.

$$\underline{u}(t) = \underline{u}(kT) \qquad kT \leq t < (k+1)T \qquad \dots (1.5)$$

where T is the sampling period and k any integer ≥ 0.

The control problem to be solved, and then implemented by the computer, may be considered in either of two ways. Assuming that a mathematical model of the process with all its non-linearities is available, the direct approach is to attempt to find a set of control policies which will transfer the system from any one state to any other in the best way possible. Research has shown the formidable difficulties inherent in this approach. Progress has been made in the theory of non-linear optimal control systems[3,4], but any solutions at present would inevitably require immense computing power.[5,34]

The second approach is to consider only small deviations of the system from some normal operating condition. This has the advantage that the system equations may be linearised about a steady-state condition and hence become much more tractable. The approach is reasonable since most engineering processes are designed to operate at one of only a small number of points rather than at an infinite number of points throughout their operating range.

1) contd.

1.2) contd.

In this thesis we consider a process which can be described by the n simultaneous first-order linear differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots x_n; \; u_1, u_2, \ldots u_r) \qquad i = 1,2,3 \ldots n$$

The equations may provide a complete description of the process or, as is more likely, may be a linearised set, valid only for small variations about a specific operating point. Without loss of generality the equilibrium state may be taken to be $\underline{x} = \underline{o}$, $\underline{u} = \underline{o}$.

In vector-matrix notation the equations are

$$\frac{d\underline{x}}{dt} = A\underline{x} + B\underline{u} \qquad \ldots (1.6)$$

where $\underline{x}$ is the n-dimensional state vector

$\underline{u}$ is the r-dimensional control vector

A is the (n×n) coefficient matrix of the process

B is the (n×r) "driving" matrix

The piecewise-constant nature of the control inputs during computer control, expressed in equation (1.5), leads to an exact solution of the equation (1.6) known as the state-transition equation,[1,6,7...]

$$\underline{x}[(k+1)T] = \phi(T)\,\underline{x}(kT) + G(T)\underline{u}(kT) \qquad \ldots (1.7)$$

1) contd.

1.2) contd.

ϕ and G are (n×n) and (n×r) matrices respectively. This equation describes the state of the process at the $(k+1)^{th}$ sampling instant in terms of the state at the k^{th} instant and the values of the control inputs over the period $[kT,(k+1)T]$.

For an index of system performance we require a function which gives a measure of the deviation of the system from its desired condition. An 'error-square' function of the state variables is the simplest suitable function in the mathematical sense. Over the period $[t_o,t_f]$ such an index would be

$$C = \int_{t_o}^{t_f} [q_1 x_1(t)^2 + q_2 x_2(t)^2 + \ldots q_n x_n(t)^2] \, dt$$

with $q_i \geqslant 0$, $i = 1,2,3 \ldots n$.

The values of the individual q's reflect the relative importance of deviations in each of the state variables.

The emphasis in computer control systems is placed on the state of the system at the sampling instants. Consequently a modified form of C is more popular in the study of such systems. The modification consists of replacing the integral by a similar sum;

1) contd.

1.2) contd.

over [0, NT]

$$C_N = \sum_{k=1}^{N} \underline{x}'(kT)\, Q\, \underline{x}(kT) \qquad \dots (1.8)$$

where Q is an (n×n) matrix, normally of the form

$$Q = \begin{bmatrix} q_1 & 0 & 0 & \dots & 0 \\ 0 & q_2 & 0 & \dots & 0 \\ 0 & 0 & q_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & q_n \end{bmatrix} \quad \begin{array}{l} \text{with } q_i \geqslant 0 \\ i = 1,2,\dots n \end{array} \qquad \dots (1.9)$$

The index is to be minimised with respect to the control input variables.

The constraints on these variables are taken to be pairs of known bounds, so that

$$a_i \leqslant u_i \leqslant b_i \quad , \quad i = 1,2,3 \dots r \qquad \dots (1.10)$$

The control problem is to find a sequence of expressions for $\underline{u}(o)$, $\underline{u}(T)$, $\underline{u}(2T)$, ... $\underline{u}[(N-1)T]$ which will minimise the index of equation (1.8), subject to the set of equations (1.7) and the input bounds of equation (1.10).

1.3) The development of a control scheme.

The principal reason for posing the control

1) contd.

1.3) contd.

problem in the form given in Section 1.2 is that in certain circumstances an analytical solution of this problem is possible. A result due to Kalman et.al,[6,7,33] shows that optimal control by computer of a linear process with unconstrained control variables can be achieved by negative feedback of the state variables. By adjoining to the performance index a quadratic function of the control variables it is possible also to allow for constraints in these variables whilst maintaining the type of solution. The method used to arrive at this solution is the technique of dynamic programming, originally developed by Bellman[8].

Section 2 of this thesis describes the dynamic programming approach to the solution of problems in the optimal control of processes by a digital computer. Additions to existing theory made by the author are presented in Section 2.3. These new results form a significant part of the base on which an alternative control scheme is developed.

It is recalled that this theory applies only to systems described by linear differential equations. Almost certainly such a system description is an approximation to the true equations, valid only for

1) contd.

1.3) contd.

small deviations from a particular operating point. Any proposed control scheme should be evaluated with this in mind. Ideally, the control scheme should simulate as far as possible the performance of a solution to the overall non-linear control problem posed by the process. A major point must be that the scheme should be capable of dealing with a change in the operating condition whilst the computer is controlling the process. In the context of the system model outlined in Section 1.2, this means that the scheme should be able to handle on-line changes in the coefficients of the performance index matrix and in the values of the control input bounds, as well as the obvious changes in the process-description matrices A and B.

The standard dynamic programming approach to discrete bounded-input control systems results in a control scheme which is naturally rigid in construction. It will be shown, for example, that to allow for changes in the operating condition would require either an exceptionally fast digital computer or large amounts of fast computer storage, or both. Sections 3.2 and 3.3 of this thesis are devoted to the development of an

1) contd.

1.3) contd.

alternative control scheme which is more flexible than the standard scheme outlined in Section 3.1. The dynamic programming approach is maintained in the new scheme.

The control scheme proposed in Section 3.3 differs from the standard scheme in two respects. Firstly, minimisation of the performance index is carried out on a single-stage rather than an N-stage basis. Over the period $0 \leqslant t \leqslant NT$ corresponding to N sampling periods, or N stages, the problem stated in Section 1.2 is to find the sequence $\underline{u}(o)$, $\underline{u}(T)$, ..., $\underline{u}[(N-1)T]$ which will minimise the index

$$\sum_{k=1}^{N} \underline{x}'(kT)\ Q\ \underline{x}(kT)$$

The single-stage scheme solves the problem of finding $\underline{u}[(k-1)T]$ to minimise the current value of the index

$$C = \underline{x}'(kT)\ Q\ \underline{x}(kT)\ ,$$

for $k = 1,2, \ldots, N$.

This sub-optimal approach removes much of the rigidity of the standard scheme since optimisation may be carried out forwards rather than backwards in time. It has been

1) contd.

1.3) contd.

advocated[33,9], and been used[10,11] before.

The argument for using a single-stage scheme in this way has been that the computational advantages outweigh the loss in optimality. A second and more formal argument springs from one of the new results of Section 2.3. This result shows that under certain conditions the single-stage scheme is identically equivalent to an N-stage scheme. The conditions could well be fulfilled in practice.

The second respect in which the proposed control scheme differs from previous theory is in that an original method is used to deal with control input bounds. The method of implementing control input bounds normally used is to modify the performance index by adding a term which represents the "cost of control". The index becomes

$$C_N = \sum_{k=1}^{N} \left(\underline{x}'(kT)\; Q\underline{x}(kT) + \underline{u}'[(k-1)T]H\; \underline{u}[(k-1)T] \right)$$

or for single-stage control

$$C = \underline{x}'(kT)Q\; \underline{x}(kT) + \underline{u}'[(k-1)T]H\; \underline{u}[(k-1)T]$$

H is a diagonal matrix (r×r), similar in form to Q. The elements of H are to be chosen so that the bounds on the

1) contd.

1.3) contd.

control inputs are satisfied. In this respect they may be considered to be similar to Lagrange multipliers.

The new method, which is based on the properties of the functions involved, gives an exact solution of the single-stage minimisation problem

$$C_{opt.} = \min_{\underline{a} \leqslant \underline{u} \leqslant \underline{b}} [\underline{x}'(kT)Q\, \underline{x}(kT)]$$

It will be shown that the use of this method allows the control system to use the total available control to combat any disturbance. It also makes possible on-line changes in the operating condition by reducing computer storage requirements and off-line computation to a minimum.

1.4) <u>Optimal control of a power-station boiler.</u>

It is now accepted that the development of large steam generating units has led to a requirement for automatic control equipment to carry out the start up and shut down functions. The speed with which the plant can be started up after an overnight shutdown is likely to be limited by the time required to carry out the necessary control operations rather than by

1) contd.

1.4) contd.

the physical characteristics of the plant being controlled[19]. The specifications of new generating stations are beginning to include digital computers whose chief function is the logical sequencing of these control operations. It is difficult to justify the purchase of an on-line control computer for dynamic control alone, but if the computer is already in use for sequencing control it is easier to justify the further use of it to attempt dynamic optimisation of the plant.

It was stated earlier that the first pre-requisite of the modern approach to control systems was the need for a better knowledge of the process dynamics. The steam generation process has been receiving considerable attention in this direction over the past ten years. Important publications include the original 1958 paper of Chien, Ergin, Ling and Lee[12], a series of papers by Nicholson which includes work on optimal control[10,13,14], and a recent paper by Anderson, Kwan and Qualtrough[15]. Each of these develops at least one mathematical model of a power boiler. Such a model is highly suitable for simulation studies to compare the value of techniques for dynamic optimal control. Results

1) contd.

1.4) contd.

obtained may be evaluated not only in terms of simple numbers, but also in terms of their practical implications.

The control studies which have been carried out on power boilers have usually concentrated on the system response to a change in load, or steam-demand. The method of representation of this load change has caused difficulties in the acceptance for control studies of either the Chien model[12] or the Nicholson model[10]. Nelson Research Laboratories, English Electric Co.Ltd., have developed a model of a natural circulation boiler[16] with a better representation of load changes which was offered to the author for control studies. The method of representation used in the NRL model is similar to, though not quite as general as, that of Anderson et.al., whose model was available in April 1968 only.

The reduced form of the NRL boiler model which has been made available for control studies is a 6^{th}-order system, with 4 control inputs. Simulation studies have been carried out in this thesis to determine the performance of optimal control techniques

1) contd.

1.4) contd.

in returning the process to equilibrium following a load disturbance. Section 4 of the thesis describes the boiler model, the control algorithms considered, and gives results and conclusions from the simulation tests. Part of the simulation has been carried out on an English Electric KDN2-LACE hybrid computer installation, but most of the worthwhile results have come from fully digital simulation on the University's Elliott 803 computer. The boiler model itself is detailed in Appendix 1 of the thesis.

The results of the tests show that the proposed control scheme, developed in the thesis, can yield boiler responses which are significantly better than those obtained by alternative schemes. A test in which the proposed control scheme performs marginally worse than one of the alternative schemes is a case where none of the schemes compared are capable of worthwhile control. The number of strict comparisons between schemes has been limited by the difficulty of finding the correct "cost of control" multipliers for the standard schemes. This difficulty is highly significant in the determination of the practical value of the control schemes. No difficulty has been encountered in the

1) contd.

1.4) contd.

tests of the proposed optimal single-stage control scheme. Due to the method of representation of load changes, in no test is there instability of a system variable, unlike the majority of a set of similar tests carried out by Nicholson[10]. The performance of the boiler model seems to confirm the view that dynamic optimisation of power boilers is a feasible proposition.

2) THE DYNAMIC PROGRAMMING APPROACH TO ON-LINE CONTROL SYSTEMS.

The application of the dynamic programming technique to the control of processes has been the subject of a considerable amount of research over the past few years[1,2,6,7,17 etc.] The technique, developed originally by R.E.Bellman, is related to the calculus of variations and to Pontryagin's maximum principle[20]. Dynamic programming is based on the Principle of Optimality[8], which states that an optimal sequence of decisions $\underline{u}(o),\underline{u}(T),\ldots\underline{u}[(N-1)T]$ has the property that, whatever the initial state $\underline{x}(o)$ and initial decision $\underline{u}(o)$, the remaining decisions $\underline{u}(T)\ldots\underline{u}[(N-1)T]$ must constitute an optimal sequence with respect to the state $\underline{x}(T)$ resulting from the first decision $\underline{u}(o)$.

Kalman[6,7] and Tou[1] have given results which show that, under certain conditions, the optimal control policy consists of a feedback matrix with constant or time-varying coefficients.

2.1) <u>Solution of the state space equation for linear systems.</u>

We consider a linear system subject to random disturbances

$$\frac{d\underline{x}}{dt} = A(t)\ \underline{x}(t) + B(t)\ \underline{u}(t) + \underline{n}(t) \qquad \text{... (2.1)}$$

where $A(t)$ is the coefficient matrix of the process

$B(t)$ is the "driving" matrix

$\underline{x}(t)$ is the state vector

$\underline{u}(t)$ is the control input vector

$\underline{n}(t)$ is a vector of independent random variables.

The system described by equation (2.1) is linear and time-varying, unless A and B are constant matrices, in which case it is linear and stationary.

If both $A(t)$ and $B(t)$ are integrable over the relevant interval $t_o \leq t \leq t_f$, then there exists a unique solution of the system equation, given $\underline{x}(t_o)$, over $t_o \leq t \leq t_f$.

Let $\phi(t,t_o)$ be an $(n \times n)$ matrix which satisfies

2) contd.

2.1) contd.

$$\frac{d}{dt}(\phi(t,t_o)) = A(t)\,\phi(t,t_o), \qquad \ldots (2.2)$$

with $\phi(t_o,t_o) = I$, the $(n \times n)$ unit matrix

and $A(t)$ the same matrix defined in equation (2.1).

If we assume that

$$\underline{x}(t) = \phi(t,t_o)\,\underline{y}(t)$$

then

$$\frac{d\underline{x}}{dt} = \frac{d}{dt}(\phi\,\underline{y}) = \phi\frac{d\underline{y}}{dt} + \frac{d\phi}{dt}\underline{y}$$

$$= \phi\frac{d\underline{y}}{dt} + A\phi\underline{y}$$

But

$$\frac{d\underline{x}}{dt} = A\phi\underline{y} + B\underline{u} + \underline{n}.$$

Therefore

$$\phi\frac{d\underline{y}}{dt} = B\underline{u} + \underline{n}\ ,$$

or, assuming that ϕ^{-1} exists,

$$\underline{y}(t) = \int_{t_o}^{t} [\phi^{-1}(\tau,t_o)(B(\tau)\underline{u}(\tau) + \underline{n}(\tau))]d\tau + \underline{y}(t_o)$$

$$\underline{y}(t_o) = \phi^{-1}(t_o,t_o)\,\underline{x}(t_o)$$
$$= \underline{x}(t_o)$$

The solution of equation (2.1) is therefore

2) contd.

2.1) contd.

$$\underline{x}(t) = \phi(t,t_o)\underline{x}(t_o) + \phi(t,t_o)\int_{t_o}^{t} \phi^{-1}(\tau,t_o)[B(\tau)\underline{u}(\tau)+\underline{n}(\tau)]\, d\tau \qquad \ldots (2.3)$$

The contribution to the solution $\phi(t,t_o)\, \underline{x}(t_o)$ is in fact the complete solution to the homogeneous equation

$$\frac{d\underline{x}}{dt} = A(t)\, \underline{x}(t) \ ,$$

which represents the free motion of the system with no control or disturbance. $\phi(t,t_o)$ is known as the state transition matrix.

If the system under consideration is time-invariant, so that A and B are constant matrices, then ϕ becomes

$$\phi(t,t_o) = e^{A(t-t_o)} \qquad \ldots (2.4)$$

The solution for $\underline{x}$ becomes

$$\underline{x} = \underline{x}(t_o)e^{A(t-t_o)} + \int_{t_o}^{t} e^{A(t-\tau)}[B\underline{u}(\tau)+\underline{n}(\tau)]\, d\tau \qquad \ldots (2.5)$$

In the general time-varying case, there is no simple expression for the state transition matrix. The expression one might expect

2) contd.

2.1) contd.

$$\phi(t,t_o) = \exp \int_{t_o}^{t} A(\tau)\, d\tau$$

is true only if A(t) and $\int_{t_o}^{t} A(\tau) d\tau$ commute for all t[21]. As a consequence it is necessary to resort to numerical integration for computing ϕ.

For the synthesis of optimal computer control systems it is simpler to use a difference equation for the system rather than the original differential equation. If we assume that the components of $\underline{x}$ are measured only every T seconds and that the values of the control inputs are held constant between these sampling instants, then the above analysis leads to the exact solution for the system

$$\underline{x}[(k+1)T] = \phi(kT)\ \underline{x}(kT)+G(kT)\underline{u}(kT)+\underline{d}(kT) \quad \ldots (2.6)$$

where k is any integer ≥ 0,

$$\phi(kT) = \phi[(k+1)T,T]$$

$$G(kT) = \int_{kT}^{(k+1)T} \phi[(k+1)T,\tau]\ B(\tau)d\tau$$

2) contd.

2.1) contd.

and $$\underline{d}(kT) = \int_{kT}^{(k+1)T} \phi[(k+1)T,\tau]\ \underline{n}(\tau)d\tau$$

For time-invariant or stationary processes, the difference equation (2.6), which is known as the state transition equation, reduces to

$$\underline{x}[(k+1)T] = \phi(T)\underline{x}(kT) + G(T)\underline{u}(kT) + \underline{d}(kT) \qquad \ldots (2.6A)$$

with

$$\phi(T) = e^{AT}$$

$$G(T) = \int_{0}^{T} e^{A(T-\tau)}\ B\ d\tau$$

and $$\underline{d}(kT) = \int_{kT}^{(k+1)T} e^{A(T-\tau)}\ \underline{n}(\tau)d\tau.$$

2.2) <u>Optimal control with unconstrained control inputs.</u>

a) <u>Linear time-varying systems.</u>

In this section the solution of the control problem posed by a linear time-varying system is derived using the technique of dynamic programming. The derivation is included to provide an example of

2) contd.

2.2) contd.

a) contd.

the general dynamic programming approach, resulting in a set of recurrence relations for the control inputs at successive stages. This method of expression of the solution of any N-stage problem is continued throughout the thesis.

Without ambiguity the notation of the state transition equation (2.6) may be simplified by omitting T to give

$$\underline{x}(k+1) = \phi(k)\ \underline{x}(k) + G(k)\ \underline{u}(k) + \underline{d}(k) \quad \ldots (2.7)$$

Starting at time $t = 0$ with initial state $\underline{x}(o)$ the control vector $\underline{u}(t)$ is required to return the system to equilibrium ($\underline{x} = \underline{o}$) by time $t = NT$ in such a way that the performance index

$$C_N = \sum_{k=1}^{N} \underline{x}'(k)\ Q(k)\ \underline{x}(k) \quad \ldots (2.8)$$

is minimised.

Q is an (nxn) diagonal matrix

2) contd.

2.2) contd.

a) contd.

$$Q(k) = \begin{bmatrix} q_1(k) & 0 & 0 & \cdots & 0 \\ 0 & q_2(k) & 0 & \cdots & 0 \\ 0 & 0 & q_3(k) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & q_n(k) \end{bmatrix} \quad \ldots (2.9)$$

with $q_i(k) \geqslant 0$ for $i = 1,2, \ldots n$ and all k.
The elements q_i of the Q matrix determine the weightings against deviations in the relevant components of the state vector. If one or more of the q_i is a function of k, equation (2.8) is seen to describe a time-weighted performance index.

The control problem is to choose the input sequence $\underline{u}(i)$, $i = 0,1,2, \ldots (N-1)$ so as to minimise C_N, for any initial state $\underline{x}(o)$. The dynamic programming approach to the solution of this N-stage decision problem is to traverse the state trajectory backwards in time from the equilibrium condition in order to determine the optimum input at each backward instant. Each input $\underline{u}(i)$ will then be independent of decisions taken at further backward (i.e. earlier) sampling instants.

2) contd.

2.2) contd.

a) contd.

Since the system equation involves random disturbances, only the expected value of the performance index can be minimised.

Let $$f_N[\underline{x}(o)] = \min_{\underline{u}(i)} E\, C_N \qquad \ldots (2.10)$$

where E stands for expected value

Imbedding this equation into the general equation for minimising the part of the index covering the intervals $[j,N]$, for $j = 0,1,2, \ldots (N-1)$,

$$f_{N-j}\,[\underline{x}(j)] = \min_{\underline{u}(i)} E\, C_{N-j}$$

$$= \min_{\substack{\underline{u}(j)\\ \underline{u}(j+1)\\ \vdots \\ \underline{u}(N-1)}} E \sum_{k=j+1}^{N} \underline{x}'(k)Q(k)\underline{x}(k) \qquad \ldots (2.11)$$

Now suppose that the first j stages are optimum. Then the contribution to the index from the remaining $(N-j)$ stages is to be equal to the contribution from the $(j+1)^{th}$ stage plus optimum contribution from the remaining $N-(j+1)$ stages.

2) contd.

2.2) contd.

a) contd.

By the Principle of Optimality,

$$f_{N-j}[\underline{x}(j)] = \min_{\underline{u}(j)} E\left\{ \underline{x}'(j+1)Q(j+1)\underline{x}(j+1) + f_{N-(j+1)}[\underline{x}(j+1)] \right\} \quad \ldots (2.12)$$

$f_{N-j}[\underline{x}(j)]$ is quadratic in $\underline{x}$, and it can be shown by induction[18] that

$$f_{N-j}[\underline{x}(j)] = \underline{x}'(j)\, P(N-j)\underline{x}(j) \quad \ldots (2.13)$$

where P is a symmetric matrix.

Substituting for f_{N-j} and $f_{N-(j+1)}$ in equation (2.12),

$$\underline{x}'(j)P(N-j)\underline{x}(j) = \min_{\underline{u}(j)} E\left\{ \underline{x}'(j+1)S[(N-(j+1)]\underline{x}(j+1) \right\} \quad \ldots (2.14)$$

with $S[(N-(j+1)]=Q(j+1)+P[N-(j+1)]$ ... (2.15)

Let $V_{N-j} = E\left\{ \underline{x}'(j+1)S[N-(j+1)]\underline{x}(j+1) \right\}$ ... (2.16)

Now, using the state-transition equation to express $\underline{x}(j+1)$ in terms of $\underline{x}(j)$, $\underline{u}(j)$, and $\underline{d}(j)$,

2) contd.

2.2) contd.

a) contd.

$$V_{N-j} = \underline{x}'\phi'S\phi\underline{x} + \underline{u}'G'SG\underline{u}$$
$$+ x'\phi'SG\underline{u}+\underline{u}'G'S\ \phi\underline{x} + E(\underline{d}'S\underline{d}) \qquad \ldots (2.17)$$

with $\underline{x}, \underline{u}, \phi, G = \underline{x}(j), \underline{u}(j), \phi(j), G(j)$

and $S = S[N - (j+1)]$

The expression for V_{N-j} involves the values of the state and control vectors, $\underline{x}$ and $\underline{u}$, at the j^{th} sampling instant only. The N-stage decision process has been reduced to a series of single-stage decisions. The optimum for each stage may be found by simple differentiation

$$\frac{dV_{N-j}}{d\underline{u}(j)} = G'S\phi\underline{x}(j) + \underline{x}'(j)\phi'SG + G'SG\ \underline{u}(j)$$
$$+ \underline{u}'(j)\ G'SG$$

In view of symmetry,

$$\frac{dV_{N-j}}{d\underline{u}(j)} = 2\ G'(j)\ S[N-(j+1)]\phi(j)\ \underline{x}(j)$$
$$+ 2\ G'(j)\ S[N-(j+1)]\ G(j)\ \underline{u}(j) \qquad \ldots (2.18)$$

Equating $\dfrac{dV_{N-j}}{d\underline{u}(j)}$ to zero gives the optimal control policy

$$\underline{u}(j) = F(N-j)\ \underline{x}(j) \qquad \ldots (2.19)$$

with

2) contd.

2.2) contd.

a) contd.

$$F(N-j) = -\ [G'(j)S[N-(j+1)]G(j)]^{-1}G'(j)S[N-(j+1)]\phi(j) \quad \ldots (2.20$$

$F(N-j)$ is clearly a feedback coefficient matrix.

The optimal control law is totally determined by the recurrence relationship betwen the P and F matrices of equation (2.20) and

$$P(N-j) = \phi'(j)S\phi(j)+\phi'(j)SG(j)F(N-j) \quad .. (2.21)$$

from equation (2.14)

Computation of the P and F matrices, starting with P(o), yields the N different F matrices required to formulate the control sequence $\underline{u}(i)$, i = o,1,2, ...(N-1). The sequence is generated in reverse time order, beginning with $\underline{u}(N-1)$ and ending with $\underline{u}(o)$.

The optimal control policy for a linear system is thus shown to consist of linear time-varying feedback of the state variables.

b) Linear time-invariant systems.

The development of an optimal control policy for linear time-varying systems does of

2) contd.

2.2) contd.

b) contd.

course also cover linear time-invariant systems. However, it is considered that an alternative development is simpler and gives more insight into the properties of the problem and of the solution.

Let the state transition equation of a linear time-invariant system be

$$\underline{x}(k+1) = \phi(T)\underline{x}(k) + G(T)\,\underline{u}(k) \qquad \dots (2.22)$$

and let the index of system performance to be minimised be

$$C_N = \sum_{k=1}^{N} \underline{x}'(k)\; Q\; \underline{x}(k) \qquad \dots (2.23)$$

where Q is a constant weighting matrix.

Concentrating on the last sampling interval [(N-1)T, NT], we have that the contribution to C_N from this interval is

$$V_N = \underline{x}'(N)\; Q\; \underline{x}(N)$$

The minimum value of V_N is given by

2) contd.

2.2) contd.

b) contd.

$$\overline{V}_N = \min_{\underline{u}(N-1)} (\underline{x}'(N)\ Q\ \underline{x}(N))$$

$$= \min_{\underline{u}(N-1)} \left\{ (\phi\underline{x}(N-1)+G\underline{u}(N-1))'Q(\phi\underline{x}(N-1)+G\underline{u}(N-1)) \right\} \qquad \ldots (2.24)$$

Again the optimum value for $\underline{u}(N-1)$ can be found by simple differentiation. Setting

$\frac{dV_N}{d\underline{u}(N-1)} = 0$ for a minimum value yields

$$G'QG\ \underline{u}(N-1) = -\ G'Q\phi\ \underline{x}(N-1), \qquad \ldots (2.25)$$

of if $G'QG$ is non-singular

$$\underline{u}(N-1) = F(1)\ \underline{x}(N-1)$$

with $F(1) = -\ [G'QG]^{-1}\ G'Q\phi$ ... (2.26)

This feedback control law is known as the law for single-stage sub-optimal control, that is how to choose $\underline{u}(k)$ to minimise the value of the summand $\underline{x}'(k+1)Q\ \underline{x}(k+1)$ at the next sampling instant.

For optimal control over the last two sampling intervals,

2) contd.

2.2) contd.

b) contd.

$$\overline{V}_{N-1} = \min_{\substack{\underline{u}(N-1)\\ \underline{u}(N-2)}} \left\{ V_N + \underline{x}'(N-1)Q\,\underline{x}(N-1) \right\} \quad \ldots (2.27)$$

$$= \min_{\substack{\underline{u}(N-1)\\ \underline{u}(N-2)}} \left\{ \begin{array}{c} [(\phi+GF(1))\underline{x}(N-1)]'Q[(\phi+GF(1))\underline{x}(N-1)] \\ +\,\underline{x}'(N-1)Q\,\underline{x}(N-1) \end{array} \right\}$$

Setting $\dfrac{dV_{N-1}}{d\underline{u}(N-2)} = 0$ gives

$$\underline{u}(N-2) = -\,[G'RG]^{-1}\,G'R\phi\,\underline{x}(N-2) \quad \ldots (2.28)$$

$$= F(2)\,\underline{x}(N-2)$$

with $R = Q + (\phi+GF(1))'\,Q(\phi+GF(1))$ ... (2.29)

Similarly, for optimal control over three intervals

$$\underline{u}(N-3) = -\,[G'WG]^{-1}\,G'W\phi\,\underline{x}(N-3)$$

with

$$W = Q + (\phi+GF(2))'\,R(\phi+GF(2)),\text{ and so on.}$$

This development of the optimal control policy for an unconstrained linear time-invariant system follows closely that given by Nicholson[10].

The analysis shows that optimal control is achieved by the time-varying feedback

$$\underline{u}(kT) = F(N-k)\,\underline{x}(kT)\ ,\ k = o,1,2\ \ldots,(N-1) \quad \ldots (2.30)$$

2) contd.

2.2) contd.

where the $r \times n$ feedback matrices F are specified by the recurrence relations

$$F(N-k) = -\ [G'Z(N-k)G]^{-1}\ G'Z(N-k)\phi \qquad \ldots (2.31)$$

$$Z(N-k) = Q + [\phi+GF(N-k-1)]'Z(N-k-1)[\phi+GF(N-k-1)] \qquad \ldots (2.32)$$

$$Z(1) = Q\ ,$$

provided that rank $(Q) \geq r$.

2.3) New results for linear time-invariant systems.

a) Equivalence of N-stage and single-stage control policies [22]

The first new result is that the series of feedback matrices $F(N-k)$ specified by the recurrence relations (2.31) and (2.32) reduces under certain conditions to a sequence of constant matrices, each of which is identical to the single-stage feedback matrix $F(1)$. The repeated use of $F(1)$ is then fully optimal for the N-stage problem.

For this equivalence property to hold it will be shown that it is sufficient for the performance index of equation (2.23)

2) contd.

2.3) contd.

a) contd.

$$C_N = \sum_{k=1}^{N} (q_1 x_1{}^2 + q_2 x_2{}^2 + \ldots q_n x_n{}^2) \text{ , } q_i \geqslant 0,$$

to contain explicit functions of not more than r of the n state variables (i.e. not more state variables appearing in the index than the number of control inputs to the system).

Without loss of generality we may take a corresponding performance index matrix Q to be

$$Q = \begin{bmatrix} q_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & q_2 & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & q_r & & \vdots \\ \vdots & & & & 0 & \vdots \\ \vdots & & & & & \ddots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \text{ , } q_i \geqslant 0 \quad \ldots (2.33)$$

Let the number of zeros on the lower part of the diagonal of Q be s, so that

$$n = r + s$$

From equations (2.31) and (2.32)

2) contd.

2.3) contd.

a) contd.

$$Z(2)-Z(1) = [\phi-G(G'QG)^{-1}G'Q\phi]'Q[\phi-G(G'QG)^{-1}G'Q\phi] \quad \dots (2.34)$$

$$= \phi'L'Q\,L\,\phi \quad \dots (2.35)$$

where $$L = I - G(G'QG)^{-1}G'Q \quad \dots (2.36)$$

Assume that the result

$L'QL$ = the null matrix ($n\times n$)

is true for $n = r+s$.

If $L'QL$ is null, then $\phi'L'QL\phi$ is also null and $Z(2) = Z(1)$. To attempt a proof by induction on s, take $n = r+s+1$. Q and G become the matrices

$$Q_{r+s+1} = \begin{bmatrix} Q_{r+s} & \vdots & \begin{matrix}0\\0\\\vdots\end{matrix} \\ \cdots\cdots & & \vdots \\ 0 \quad 0 & \dots & 0 \end{bmatrix}, \quad G_{r+s+1} = \begin{bmatrix} G_{r+s} \\ \cdots\cdots\cdots\cdots \\ g_1 \quad g_2 \quad \dots \quad g_r \end{bmatrix} \quad \dots (2.37)$$

By forming appropriate products of the banded matrices, we find that

$$G'_{r+s+1}\,Q_{r+s+1}\,G_{r+s+1} = \left[G'_{r+s}\;Q_{r+s}\;G_{r+s}\right] \quad \dots (2.38)$$

and hence that

$$L_{r+s+1} = (I - G[G'QG]^{-1}G'Q)_{r+s+1}$$

2) contd.

2.3) contd.

a) contd.

$$= \begin{bmatrix} & L_{r+s} & \vdots & 0 \\ & & \vdots & 0 \\ & & \vdots & \vdots \\ & \cdots\cdots & \vdots & 0 \\ \alpha_1 & \alpha_2 \cdots & \alpha_{r+s} & 1 \end{bmatrix} \qquad \ldots (2.39)$$

where the α's are functions of the g_i of G_{r+s+1} and of the elements of $(G[G'QG]^{-1}G'Q)_{r+s}$.

$$\text{Finally, } (L'QL)_{r+s+1} = \begin{bmatrix} & (L'QL)_{r+s} & \vdots & 0 \\ & & \vdots & 0 \\ & & \vdots & \vdots \\ & \cdots\cdots\cdots & \vdots & \vdots \\ 0 & 0 & \cdots\cdots & 0 \end{bmatrix} \qquad \ldots (2.40)$$

The first part of the induction proof shows that if $(L'QL)_{r+s}$ is a null matrix, then $(L'QL)_{r+s+1}$ is also null. To complete the proof take $s = o$.

$$L_{n=r} = (I - GG^{-1}Q^{-1}G'^{-1}G'Q)_{n=r} \;, \qquad \ldots (2.41)$$

which is a null matrix provided that G and Q are both non-singular. This "square system" result is given by Tou[1].

It has therefore been shown that $Z(2) = Z(1)$, for $n = r, r+1, r+2, \ldots$, for the

2) contd.

2.3) contd.

a) contd.

set of performance index matrices Q defined by equation (2.33). The recurrence relations for Z show that in this case

$$Z(N) = Z(N-1) = \dots = Z(2) = Z(1),$$

so that the N-stage sequence of feedback matrices $F(N-k)$ reduces to a sequence of constant matrices. The optimal control policy is given by

$$\underline{u}(kT) = - [G'QG]^{-1} G'Q\phi \, \underline{x}(kT) \;, \; k=o,1,2\dots N \quad \dots (2.42)$$

for any number of stages, N.

To sum up, it has been shown that if not more state variables appear in the index than there are control inputs to the system, an N-stage control scheme simplifies identically to the corresponding single-stage scheme.

In large industrial systems it is quite likely that this condition on Q will be met, since the values of many of the state variables will be of no interest provided they remain bounded. The knowledge of possible simplification could also influence the choice of system model and performance index.

2) contd.

2.3) contd.

b) <u>On the singularity of the matrix product G'QG.</u>

The results of the theoretical synthesis of N-stage systems, and not least the equivalence property stated above, depend on this stage on the non-singularity of the matrix product G'QG. A discussion of this dependence seems to have been avoided by previous authors. It is considered necessary to attempt to establish whether solutions for the feedback matrix F(N-k) exist even when G'QG is singular.

It has been shown that provided the matrix product G'QG is non-singular the unbounded control problem has a solution in the form of a sequence of matrices, the appropriate matrix from the sequence to be selected at each sampling instant.

When dealing with processes involving anything more than a very small number of variables, it is likely that one or more of the components of the state vector $\underline{x}$ will not appear explicitly in the index of performance

2) contd.

2.3) contd.

b) contd.

$$C_N = \sum_{k=1}^{N} \underline{x}'(k)\ Q\ \underline{x}(k)$$

The matrix Q will then contain one or more zeros on the principal diagonal, and hence will be singular. In this case the matrix $G'QG$ may or may not also be singular. If it is, no inverse exists, and equation (2.26) for $F(1)$

$$F(1) = -\ [G'QG]^{-1}\ G'Q\phi$$

does not hold. It would of course be possible to ensure non-singular $G'QG$ by the device of replacing the relevant zeros by very small positive numbers.

We wish to determine whether a solution of the original equation for $F(1)$

$$[G'QG]\ F(1) = -\ G'Q\phi \qquad \ldots (2.43)$$

exists when the matrix $G'QG$ is singular and therefore has no inverse. Let us assume that the number of non-zero elements on the principal diagonal of Q is m. This is the number of state variables appearing explicitly in the performance index.

Physical considerations give immediately that

2) contd.

2.3) contd.

b) contd.

$m \leqslant n$, the total number of state variables, and that the number of control inputs, r, satisfies the relation $r \leqslant n$.

Q is an nxn matrix and G an nxr matrix.

The rank[23] of the matrix Q is m.

The rank of $G \leqslant r$

The rank of a product of matrices AB cannot exceed the rank of either factor

$$\text{rank}(AB) \leqslant \min(\text{rank } A, \text{rank } B)$$

$G'QG$ is an rxr matrix; if rank $(G'QG) < r$ the matrix is singular; if rank $(G'QG) = r$ the matrix is non-singular and possesses a unique inverse.

Let the rank of G be ρ, and consider the possibilities

1) $\underline{m = n}$

$$\text{rank}(G'QG) \leqslant \min(\text{rank } G', \text{rank } Q)$$
$$\leqslant \rho$$
$$\text{rank}(G'QG) \leqslant \rho$$

2) $\underline{m < n}$

(a) $m \geqslant r$

Again, rank $(G'QG) \leqslant \rho$

2) contd.

2.3) contd.

b) contd.

(2) contd.

(b) $m < r$

$$\text{rank}\ (G'QG) \leqslant m \quad , \quad < r$$

$G'QG$ is singular.

In cases (1) and (2a) it is probable that $\text{rank}(G'QG) = \rho$, so that $G'QG$ is non-singular unless $\rho < r$.

In practice if rank G (that is ρ) $< r$ we can select any ρ columns of G and express each of the remaining $(r-\rho)$ columns as a sum of multiples of these ρ columns. This means that only ρ of the r control input variables are independent. The problem should therefore be re-formulated in terms of ρ independent control variables.

For case (2b), the condition that there should be a solution of equation (2.43) for F(1) is that the augumented matrix $[G'QG : G'Q\phi]$ should have the same rank as $G'QG$[23]. The condition is satisfied if rank $G \geqslant m$. A set m of the r control variables will then provide a solution, the remaining $(r-m)$ variables having been given arbitrary values. Not all sets of $(r-m)$ of the

2) contd.

2.3) contd.

b) contd.

variables may be given arbitrary values; the criterion is that the coefficients of the remaining m control variables should give a non-zero minor of order m in $G'QG$.

Hence it has been shown that a solution of the equation for $F(1)$

$$[G'QG]\ F(1) = -\ G'Q\phi$$

exists even when the matrix $G'QG$ is singular. It may on occasion be necessary to re-formulate the problem in order to arrive at a solution, in particular when the control variables are not all independent.

$F(1)$ is the appropriate feedback matrix for single-stage control. The equations for the feedback matrices $F(2)$, $F(3)$ ... $F(N)$ are similar to the equation for $F(1)$, but in place of Q they involve matrices $Z(2)$, $Z(3)$ ... (equation (2.32)). These matrices are seen to have the same rank as Q, so that the above rank analysis holds for non-equivalent N-stage control as well as for single-stage control.

3) OPTIMAL CONTROL WITH BOUNDED CONTROL INPUTS.

The specification to be met by any physical control system will inevitably include some sort of constraints on the control inputs. If this were not so an optimal system would quite rightly demand infinitely large control for an infinitely short period to drive the system back to equilibrium.

The form in which constraints are specified will vary from problem to problem. Usually, "hard" constraints or bounds on the control inputs will be included. It is often argued that these bounds should be "soft" or elastic constraints, since small violations will not matter, but this approach may lead to difficulties with the degree of softness. The process engineer may also find bounds more easily specified than any other form of control input constraint.

3.1) The Kalman-Tou method.

Kalman and Koepcke[6], and Tou[1] have given a method which has become the standard method[24] of dealing with control input constraints in optimal computer control systems. The essence of the method, as outlined in the introduction, is that the index of system performance

$$C_N = \sum_{k=1}^{N} \underline{x}'(k) Q \, \underline{x}(k)$$

is modified to include a quadratic function of the

3) contd.

3.1) contd.

control inputs

$$C_N = \sum_{k=1}^{N} \left\{ \underline{x}'(k)Q\,\underline{x}(k) + \underline{u}'(k-1)H\,\underline{u}(k-1) \right\} \quad \ldots (3.1)$$

where

$$H = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_r \end{bmatrix} \quad \text{with } \lambda_i > 0 \quad \ldots (3.2)$$

The addition of this term to the performance index makes only a small change to the dynamic programming solution of the control problem outlined in Section 2.2. In particular the Nicholson-type solution for linear time-invariant systems becomes, for single-stage control,

$$\underline{u}(N-1) = F(1)\,\underline{x}(N-1)$$

with, instead of $F(1) = -[G'QG]^{-1}\,G'Q\phi$,

$$F(1) = -[G'QG+H]^{-1}\,G'Q\phi \quad \ldots (3.3)$$

The extension to N-stage control follows a recurrence relation pattern similar to that of the N-stage modified index solution. For example

$$\underline{u}(N-2) = -[G'RG]^{-1}\,G'R\phi\underline{x}(N-2)$$

with

3) contd.

3.1) contd.

$$R = Q + F(1)' \, H \, F(1) + (\phi + GF(1))' Q (\phi + GF(1)) \quad \dots (3.4)$$

The effect of the term $\underline{u}'(k-1)H\,\underline{u}(k-1)$ is to add to the performance index a quantity which varies as the square of the amount of control action used to return the system to equilibrium after a disturbance. If this method is used to implement bounds on the control inputs, the λ_i of the H matrix are chosen so that each of the control inputs remains in its allowable region, $|u_i| \leqslant M_i$, $i = 1,2, \dots r$.

The method outlined above for the optimal control by computer of systems with bounded control inputs is theoretically very neat and simple, which is an advantage. It is the opinion of the author, however, that the method is of doubtful practical value.

Let us consider the use of this method in practice. The significant difficulties arise from the selection of the values of the λ elements of the H matrix. Even with a good iterative scheme the work involved in calculating the best values for the λ's is considerable. (An example involving four control inputs is detailed in Section 4 of this thesis). This work may be done either off-line using a digital or a hybrid

3) contd.

3.1) contd.

computer, or on-line using the control computer whilst it is controlling the process.

If selection is made off-line on a "once and for all" basis, response tests must be made with the largest disturbance the system is expected to encounter. Full available control will therefore be used only when the largest disturbance occurs, and with smaller disturbances the system will not be optimum in the original sense. Alternatively a number of sets of $\underline{\lambda}$ could be calculated corresponding to different levels of disturbance. The sets would be stored in the computer and when a disturbance occurred it would be monitored in some way and the appropriate set of $\underline{\lambda}$ selected, with the aid of an interpolation routine. The sequence of feedback matrices $F(1)$, $F(2)$ ... $F(N)$ could then be calculated and finally appropriate control applied to the process.

It would be preferable if selection of $\underline{\lambda}$ could be carried out on-line. Let us assume that an estimate of the disturbance can be made and the estimate included in the system equations. Now, one iteration on $\underline{\lambda}$ involves calculating the λ-dependent sequence of feedback matrices, simulating the corresponding system response and using this to improve the current estimate of $\underline{\lambda}$. When a satisfactory

3) contd.

3.1) contd.

$\underline{\lambda}$ has been computed, the corresponding sequence of feed-back matrices can be applied immediately to the process. It seems unlikely that the application of the on-line method with its consequent speed requirements would be a practical proposition in a process control installation.

The preceding remarks on the selection of $\underline{\lambda}$ have not taken into account the likelihood that the system model is only an approximation, valid for small deviations about one particular operating point. If this is so, the off-line selection method would involve the computation of several sets of $\underline{\lambda}$ for each of several sets of Q,ϕ and G matrices corresponding to the different operating points. All this data must be stored in the control computer, suitably for fast access since even when $\underline{\lambda}$ has been selected it would remain to calculate the sequence of feedback matrices.

Both of the $\underline{\lambda}$-selection methods discussed hinge on the ability of the system to recognise and estimate the size of a disturbance. If this cannot be done satisfactorily, then the practical value of the Kalman-Tou method is further reduced.

The theoretical value of the method is difficult to estimate. A control policy evolved by this method is only sub-optimal, in the original sense, since the solution

3) contd.

3.1) contd.

of the problem of minimising

$$\sum_{k=1}^{N} \left\{ \underline{x}'(k)Q\,\underline{x}(k) + \underline{u}'(k-1)H\,\underline{u}(k-1) \right\}_{|\underline{u}| \leq \underline{M}}$$

is not identical with the solution of the problem of finding

$$\min_{|\underline{u}| \leq \underline{M}} \sum_{k=1}^{N} \underline{x}'(k)Q\,\underline{x}(k)$$

A comparison made recently by Fuller[25] for a number of low-order continuous systems shows that the technique may lead to performance index values 30 to 50% greater than the true optimum.

Another consideration is that the addition of the H matrix, whilst removing any singular G'QG difficulties, also destroys the equivalence theorems between N-stage and single-stage control.

3.2) <u>Single-stage control.</u>

In the previous section the Kalman-Tou method of dealing with control input constraints was derived and discussed. It was concluded that the method had practical

3) contd.

3.2) contd.

shortcomings. In the remainder of this section of the thesis an alternative method is derived which overcomes most of the difficulties associated with the standard method.

One way of simplifying the application of the theoretical solution of the computer control problem is to approximate to the N-stage optimal policy by a single-stage optimal policy. If the system equations in discrete form are

$$\underline{x}(k+1) = \phi\ \underline{x}(k) + G\ \underline{u}(k) \qquad \dots (3.5)$$

and the performance index is

$$C_N = \sum_{k=1}^{N} \left\{ \underline{x}'(k)Q\ \underline{x}(k) + \underline{u}'(k-1)H\ \underline{u}(k-1) \right\}, \qquad \dots (3.6)$$

then the optimal control in the last interval $t = [(N-1)T,\ NT]$ is

$$\underline{u}(N-1) = -\ [G'QG+H]^{-1}\ G'Q\phi\ \underline{x}(N-1) \qquad \dots (3.7)$$

Choosing the feedback matrix defined by this equation at any time $t = kT$ will guarantee a minimum of the contribution to the index at the next sampling instant,

$$C(k+1) = \underline{x}'(k+1)Q\ \underline{x}(k+1) + \underline{u}'(k)H\ \underline{u}(k) \qquad \dots (3.8)$$

3) contd.

3.2) contd.

The system response achieved by repeated use of the single-stage feedback matrix approximates to the "optimal" response achieved by applying the full N-stage series of feedback matrices[33]. An alternative to this is the repeated application of $F(\infty)$, the infinite-stage matrix[7]. The use of a constant feedback multiplier matrix has much to recommend it. Both storage and speed requirements of the control computer will be significantly reduced, making on-line control a more attractive proposition.

The alternative control scheme developed in this section is to be based on the use of a single-stage index, but not the single-stage version of the Kalman-Tou method, nor the constant feedback matrix arrived at by the convergence of an infinite-stage scheme. It is the contention of the author that there is no need to introduce the artificial H matrix to deal with control input bounds. It is possible to solve the optimal control problem posed by a single-stage index with no devices at all.

If the system to be considered is described by the state transition equation

$$\underline{x}(k+1) = \phi\ \underline{x}(k) + G\ \underline{u}(k)$$

and the single-stage index is

3) contd.

3.2) contd.

$$C(k+1) = \underline{x}'(k+1)\ Q\ \underline{x}(k+1),$$

then

$$C(k+1) = [\phi\underline{x}(k)+G\underline{u}(k)]'Q[\phi\underline{x}(k)+G\underline{u}(k)] \qquad \dots (3.9)$$

Let the control input bounds be described by

$$\underline{a} \leqslant \underline{u} \leqslant \underline{b} \qquad \dots (3.10)$$

We are required to minimise C(k+1) subject to the bounds $\underline{a}$ and $\underline{b}$.

At time t = kT, the matrices G, Q and ϕ and the value of the state vector $\underline{x}$ are all fixed. It remains only to minimise C(k+1) with respect to its one remaining variable, $\underline{u}$.

$C(k+1) = (\phi\underline{x}+ G\underline{u})'Q(\phi\underline{x}+G\underline{u})$ is a positive semi-definite quadratic form in the components of the control input vector $u_1, u_2, \dots, u_r$. The control problem therefore becomes a problem in the quadratic programming field.

We already know that the solution in the absence of constraints is

$$\underline{u}(k) = -[G'QG]^{-1}\ G'Q\ \phi\ \underline{x}(k) \qquad \dots (3.11)$$

In addition, from the new results of Section 2.3, a solution for $\underline{u}$ is still possible if G'QG is singular, and should the conditions of the equivalence theorem be fulfilled then equation (3.11) is the solution of the N-stage unbounded

3) contd.

3.2) contd.

problem as well as the single-stage problem.

It should be mentioned here that a previous author, Nicholson, also avoids the H matrix approach to single-stage systems[10]. His method is developed from an original suggestion in a paper by Kalman and Bertram[33]. Nicholson's method of dealing with input bounds is to calculate the free minimum by equation (3.11) and then to limit the individual components of u according to their bounds a and b. [Fig.13 of ref.10]. He calls this "including the constraints outside the linear range of equation (96)" [equation (3.11) of this thesis]. The method gives a fast but inaccurate solution in every case except where the mappings of constant C(k+1) in u-space consist of a family of hyperspheres, when the solution is accurate. This case only occurs when in the expansion

$$C(k+1) = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_{ij}\, u_i\, u_j + \sum_{i=1}^{r} \beta_i(k) u_i + \gamma(k), \quad \ldots (3.12)$$

the α_{ij} satisfy $\alpha_{ij} = 0, \quad i \neq j$,

$$\alpha_{11} = \alpha_{22} = \ldots = \alpha_{rr}$$

The error involved in Nicholson's method depends on the relative sizes of the α_{ij}. Large ratios of the u_i^2 terms, α_{ii}/α_{jj}, could lead to a "solution" far from the

3) contd.

3.2) contd.

true optimum.

3.3) A computational procedure to solve the quadratic programming problem.

It has been shown that the bounded input single-stage control problem results in a type of quadratic programming problem. Several published methods were considered for the development of an algorithm to solve the problem. The four methods listed below are examples of the different types of method considered.

1) Box's method[26] of using the transformation $u_i = a_i+(b_i-a_i)\sin^2 v_i$ to yield an unconstrained minimisation problem.

2) Wolfe's modified simplex method[27]. A simple linear transformation $v_i = u_i + a_i$ would give a notation nearer standard.

3) Fiaccio and McCormick's penalty function technique[28], using some approximation to the function near the bounds.

4) Hadley's method of Lagrange multipliers and "active constraints"[29].

Methods proposed for the solution of quadratic programming problems can be divided into two classes.

3) contd.

3.3) contd.

Class A consists of methods which transform the problem to remove the constraints, such as Box, and Fiaccio and McCormick, and class B methods which use the constraints to solve the problem, such as Wolfe, and Hadley. There also exist methods such as Hildreth[27] which remove some constraints and retain others.

In the choice of an algorithm for on-line computation of the solution of the control problem, it was considered advisable

a) to retain as far as possible the feedback nature of the solution

and b) to attempt to minimise the maximum time taken to find the solution.

These two conditions tend to rule out class A methods, which are more suitable for off-line computing on larger one-off problems. The class B methods (with the exception of Hadley's method) are again more suitable to larger, more complicated problems. None of the methods satisfy condition (a), not surprisingly, and not all methods guarantee even convergence in a finite number of steps.

The on-line control problem is distinguished from normal problems in the quadratic programming field by the relatively small number of variables, the

3) contd.

3.3) contd.

simplicity of constraints, and the necessity for speed guarantees. It seemed worthwhile therefore to look at the problem in a different way and to attempt to build up a new method more suitable for the particular application. The method which has been developed is based on a geometrical view of the function in variable-space, yielding certain properties of the function and of its minimum.

We distinguish immediately between the free or global minimum, which is known to be given by

$$\underline{u}(k) = - [G'QG]^{-1} G'Q\phi \, \underline{x}(k) \qquad \ldots (3.11)$$

and the constrained minimum, which is the solution of the problem.

The function, C(k+1) of equation (3.12), is a positive semi-definite quadratic form in the control variables $u_1, u_2, \ldots, u_r$, and is therefore convex. It follows that any local minimum is the global minimum. From this we have properties 1 and 2 of the solution.

<u>Property 1</u> - If the free minimum satisfies all the bounds on $\underline{u}(k)$, then it is the solution of the problem.

<u>Property 2</u> - If the free minimum violates any of the bounds

3) contd.

3.3) contd.

Property 2 (contd) - on $\underline{u}(k)$, then the constrained solution must lie on at least one of the bounds.

Proof. Assume that the free minimum violates one or more of the bounds, $\underline{a} \leqslant \underline{u} \leqslant \underline{b}$, and that the constrained solution lies totally inside the bounds. Then the solution must also be the global minimum, which contradicts the original assumption.

Properties relating the free and constrained minima are clearly important, since the free minimum is given by a simple matrix equation (3.11). A further relational property which applies is

Property 3 - If the free minimum violates a number m of the 2r bounds, then the constrained solution lies on at least one of these m bounds.

Proof. The mapping in $\underline{u}$-space of surfaces of constant $C(k+1)$ is a family of r-dimensional hyperellipses. The hyperellipses are cut by hyperplanes which represent the bounds. Let the free minimum be a point which lies outside m of the 2r hyperplanes which define the allowable control region. Then a straight line joining the free minimum to any

3) contd.

3.3) contd.

Proof (contd) interior point of the allowable region must cross all the m hyperplanes. The point at which the last of these m hyperplanes is crossed is a point on the boundary of the region. Since the gradient of the function value along the line, $\frac{dC}{d\ell}$, must be always positive, this boundary point will yield a lower value of C than any interior point on the line. (Interior points are taken to include boundary points on any of the (2r-m) bounds not violated by the free minimum). The constrained minimum must therefore lie on at least one of the m bounds violated by the free minimum.

An illustration of this for a problem with two control variables is given in Fig.3.1.

Fig.3.1 also illustrates the fact that the constrained minimum point does not, in general, lie on all the m violated bounds. Besides confirming the failure of Nicholson's method of locating the solution (page 52) the figure shows that the computational procedure

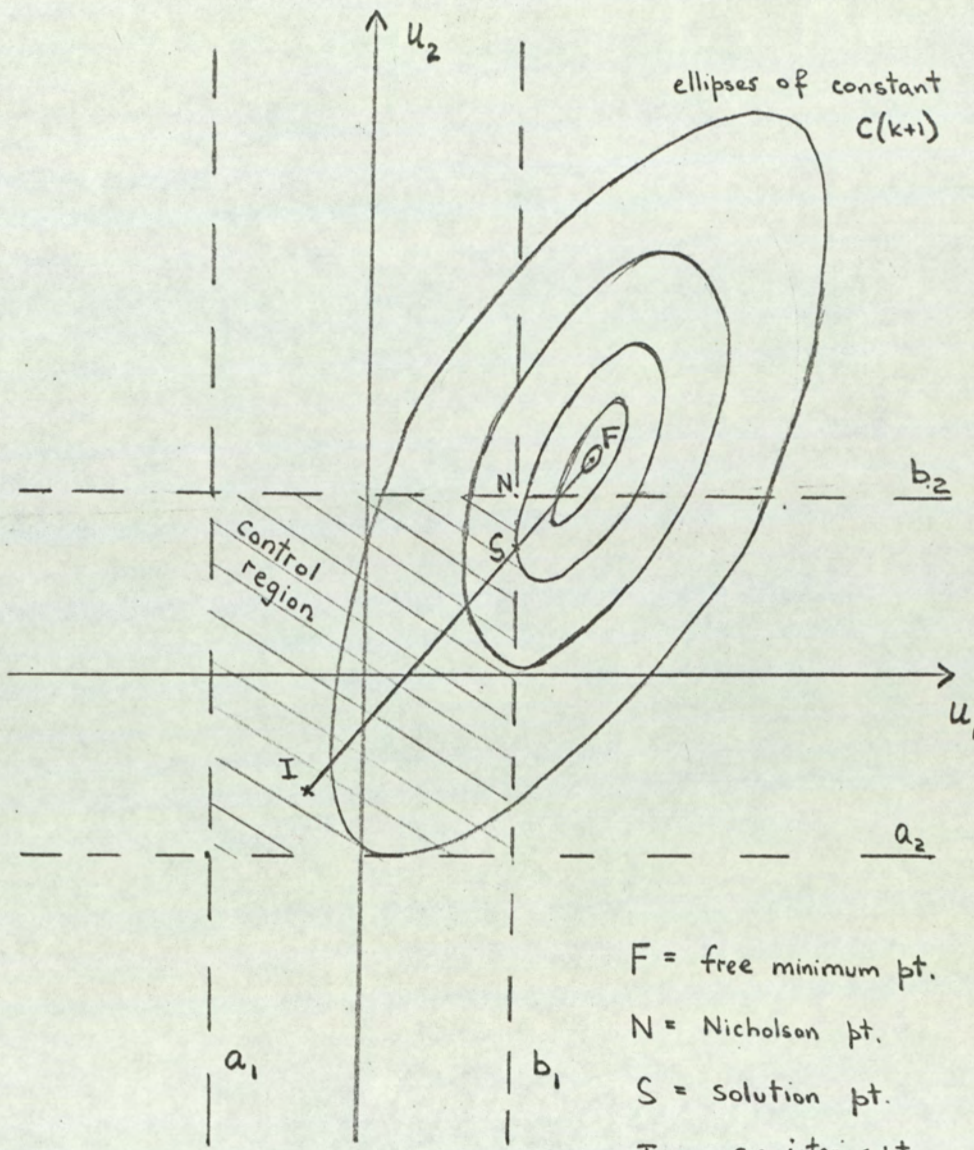


FIG. 3.1 - ILLUSTRATION IN CONTROL VARIABLE SPACE OF PROBLEM WITH TWO VARIABLES.

3) contd.

3.3) contd.

suggested by the author in reference 30 would not in general locate the correct minimum point.

This computational procedure is based on a further property of the function, which is still useful:-

Property 4 - The local minimum on a particular bound, $u_1 = b_1$ (say), is given by

$$\underline{u}^1 = -[G^{1\prime}QG^1]^{-1}G^{1\prime}Q(\phi\underline{x}+b_1\underline{c}_1) \quad \ldots (3.13)$$

where $\underline{u}^1$ is the vector of control inputs whose values remain free;

$$\underline{u}^1 = [u_2 \; u_3 \; \ldots \; u_r] \; ,$$

G^1 is the matrix G minus its first column:

$$G^1 = \begin{bmatrix} g_{12} & g_{13} & \cdots & g_{1r} \\ \vdots & & & \vdots \\ g_{n2} & g_{n3} & \cdots & g_{nr} \end{bmatrix}$$

and $\underline{c}_1$ is the vector whose components constitute the first column of the matrix G:

$$\underline{c}_1 = [g_{11} \;\; g_{21} \; \ldots \; g_{n1}]'$$

The property follows immediately from the expression of the function value on the bound as

3) contd.

3.3) contd.

$$C = [\phi\underline{x} + G^1\underline{u}^1 + b_1\underline{c}_1]'Q[\phi\underline{x}+G^1\underline{u}^1+b_1\underline{c}_1]$$

From the properties of the function already stated it is clearly possible to develop a computational procedure to locate the constrained minimum, by effectively reducing the dimension of the problem at each stage (.e.g. a problem where 3 of 5 components of the free minimum vector lay outside bounds could be expressed as 3 separate problems in 4 control variables, instead of one problem in 5 variables). However it seemed that more investigation of one particular bound would be worthwhile. The relevant bound is that one bound of all the bounds violated by the free minimum whose local minimum (c.f. property 4), possesses the largest function value. A further property of the solution, concerning this bound, is seen to apply:-

Property 5 - If the constrained solution lies on one only of the bounds violated by the free minimum, then this bound is the one whose local minimum has the largest function value.

The proof of property 5 follows immediately from the proof of property 3 if the constrained solution is the

3) contd.

3.3) contd.

local minimum on the bound. If the solution involves bounds not violated by the free minimum, the proof is more complex. Property 5 as it stands is not very useful in developing a computational procedure to locate the solution. If it were possible to extend the property by proving that the constrained minimum always lay on the one bound of all bounds violated by the free minimum whose local minimum had the largest function value, the property would become computationally important. By comparing the local minima on each of the violated bounds, it would then be possible to reduce one problem of r^{th} order to one problem of $(r-1)^{th}$ order to one of $(r-2)^{th}$ order, and so on. Such a proof has been sought, but notfound, and it seems likely that the suggested extension to property 5 is not in fact a property of the solution of the problem.

The computational procedure which is used in Section 4 of this thesis is therefore based on properties 1 to 5 as stated above. As an example, for a problem in four control variables, the procedure carried out is as follows:-

Step 1 Compute the free minimum from the matrix equation

$$\underline{u} = -\,[G'QG]^{-1}\,G'Q\phi\underline{x}$$

3) contd.

3.3) contd.

Step 2 For each of the bounds violated by $\underline{u}$, compute the local minimum and treat this as the free minimum of a problem with three variables

Step 3 Reduce each three variable problem to a number of two variable problems, using the same technique.

Step 4 Use property 5 to solve each of the two variable problems.

Step 5 Compare each feasible solution to find the one with the lowest function value, which is the solution of the problem.

The logical ordering of the operations is by repeated vertical scanning of steps 2 to 5. Each step is nested within the previous step.

For a problem with r control inputs, the maximum number of individual computations of minima needed to find the solution is 2^r-1. Each minimum point is given by an equation similar to equation (3.13), its computation therefore involving only simple matrix operations. The time taken at each sampling instant to find the optimal control vector is therefore mainly dependent not on the order of the system, but on the number of control inputs.

3) contd.

3.3) contd.

The computational procedure in its present form is probably suitable for processes with reasonably small numbers of inputs. For large numbers of inputs, it would be necessary either to improve the method by extending the theory, if this is possible, or to develop another method. One possible method, suggested by Lighthill in a private communication[31], is to replace the rectangular parallelepiped $|u_i| \leq b_i$ by the constraint

$$\sum_{i=1}^{r} \left(\frac{u_i}{b_i}\right)^{2N} \leq 1$$

He suggests that N = 5 would give a close enough approximation. Computation of the optimal control vector would then involve the solution of the (r+1) simultaneous non-linear algebraic equations

$$\sum_{j=1}^{r} \alpha_{ij} u_j + \beta_i + 2N\lambda \frac{u_i^{2N-1}}{b_i^{2N}} = 0, \quad i = 1,2, \ldots r,$$

and $$\sum_{i=1}^{r} \left(\frac{u_i}{b_i}\right)^{2N} = 1$$

(Note - the bounds are taken here to be $|\underline{u}| = |\underline{b}|$)

3) contd.

3.4) <u>Conclusions on the proposed technique</u>.

The method proposed by Lighthill is an alternative way of solving the non-linear programming problem, and does not affect the approach to the solution of bounded-input control problems advocated in this section. The technique for optimal computer control of linear or linearised multivariable control systems put forward here has the following properties:-

a) The control policy conforms with tradition in that control is achieved by feedback unless or until saturation occurs.

b) The computing load is spread evenly over a control period [0,NT], since the minimisation carried out at each sampling instant will take a near-constant length of time. The Kalman-Tou method involves a large computing load at $t = 0$ followed by little work at all other sampling instants.

c) The computational procedure is unaffected by changes in any of the parameters of the system model, the performance index, or the bounds on the control inputs. It would probably be best to evaluate off-line the system ϕ and G matrices corresponding to each set-point, but even this could be done on-line if it were

3) contd.

3.4) contd.

c) contd.

desired to adapt to changes in the parameters of the A and B matrices. The technique offers maximum flexibility for use with non-linear processes, since changes in set-point present no problems.

d) The technique is only sub-optimal over a period [0,NT] since it is based on a single-stage index. Varied with respect to this index, however, it is completely optimal. In contrast to the Kalman-Tou method, the technique automatically makes full use of all the available control action.

4) OPTIMAL CONTROL OF A POWER-STATION BOILER.

4.1) The boiler model.

The equations used in this thesis to represent the dynamics of a natural circulation boiler were obtained by Harris, Leigh, Mudge and Sutton of English Electric Co.Ltd.[16] The equations are similar to equations given in the well-known paper of Chien, Ergin, Ling and Lee,[12] but use enthalpy throughout rather than a mixture of enthalpy and pressure. The complete equations and the assumptions under which they were obtained are listed in Appendix 1.

The process is represented by a set of six simultaneous linear first-order differential equations, suitable for the study of small perturbations about a given operating point. There are four control inputs. In vector-matrix notation the equations are

$$\frac{d\underline{x}}{dt} = A\underline{x}(t) + B\underline{u}(t) + \underline{n}(t) \qquad \ldots (4.1)$$

The 6×6 matrix A and the 6×4 matrix B given below are functions of the boiler parameters and of the operating level.

4) contd.

4.1) contd.

$$A = \begin{bmatrix} -a_{11}\alpha & -a_{12}\beta & 0 & \gamma-\xi\mu & -1 & 0 \\ a_{21}\alpha & -\beta & 0 & \gamma-\xi\mu & -1 & 0 \\ -a_{31}\alpha & a_{32}\beta & 0 & -a_{34}\gamma+\xi\mu & 1 & 0 \\ 0 & 0 & 0 & -\delta-\xi & 1 & 0 \\ 0 & 0 & 0 & \delta & -1 & 0 \\ 0 & 0 & 0 & a_{64}\gamma+\eta\xi & 0 & -\epsilon \end{bmatrix} \quad \ldots (4.2)$$

$$B = \begin{bmatrix} -1 & \mu & 0 & 0 \\ -1 & \mu & 0 & 0 \\ 1 & b_{32}\mu & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (4.3)$$

The disturbance vector $\underline{n}(t)$ represents changes in steam demand.

The components of $\underline{n}$ are related to a change in the area of opening of the throttle valve, N,

$$\underline{n} = [\ -\mu N \quad -\mu N \quad \mu N \quad -N \quad 0 \quad \eta N]' \qquad \ldots (4.4)$$

The change in steam flow from the boiler is given by

$$n = N + \xi x_4 \ , \qquad \ldots (4.5)$$

as stated in Appendix 1. The change is therefore a dependent variable[15].

The independent variable of the equations, t, is

4) contd.

4.1) contd.

proportional to time.

The state vector, $\underline{x}(t)$, the control vector, $\underline{u}(t)$, and the disturbance vector $\underline{n}(t)$ are dimensionless. Components of the vectors are related to variables of the boiler as follows:-

x_1 to the sum of mass-flow rates per sq.ft. of cross-sectional area in the riser and downcomer tubes

x_2 to the density of the liquid-vapour mixture leaving the riser tubes

x_3 to the mass of liquid in the drum

x_4 to the saturated vapour density corresponding to the drum pressure

x_5 to the enthalpy of drum and downcomer liquid

x_6 to the superheater outlet temperature

u_1 to the heat-input rate from the riser tube walls into the boiling liquid

u_2 to the feedwater mass-flow rate

u_3 to the enthalpy of the feedwater

u_4 to the heat-input rate from hot gases into the superheater tube walls

N to the area of the throttle valve opening

4) contd.

4.1) contd.

For this investigation, data is available on a particular natural circular boiler, a 100,000 lb/hr. industrial power-station boiler. An English Electric digital computer program exists which transforms such data into numerical values for the parameters α, β, γ ... a_{64}, b_{32} of the A and B matrices. For small perturbations about the normal operating point of this boiler, the values of these parameters are

$\alpha = -2.29066$	$a_{11} = -10.4878$
$\beta = 1.67817$	$a_{12} = -6.41307$
$\gamma = 2.02097$	$a_{21} = 0.33368$
$\delta = 1.09005$	$a_{31} = 0.34881$
$\epsilon = 0.042829$	$a_{32} = 1.04536$
$\eta = -0.128955$	$a_{34} = 1.02089$
$\mu = 5.68189$	$a_{64} = 0.00091809$
$\xi = 0.93696$	$b_{32} = 0.71946$

The numerical forms of the relationships between the boiler and equation variables are

4) contd.

4.1) contd.

$$x_1 = -\,0.26004\ \frac{\delta W}{W} \qquad u_1 = 2.9746 \times 10^{-2}\ \frac{\delta Q_B}{Q_B}$$

$$x_2 = 0.29418\ \frac{\delta \rho}{\rho} \qquad u_2 = 9.228 \times 10^{-4}\ \frac{\delta w_i}{w_i}$$

$$x_3 = 0.43122\ \frac{\delta M}{M} \qquad u_3 = 6.2190 \times 10^{-4}\ \frac{\delta h_i}{h_i}$$

$$x_4 = 0.22013\ \frac{\delta \rho_B}{\rho_B} \qquad u_4 = 4.9127 \times 10^{-1}\ \frac{\delta Q_{gs}}{Q_{gs}}$$

$$x_5 = 0.97025\ \frac{\delta h_w}{h_w} \qquad n = 3.0668 \times 10^{-2}\ \frac{\delta w_B}{w_B}$$

$$x_6 = 23.349\ \frac{\delta T_s}{T_s} \qquad t = \frac{\text{real time}}{7.35}$$

The system A and B matrices become

$$A = \begin{bmatrix} -24.024 & 10.762 & 0 & -3.3028 & -1 & 0 \\ -0.76434 & -1.6782 & 0 & -3.3028 & -1 & 0 \\ 0.79901 & 1.7543 & 0 & 3.2605 & 1 & 0 \\ 0 & 0 & 0 & -2.0270 & 1 & 0 \\ 0 & 0 & 0 & 1.0900 & -1 & 0 \\ 0 & 0 & 0 & -0.11897 & 0 & -0.042829 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 5.6819 & 0 & 0 \\ -1 & 5.6819 & 0 & 0 \\ 1 & 4.0879 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4) contd.

4.1) contd.

Fig.4.1 shows the uncontrolled transient reponse of this model to a + 10% change in the throttle valve opening. The individual response curves of boiler variables are given as percentage variations from normal levels rather than in absolute units. Agreement between these and earlier published boiler response curves[10,15] is considered to be reasonable. The drum level response exhibits the expected initial rise followed by a rapid fall. The fall is limited by the return of the steam mass-flow rate to zero, as observed by Anderson [15]. It was not possible to determine whether or not the rate settled at a small negative value because of the limited accuracy of the calculations. The superheater outlet temperature seems peculiarly insensitive to changes in steam demand. The time scale of the responses compares with that of responses obtained by Nicholson[10] for a similar boiler.

4.2) <u>On-line computer control</u>.

For control system studies the equation of the model, equation (4.1), is expressed in discrete form

$$\underline{x}[(k+1)T] = \phi(T)\ \underline{x}(kT) + G(T)\ \underline{u}(kT) + \underline{d}(kT) \quad \ldots (4.6)$$

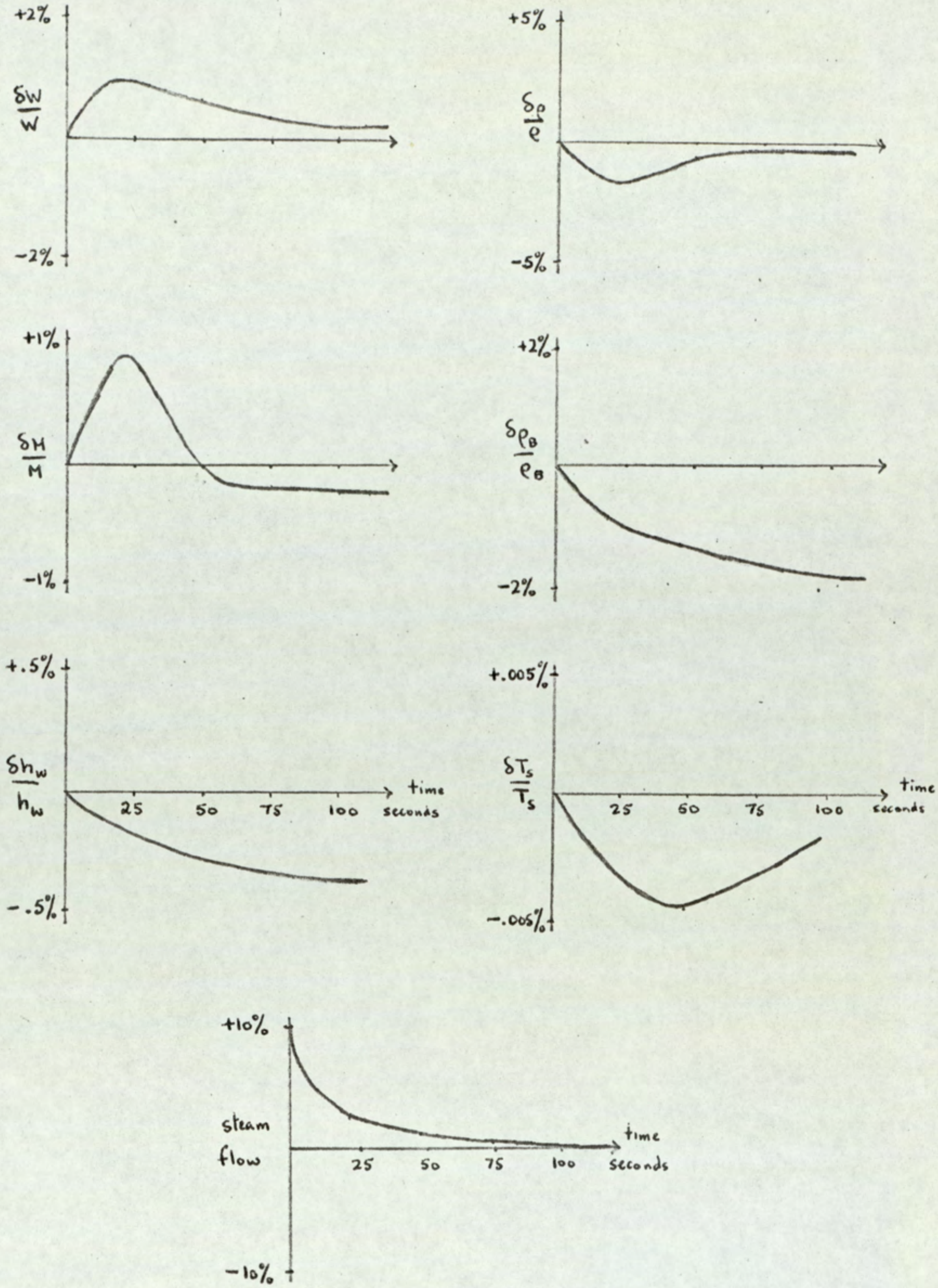


FIG 4.1 - SYSTEM FREE RESPONSE TO +10% STEAM DEMAND STEP.

4) contd.

4.2) contd.

The control input vector $\underline{u}(t)$ is assumed to be updated every T seconds.

The matrices ϕ and G and the vector $\underline{d}$ were originally obtained from the system equations (4.1) to (4.4) by analytic integration using elementary Laplace transform techniques. This exercise was carried out to obtain the maximum information about the discrete system equations. Numerical integration using the given data serves as a check on the analytical results, and has been used for all later work.

The normal method of obtaining numerical values for the ϕ and G matrices is to use the series[7,10,15]

$$\phi(T) = e^{AT} = \sum_{r=o}^{\infty} \frac{A^r\ T^r}{r!}$$

$$G(T) = \int_o^T \phi(\tau)\ B\ d\tau = \sum_{r=o}^{\infty} \frac{A^r\ T^{r+1}}{(r+1)!}\ B$$

To aid convergence, it has been found convenient to compute the following matrices

$$\phi(^T/M) = \sum_{r=o}^{\infty} \frac{A^r(^T/M)^r}{r!}$$

4) contd.

4.2) contd.

and $$F_M = \sum_{r=o}^{\infty} \frac{A^r(^T/_M)^{r+1}}{(r+1)!}$$

with M a suitable integer (in this case between 10 and 32). These calculations can be carried out simultaneously. The relations between $\phi(^T/M)$, F_M and $\phi(T)$, $G(T)$, $\underline{d}(T)$ are

$$\phi(T) = \left[\phi(^T/M)\right]^M \qquad \ldots (4.7)$$

$$G(T) = \left[A^{-1}(AF_M+I)^M - A^{-1}\right] B$$
$$= S_M B$$

where $$S_M = \left[MI + \frac{M(M-1)}{2!}(AF_M) + \ldots + (AF_M)^{M-1}\right] F_M \qquad \ldots (4.8)$$

and $$\underline{d}(T) = S_M \underline{n} \qquad \ldots (4.9)$$

The simpler algorithm for G(T),

$$G(T) = A^{-1}[\phi(T) - I]B$$

is not successful in a case such as this where A is a singular matrix.

The principal objective of this part of the work is to test the value of several different control schemes rather than to test the boiler model. Certain conditions of the tests are therefore kept constant.

4) contd.

4.2) contd.

The change in steam demand which disturbs the system from equilibrium is taken to be + 10% change in throttle valve opening, applied as either a continuous or a randomly varying disturbance. The sampling period is taken to be T = 3, corresponding to 22 seconds real time. Under these conditions the numerical forms of ϕ, G and $\underline{d}$ are

$$\phi = \begin{bmatrix} -3.67.10^{-5} & 1.05.10^{-3} & 0 & -1.50.10^{-1} & -2.25.10^{-1} & 0 \\ -7.49.10^{-5} & 2.15.10^{-3} & 0 & -2.85.10^{-1} & -4.27.10^{-1} & 0 \\ 7.83.10^{-5} & 1.04 & 1 & 1.15.10^{-1} & 2.57.10^{-1} & 0 \\ 0 & 0 & 0 & 9.80.10^{-2} & 1.50.10^{-1} & 0 \\ 0 & 0 & 0 & 1.64.10^{-1} & 2.52.10^{-1} & 0 \\ 0 & 0 & 0 & -8.59.10^{-2} & -7.11.10^{-2} & 8.79.10^{-1} \end{bmatrix}$$

$$G = \begin{bmatrix} -1.03 & 0.677 & -0.807 & 0 \\ -1.92 & 1.28 & -1.49 & 0 \\ 1.51 & 28.4 & 1.25 & 0 \\ 0.788 & 0.788 & 0.638 & 0 \\ 0.695 & 0.695 & 1.44 & 0 \\ -0.183 & -0.183 & -0.112 & 2.82 \end{bmatrix}$$

$$\underline{d} \times 10^{3} = [-2.04 \ \ -3.91 \ \ 2.83 \ \ -2.42 \ \ -2.13 \ \ -0.552]'$$

The performance of a control scheme in returning the system to equilibrium is judged by the resulting value of the index

4) contd.

4.2) contd.

$$C = \sum_{k=1}^{N} \underline{x}'(k)\ Q\ \underline{x}(k) \qquad \dots (4.10)$$

For the purposes of comparison the upper limit N is normally given the value 10, corresponding to a total optimisation period of 220 seconds.

Q is as usual a diagonal matrix with all elements either positive or zero, determining the weighting against errors in each of the boiler state variables. Most of the simulation tests carry weightings against errors in drum pressure, drum level, and steam temperature to follow practical conditions.

Constraints on the control inputs are set as ± 10% maximum deviations from steady-state values. The bounds are expressed numerically by

$$|\underline{u}| \leqslant [2.97.10^{-3} \quad 9.23.10^{-5} \quad 6.22.10^{-5} \quad 4.91.10^{-2}]'$$

In this form the control problem is an example of the type of problem considered in the earlier part of the thesis. Simulation of the response of the boiler model to changes in steam demand under the different control schemes analysed earlier should therefore provide some practical evidence of their indivdual merits.

4) contd.

4.3) Control algorithms.

The remainder of the thesis describes the results of attempts at optimal control of the boiler model following a change in steam demand. Three different control algorithms are used. They are

a) 10-stage optimal control based on the modified Kalman-Tou index

$$C = \sum_{k=1}^{10} [\underline{x}'(k)Q\,\underline{x}(k) + \underline{u}'(k-1)H\,\underline{u}(k-1)] \quad \ldots (4.11)$$

b) Single-stage control with a similar modified index

c) Optimal single-stage control with an unmodified index, as developed in this thesis.

The three algorithms are discussed in detail in Section 3 of the thesis.

A change in steam demand is not strictly a random disturbance, since, if it is continuous, its value will generally be known beforehand. It is considered therefore that the detailed algorithms used should allow for knowledge of the size of the disturbance. The control policies (b) and (c) are easily modified to include this knowledge:-

4) contd.

4.3) contd.

b) For single-stage control corresponding to the index

$$C = \underline{x}'(k)\ Q\ \underline{x}(k) + \underline{u}'(k-1)\ H\ \underline{u}(k-1),$$

the optimal control vector is simply

$$\underline{u}(k) = -\ [G'QG+H]^{-1}\ G'Q(\phi\underline{x}(k)+\underline{d}) \qquad \ldots (4.12)$$

c) For optimal single-stage control, the control vector $\underline{u}(k)$ is chosen to yield the index value

$$C_{k+1} = \min_{|\underline{u}| \leq \underline{b}} \left[\{\phi\underline{x}(k)+\underline{d}+ G\underline{u}(k)\}'Q\{\phi\underline{x}(k)+\underline{d}+G\underline{u}(k)\}\right] \ldots (4.13)$$

The mechanics of this choice are detailed in Section 3.3 of the thesis.

a) The modification of the standard Kalman-Tou N-stage control policy to include a continuous disturbance is not simple. To the knowledge of the author, the recurrence relations for this type of control are not stated in the literature. Nicholson[10] develops a few stages of control under a continuous disturbance with an unmodified index, but does not extend this to deal with the Kalman-Tou index of equation (4.11).
The final stage of such a policy will correspond to the single-stage policy of (b),

$$\underline{u}(N-1) = -\ [G'QG+H]^{-1}\ G'Q[\phi\underline{x}(N-1)+\underline{d}]$$
$$= F_1\ \underline{x}(N-1) + D_1\ \underline{d}\ , \qquad \ldots (4.14)$$

4) contd.

4.3) contd.

a) contd.

with $F_1 = -[G'QG+H]^{-1}\ G'Q\phi$

$D_1 = -[G'QG+H]^{-1}\ G'Q$

For the previous stage,

$$\underline{u}(N-2) = -[G'\{Q+F_1'HF_1 + (\phi+GF_1)'Q(\phi+GF_1)\}\ G+H]^{-1} \begin{bmatrix} -G'\{Q+F_1'HF_1+(\phi+GF_1)'Q(\phi+GF_1)\}\ (\phi\underline{x}(N-2)+\underline{d}) \\ +G'\{F_1'HD_1 + (\phi+GF_1)'Q(I+GD_1)\}\ \underline{d} \end{bmatrix},$$

where I is the r × r unit matrix; or

$$\underline{u}(N-2) = +F_2\ \underline{x}(N-2) + D_2\underline{d} \qquad \ldots (4.15)$$

where $F_2 = -[G'R_2G + H]^{-1}\ G'R_2\ \phi,$

$D_2 = -[G'R_2G + H]^{-1}\ G'S_2$,

$R_2 = Q + F_1'HF_1 + (\phi+GF_1)'\ Q(\phi+GF_1),$

$S_2 = R_2 + F_1'HD_1 + (\phi+GF_1)'Q(I+GD_1)$

This treatment has been continued to yield recurrence relations for the control vector at any stage,

$$\underline{u}(N-k) = F_k\ \underline{x}(N-k) + D_k\underline{d} \qquad \ldots (4.16)$$

with
$$\left.\begin{aligned} F_k &= -[G'R_kG+H]^{-1}\ G'R_k\phi \\ D_k &= -[G'R_kG+H]^{-1}\ G'S_k, \end{aligned}\right\} \qquad \ldots (4.17)$$

$$\left.\begin{aligned} R_k &= Q+F_{k-1}'HF_{k-1}+(\phi+GF_{k-1})'R_{k-1}(\phi+GF_{k-1}) \\ S_k &= R_k+F_{k-1}'HD_{k-1}+(\phi+GF_{k-1})'(S_{k-1}+R_{k-1}GD_{k-1}) \end{aligned}\right\}\ldots(4.18)$$

and $R_1 = S_1 = Q.$

4) contd.

4.3) contd.

This set of recurrence relations completes the statement of the three control algorithms whose performance is to be compared. Algorithm (a) requires the complete control policy of N-1 feedback matrices to be computed in advance, whereas algorithms (b) and (c) require N-1 applications of single-stage policies.

4.4) Simulation on a hybrid computer.

It is generally accepted that simulation tests of experimental computer control schemes are best carried out on an analogue-digital computer system. Such a computer installation enables the process to be simulated on an analogue computer and control to be effected by a digital computer, thus approaching as nearly as possible the conditions applying in practice.

A system of this type was made available, for a limited period, by English Electric Co., Kidsgrove. The system consists of a LACE Mk.II analogue computer, a KDN2 digital computer, analogue-to-digital and digital-to-analogue conversion equipnent and a multiplexor scanning system. A full description is given in Sutton and Tomlinson[I1].

A considerable amount of work was put into

4) contd.

4.4) contd.

tests using the KDN2-LACE installation, more than was justified by the results. The contents of this section are limited to a discussion of the main points of the study together with some conclusions.

An analogous electrical circuit was built up to simulate the boiler on the LACE analogue computer. The twenty scan points on the multiplexor were allocated in three blocks of six with two points unused. The six points of each block were connected, in order, to the six LACE amplifier outputs representing the state variables x_1 to x_6. At a scan rate of 50 points/sec. the number of blocks of six state variables scanned was 15/2 per sec. The sampling interval was chosen to be 3 seconds, so that interrogation of the state of the system occupied less than 5% of the sampling interval.

The KDN2 is basically a process control computer and therefore is designed principally for fixed-point arithmetical operations and machine-code programming. Because the arithmetical operations are fixed-point a knowledge of the values of the elements of the system matrices was necessary before a control program could be written for the KDN2. Control

4) contd.

4.4) contd.

algorithm (c), the optimal single-stage control policy, had not at the time (1965) been developed. It was therefore decided to begin tests with the conventional single-stage policy, algorithm (b). A limited version of the general problem was taken to reduce the programming task involved. The performance index for the tests included weightings on the drum level, drum pressure and steam temperature variables only, i.e. x_3, x_4 and x_6. The performance matrix Q was

$$Q = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & & & & \cdot \\ \cdot & & \lambda_5 & & & \cdot \\ \cdot & & & 1 & & \cdot \\ \cdot & & & & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \lambda_6 \end{bmatrix}$$

and the "cost of control" H matrix,

$$H = \begin{bmatrix} \lambda_1 & \cdot & \cdot & 0 \\ \cdot & \lambda_2 & & \cdot \\ \cdot & & \lambda_3 & \cdot \\ 0 & \cdot & \cdot & \lambda_4 \end{bmatrix}$$

It was found necessary to decide on approximate values for $\lambda_1 - \lambda_6$ before writing the control program because of scaling difficulties with the digital computer, which has an 18-bit word length. The pilot test involved

4) contd.

4.4) contd.

a performance index with equal weightings against deviations in M, ρ_B, and T_s. Equal weightings against deviations in each of the control inputs were also chosen, the values of $\lambda_1 - \lambda_4$ being chosen to give about $^1/50$ of the contribution to the index of the same deviation in a state variable. These values may conveniently be expressed as two vectors,

$$\underline{q} = [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1]$$

and $$\underline{\lambda} = {}^1/_{50}[1 \quad 1 \quad 1 \quad 1]$$

This notation will be continued for later tests.

A KDN2 program of 1750 instructions was required to calculate the elements of the feedback matrix

$$F = - [G'QG + H]^{-1} \; G'Q\phi \;,$$

and a further 200 instructions for the controlprogram

$$\underline{u}(k) = F \; \underline{x}(k)$$

The two functions of the KDN2 were

a) to accept a set of $\lambda_1 - \lambda_6$ and compute the corresponding F matrix, and then

b) to interrogate the state of the boiler model once every 3 seconds, up-dating the values of the control inputs by execution of the control program.

4) contd.

4.4) contd.

The area of opening of the throttle valve was represented by a voltage, connected through a switch to the appropriate amplifiers of the analogue circuit. In all the simulation tests this voltage, controlled by the KDN2, had a constant level corresponding to a 10% change in load and a sign which varied randomly, changes in sign occurring at the sampling instants.

A series of twenty tests was carried out with different values of $\lambda_1 - \lambda_6$. A section from one of the tests is shown in Fig.4.2. The general behaviour of the system was satisfactory, the response curves exhibiting the expected trends and levels. It was found, however, that the initial $\underline{\lambda}$ selected,

$$\underline{\lambda} = \frac{1}{50}[1 \quad 1 \quad 1 \quad 1]$$

gave control input values which bore little relation to the bounds. The machine code programs left little scope for large changes in individual λ's: it was possible to put any $\lambda_i = 0$ or to change the size of a λ_i by a factor of 2 or 3, but greater changes resulted in a complete loss of significance in both the control and F programs. The quantitative results may be summarised as

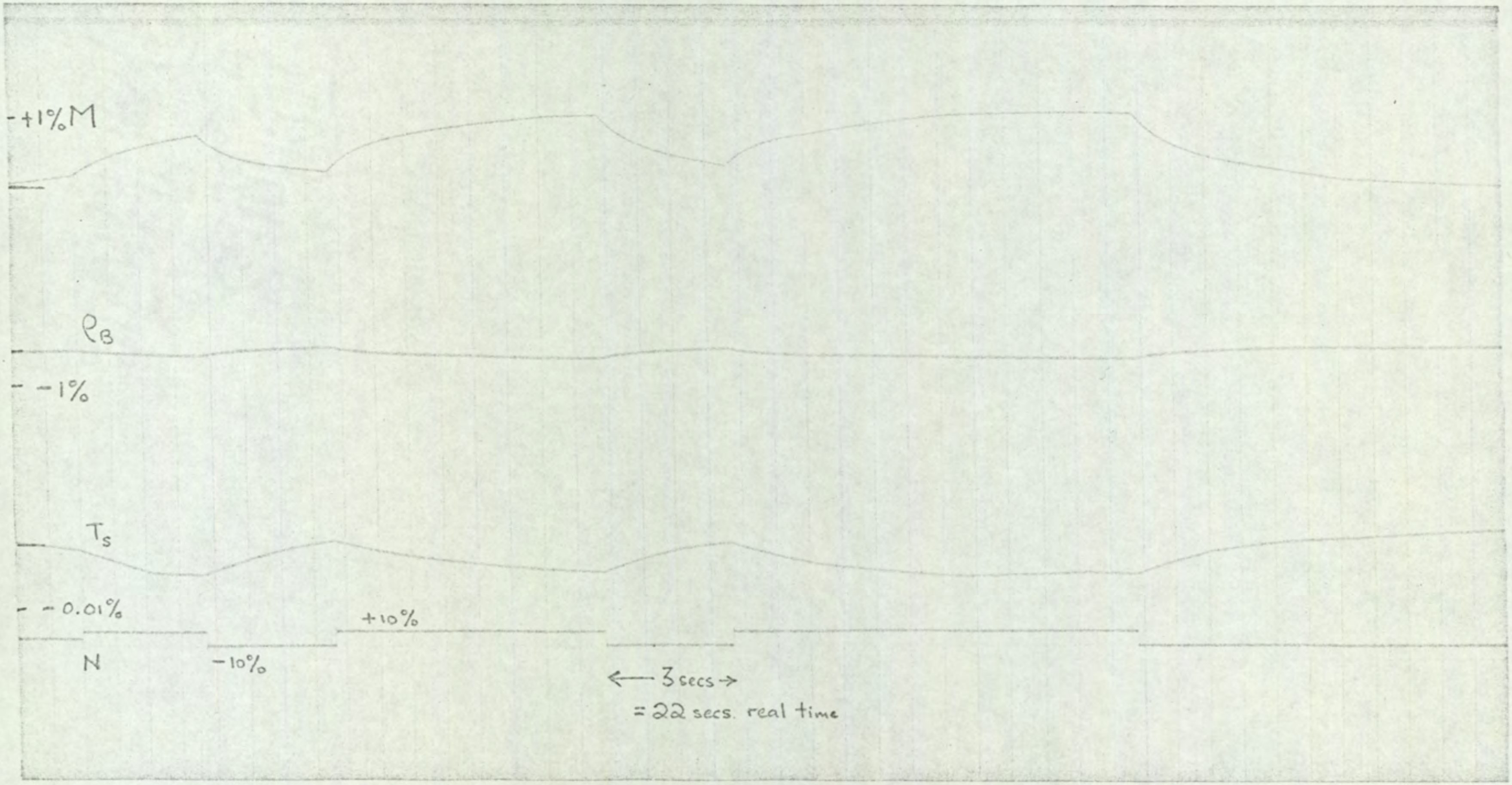


FIG. 4.2 - RESULTS FROM A KDN2-LACE SIMULATION TEST

4) contd.

4.4) contd.

a) the $\lambda_i = 0$ cases

(i) $\lambda_5 = 0$; no weighting against errors in x_3 (drum-level)

x_3 variation three times normal,

u_2 and u_3 (feedwater mass-flow rate and enthalpy) effectively zero

(ii) $\lambda_6 = 0$; no weighting against errors in x_6 (steam-temperature)

x_6 variation three to four times normal,

u_4 (superheater heat input rate) zero.

b) $\lambda_i \neq 0$;

the allowable variation in the λ's was too small to show significant variations either in the state variable responses or in the control input values.

The major difficulty encountered in this study was the near-complete absence of software from the KDN2 specification. No trouble was caused by the computer installation, one of the first of the hybrid type, which worked very well. To write the relevant floating-point subroutines for the KDN2 was not considered worthwhile, since the installation was available only for a limited period. The cause of the difficulty with software was the necessity to experiment with the values of $\underline{\lambda}$ to achieve

4) contd.

4.4) contd.

a set which would approximate to the bounds on the control inputs. To gain useful results on the relative performances of different control schemes using the KDN2-LACE would have posed even more problems. Since this was the main purpose of the study, it seemed better to transfer all the simulation to a scientific digital computer where the appropriate software would be available.

4.5) Simulation on a digital computer.

Simulation of a process model on a digital computer to some extent idealises the situation. For example, no allowance is likely to be made for the time taken to scan the state of the model. This factor alone has been known to cause difficulties. However, in this case, the series of tests on the KDN2-LACE computer did demonstrate the good behaviour of the process model under computer control. The dangers of total digital simulation with the boiler model would seem therefore to be reduced.

The computer requirement for these studies was a machine with automatic floating point arithmetic and a set of procedures for matrix operations, preferably available with an ALGOL or FORTRAN compiler. The University's Elliott 803 meets these conditions. The speed

4) contd.

4.5) contd.

of the computer was not considered important.

The matrix differential equation governing the behaviour of the boiler model is

$$\frac{d\underline{x}(t)}{dt} = A\underline{x}(t) + B\underline{u}(t) + \underline{n}(t)$$

In principle the transient response of the model may be obtained by standard numerical integration routines. With this model, as with another[15], the consequent computational load was found to be excessive. The model responses were therefore obtained using the difference equation

$$\underline{x}[(k+1)T] = \phi(T)\ \underline{x}(kT) + G(T)\ \underline{u}(kT) + \underline{d}(kT)$$

Elliott Algol programs have been written for simulation tests using each of the three control algorithms,

a) 10-stage optimal control according to the Kalman-Tou index

b) single-stage control with the same modified index

c) optimal single-stage control according to the original index

The test program for (a) requires, as data, the $\underline{q}$ and $\underline{\lambda}$ - dependent feedback matrices, which are evaluated in separate program.

No extra programs are required for algorithms

4) contd.

4.5) contd.

(b) and (c). Data input to (b) includes $\underline{q}$ and $\underline{\lambda}$. The program for (b) consists of about 25 Algol statements, replacing the 2000 instructions required for the same program by the KDN2 computer.

The main feature of the program for (c) is the declaration of a procedure for the solution of the quadratic programming problem to which the control problem reduces. This procedure, which was developed according to the theory of Section 3.3, is the only non-standard piece of programming involved, and is therefore reproduced in Appendix 2 of the thesis.

All three programmed algorithms allow for monitoring of the disturbance and are therefore as set down in Section 4.3. Initial tests were carried out with a continuous + 10% demand in steam flow, treated as a random disturbance. It was found that each of the control schemes maintained the system at a similar off-set state, the amount of off-set being the effect of the disturbance over the first sampling interval, during which there is no control. This was to be expected.

The discussion of the results of the simulation tests is divided into two sections, depending on the number of state variables appearing in the

4) contd.

4.5) contd.

particular performance index of a test (i.e. the number of non-zero elements in q).

a) Tests with some zero elements in q

A performance index with practical application would include weightings against deviations in drum level, drum pressure, and steam temperature, i.e. x_3, x_4, and x_6. Such an index was used for the KDN2-LACE tests. The other state variables may not be considered as important as these three in assessing the performance of the boiler, and might not appear in the index.

It was expected that the optimal single-stage scheme would perform well with low-order indices of this type, because of the equivalence theorem of Section 2.3(a). The theorem does not strictly apply to bounded control, but it was thought that the equivalence property would to some extent carry over from unbounded control. In fact this has been verified, and the tests show that algorithm (c) gives the best results for low-order indices in all respects.

If an index were chosen with weightings against three only of the state variables, the matrix product G'QG would be singular. The analysis of Section 2.3(b)

4) contd.

4.5) contd.

a) contd.

shows that it would then be necessary, for algorithm (c), to set one of the four inputs to an arbitrary value (probably zero), and optimise with respect to the remaining three inputs. To avoid restricting the comparison on use of control with algorithms (a) and (b), where this singularity is unimportant, all the indices used involve at least four state variables.

The base index considered is an index with equal weightings against x_3, x_4 and x_6, and a much smaller weighting against x_2:

$$\underline{q} = [0 \quad 0.001 \quad 1 \quad 1 \quad 0 \quad 1],$$

which approximates to an index involving only the three important state variables.

The responses of the system to a continuous + 10% load disturbance under each of the three control schemes, with this index, are shown in Fig.4.3.

The response under algorithm (c) needed to be computed once only, since the algorithm has been constructed to solve on-line the problem posed by amplitude constraints on the inputs. The response is seen to be good, with full use being made of the available control action. The value of the performance index is

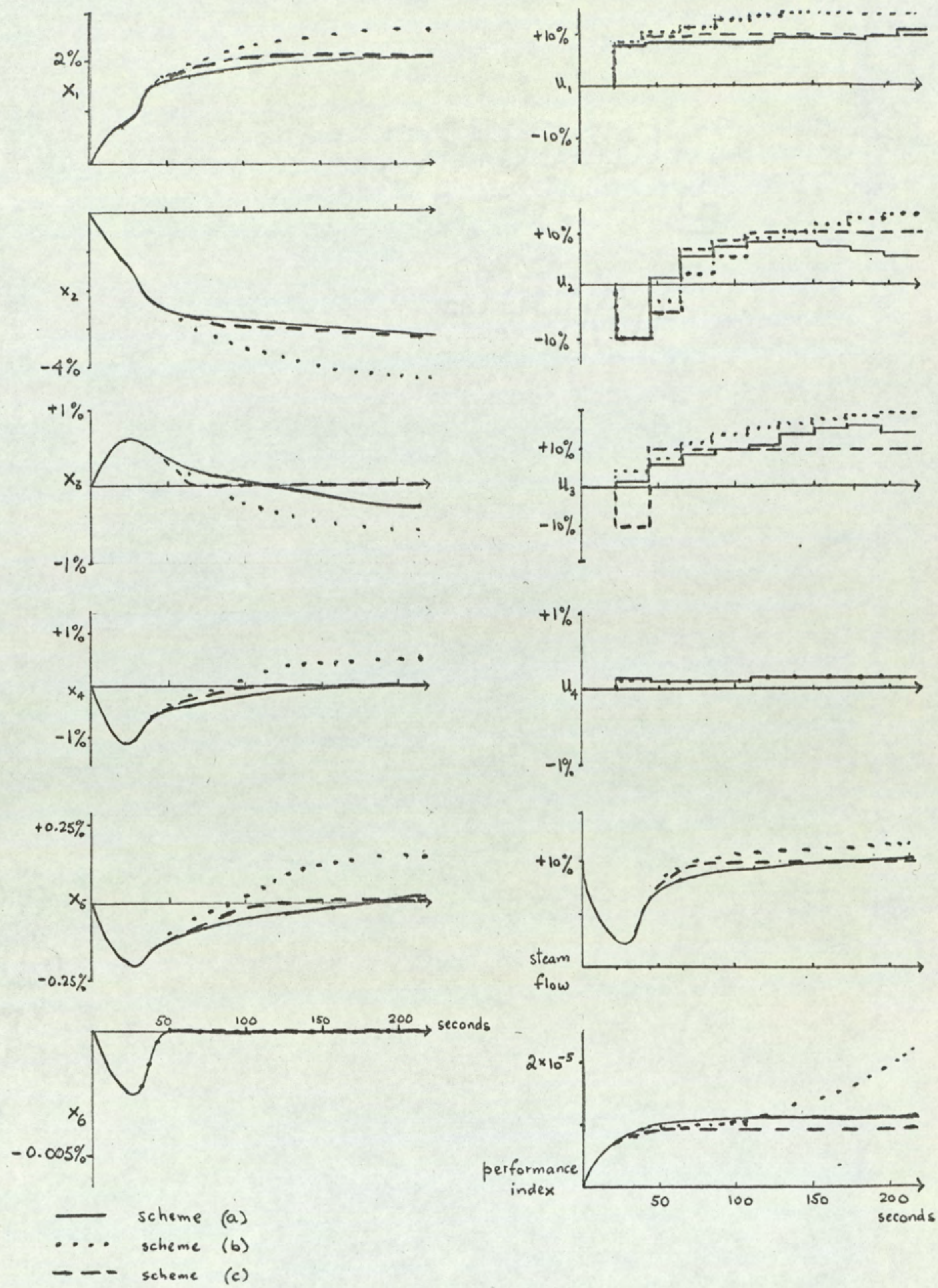


FIG. 4.3 – SYSTEM RESPONSES, $\underline{q} = [0 \quad 0.001 \quad 1 \quad 1 \quad 0 \quad 1]$

4) contd.

4.5) contd.

a) contd.

$$C = 1.017 \times 10^{-5},$$

of which 0.792×10^{-5} is contributed by the uncontrolled first interval.

It will be observed that the other control schemes have been allowed some latitude to exceed the set-down values of the input bounds. The reason for this was the difficulty of adjusting the elements of $\underline{\lambda}$ to maintain reasonable responses whilst keeping the input values within their bounds. For algorithm (b), 15 successive sets of $\underline{\lambda}$ were used to compute 15 different responses. The best of these is illustrated in Fig.4.3. The same $\underline{\lambda}$ was found to give the best of the responses achieved under the N-stage algorithm (a),

$$\underline{\lambda} = \frac{1}{100}\,[1.5 \quad 5 \quad 0.2 \quad 0]$$

The values of the performance index achieved were, for scheme (a),

$$C = 1.191 \times 10^{-5},$$

and for scheme (b),

$$C = 2.170 \times 10^{-5}.$$

The contributions to the index during the controlled part of the interval, [T, 10T], were

4) contd.

4.5) contd.

a) contd.

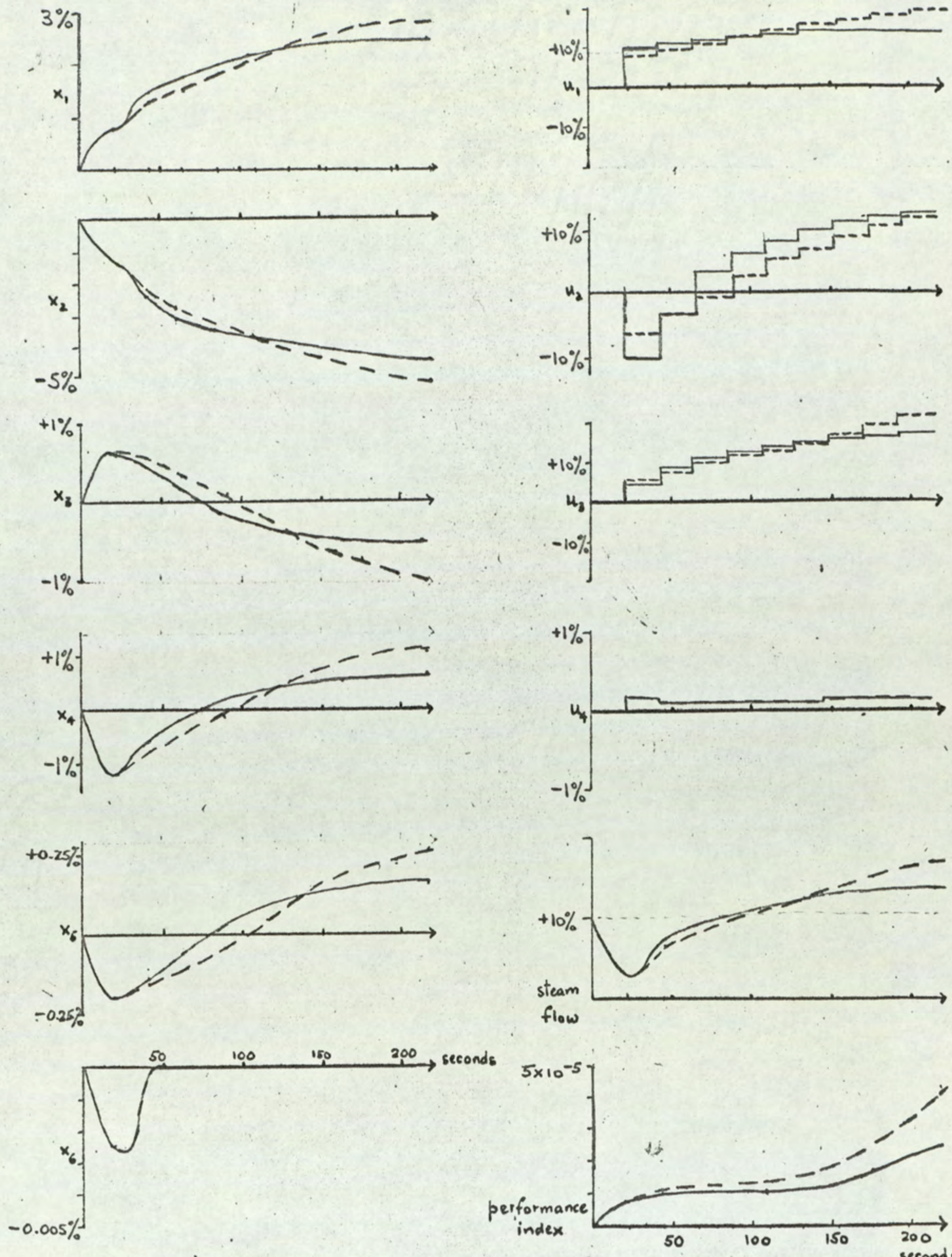
$$C_a{}' = 0.399 \times 10^{-5}$$
$$C_b{}' = 1.378 \times 10^{-5}$$
$$C_c{}' = 0.225 \times 10^{-5},$$

showing a significant advance by the optimal single-stage control scheme (c) over the best offered by a conventional control scheme.

The difficulty in selecting a set of $\underline{\lambda}$ for conventional schemes is illustrated by the comparison shown in Fig.4.4, between the system responses of the selected $\underline{\lambda}$ and of a multiple of the same $\underline{\lambda}$, viz.

$$\underline{\lambda} = \frac{\sqrt{2}}{100}\,[1.5 \quad 5 \quad 0.2 \quad 0]$$

It would be thought that with the same relative weightings between inputs and a larger overall weighting on the use of control, less control action would be used. However, the system responses show that in the second case the model is not so well controlled, with the result that more and more control action is used as time progresses. Despite this extra control, the final value of the index for the second $\underline{\lambda}$, $C = 4.042 \times 10^{-5}$ is much higher than that



——— $\underline{\lambda} = \frac{1}{100}[1.5 \quad 5 \quad 0.2 \quad 0]$

– – – $\underline{\lambda} = \frac{\sqrt{2}}{100}[1.5 \quad 5 \quad 0.2 \quad 0]$

FIG. 4.4 — SYSTEM RESPONSES, $\underline{q} = [0 \quad 0.001 \quad 1 \quad 1 \quad 0 \quad 1]$

SINGLE-STAGE SCHEME (b)

4) contd.

4.5) contd.

a) contd.

for the original $\underline{\lambda}$.

Returning to the three responses of Fig.4.3, an important difference between the control schemes is noted over the existence of off-set equilibrium values of the state variables. Nicholson[10], who used mainly a scheme of type (a), went to considerable lengths to minimise these steady-state errors. It is noted that steady-state errors in two of the three main controlled variables occur again here under schemes (a) and (b), whereas under scheme (c) the values of x_3 and x_4 are reduced to zero within 5 of the 10 sampling intervals. One immediate consequence is a better steam-flow response. The inference is that the optimal single-stage scheme (c), by its nature, has a greater capability to reduce the occurrence of steady-state errors.

Further simulation involving low-order $\underline{q}$ indices has been confined to tests of the performance of the optimal single-stage control scheme developed in this thesis. Fig.4.5 shows the effect of increasing the weighting on one particular variable by a factor of 10, in one case variations in drum level and in the

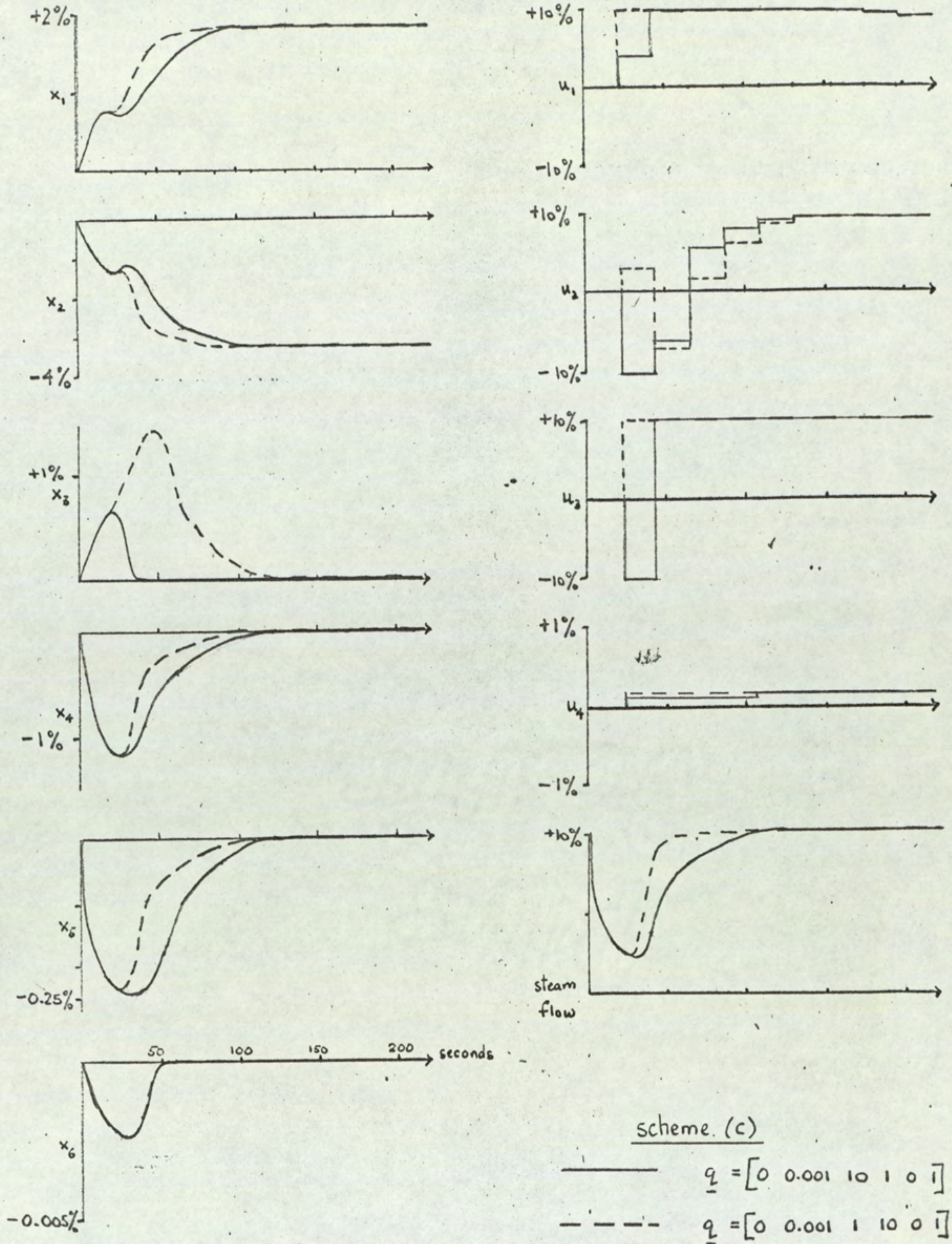


FIG. 4.5 - SYSTEM RESPONSES UNDER OPTIMAL SINGLE-STAGE SCHEME

4) contd.

4.5) contd.

a) contd.

other variations in drum pressure. Satisfactory responses are achieved in both cases. As an example of the computer output results from which the system responses have been plotted, the computed results from one of these tests are included in Appendix 2.

b) Tests with all elements of $\underline{q}$ non-zero.

The performance indices of the tests described in section (a) were in line with suggestions made by staff of English Electric Co., who developed the boiler model. Nevertheless it was considered necessary to attempt to carry out simulation tests under conditions which were unfavourable to the optimal single-stage control scheme. Such conditions were found to occur under a performance index with equal weightings against each of the six state variables,

$$\underline{q} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1].$$

The system response corresponding to a continuous + 10% change in steam demand controlled by the optimal single-stage algorithm (c) was found to demand small values of the control inputs, reaching saturation on only one input, u_3. This situation is favourable for the standard schemes (a) and (b) since only one element

4) contd.

4.5) contd.

b) contd.

of $\underline{\lambda}$ need be non-zero. (Note - if all elements of $\underline{\lambda}$ were zero, the problem would reduce to unbounded control, and scheme (a) must then be fully optimal, schemes (b) and (c) becoming identical). The value of $\underline{\lambda}$ required was found by trial and error to be

$$\underline{\lambda} = \frac{1}{100}\,[0 \quad 0 \quad 0.5 \quad 0]$$

The system responses under the three control schemes are plotted in Fig.4.6. The state variable responses under the two single-stage schemes are effectively identical, as is the value of the performance index,

$$C = 1.363 \times 10^{-4}.$$

The N-stage control scheme (a) gives a marginally better value of the index,

$$C = 1.356 \times 10^{-4}.$$

The immediate conclusion is that the performance of schemes (a), (b) and (c) are comparable on this test. However, the steam flow curve shows that of the extra 10% demand for steam, only between 0.3 and 0.4% is supplied in the steady state. The complete set of response curves is quite close

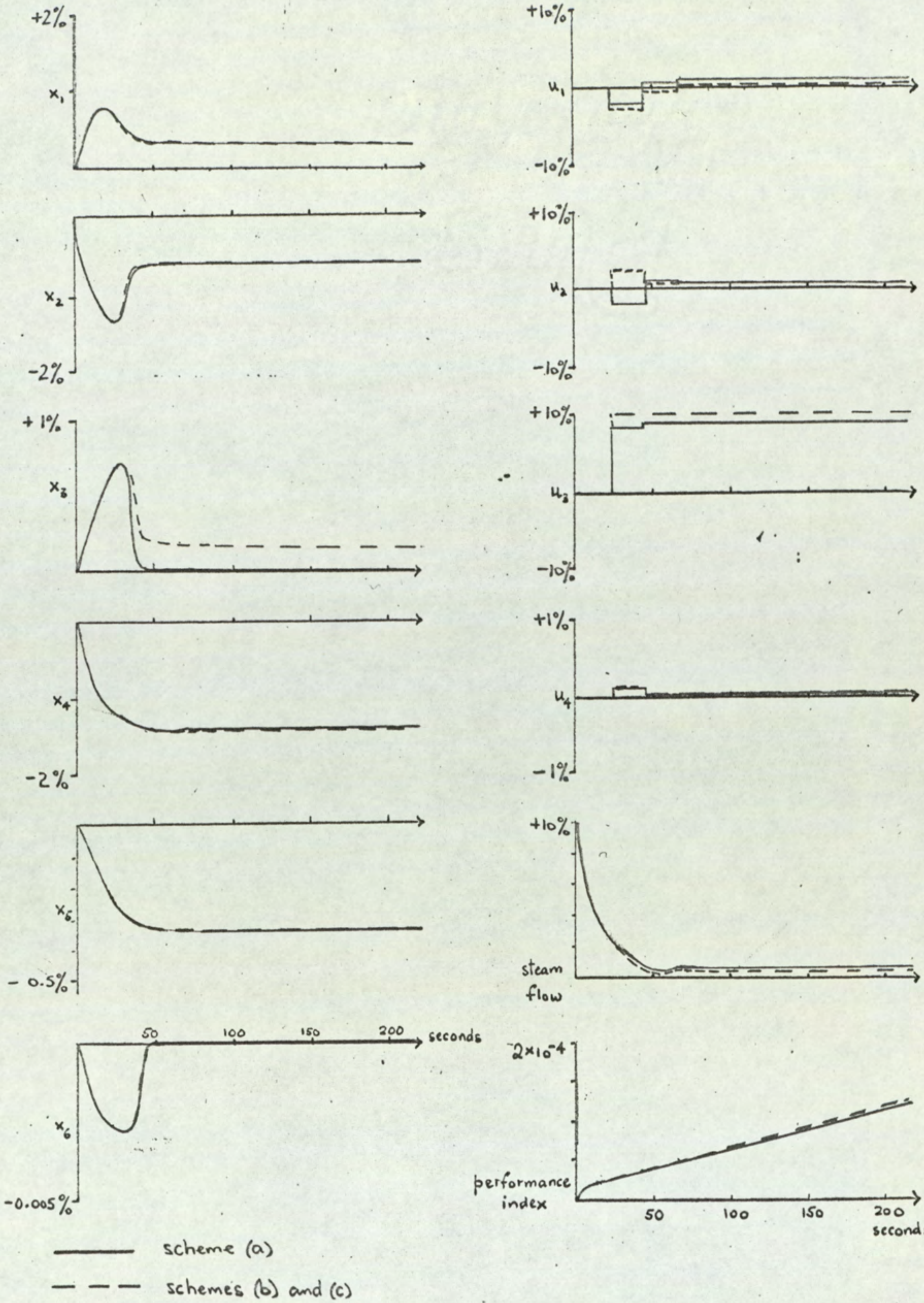


FIG. 4.6 - SYSTEM RESPONSES, $\underline{q} = [1\ 1\ 1\ 1\ 1\ 1]$

4) contd.

4.5) contd.

b) contd.

to the set of Fig.4.1 depicting the free response of the boiler model to a +10% change in demand under no control. It becomes clear that the reason for the small values of the control inputs demanded by all of the schemes is their inability to effect worthwhile control of the model according to an index weighted equally against deviations in each of the state variables. The value of this tests as a comparison between schemes is considered therefore to be reduced.

Any practical performance index would be formulated with high regard to reducing the steady-state error between load demand and load supply. Since the method of representation of steam flow in this model involves x_4, the drum pressure, further tests of schemes (a) and (c) were run with an increased weighting against x_4:

$$\underline{q} = [1 \quad 1 \quad 1 \quad 10 \quad 1 \quad 1]$$

Responses under both schemes showed a large improvement to supplying 9.5 and 9.4% respectively of the 10% demand, accompanied by larger deviations of state variables x_1, x_2 and x_3. The difference between

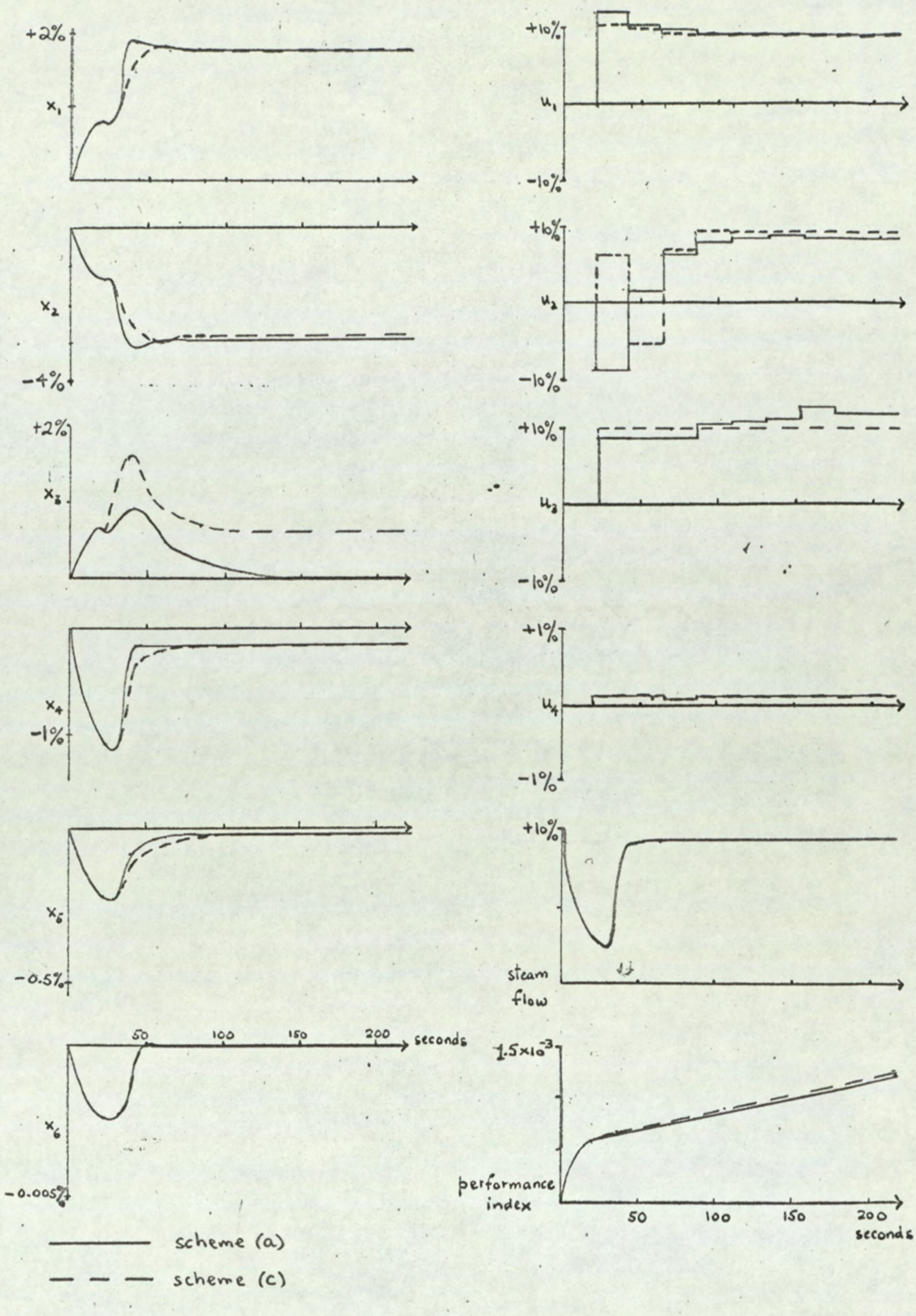


FIG. 4.7 - SYSTEM RESPONSES, $\underline{q} = [1\ 1\ 1\ 10\ 1\ 1]$

4) contd.

4.5) contd.

b) contd.

the final index values,

$$C_a = 1.207 \times 10^{-3} \quad \text{and } C_c = 1.248 \times 10^{-3},$$

may be accounted for in part by the relaxation of the input bounds for scheme (a), the N-stage control policy, and in part by the better control of x_3. The relaxation was found necessary due to the usual difficulty in selecting the elements of $\underline{\lambda}$. The final $\underline{\lambda}$ selected after 8 runs of scheme (a) (4 hrs. computing time) was

$$\underline{\lambda} = \frac{1}{100}\,[2 \quad 7 \quad 0.35 \quad 0]$$

The two sets of response curves are compared in Fig.4.7. Neither control scheme is still quite able to cope fully with the load demand, but a small second increase in the weighting against x_4 would clearly remedy this deficiency.

(c) Timing.

The computer time required for 10-stage runs by each of the control programs is as follows,

a) N-stage control - 20 minutes to produce the required feedback multiplier matrices corresponding to given $\underline{q}$ and λ, plus 7 minutes

4) contd.

4.5) contd.

a) contd.

run time covering the optimisation period [0 , 10T]

b) Single-stage control - 1 minute to produce the constant feedback multiplier matrix for given $\underline{q}$ and $\underline{\lambda}$, plus 7 minutes run time

c) Optimal single-stage control - Average 16 minutes run-time to obtain the system responses corresponding to a given $\underline{q}$.

The time taken by the minimisation procedure in scheme (c) averages $\frac{16-7}{9}$ = 1 minute per stage of the calculation. A reasonable timing conversion factor from Elliott 803 Algol to the machine-level code of a modern control computer might be $\frac{1}{200}$, reducing the minimisation time to less than $\frac{1}{2}$ second per sampling interval.

For the case with the index

$\underline{q} = [0 \quad 0.001 \quad 1 \quad 1 \quad 0 \quad 1]$, the total computer time taken by each scheme to achieve the responses of Fig.4.3 was as follows:

c) - 16 minutes

b) - 15 × 8 = 120 minutes

4) contd.

4.5) contd.

(a) - using scheme (b) as a first estimate, $120 + 3 \times 27 \simeq 200$ minutes; this method was found to be faster than attempting a number of runs at 27 minutes each with no informed first estimate.

4.6) Conclusions on the simulation study.

The performance of the boiler model during the study was in general satisfactory. The method of representation of changes in steam demand did not require that the load supplied must equal the load demanded; as a result, stability of all system variables was a common feature of all the tests. The part of the model involving steam temperature was relatively insensitive to changes in steam demand, and contributed little to the control studies. This part of the model might bear further investigation.

The KDN2-LACE tests demonstrated only the feasibility of computer control without yielding quantitative results. The tests showed the necessity, for research studies in computer control, of suitable computer software. Early tests on the Elliott computer verified this, and also demonstrated the necessity

4) contd.

4.6) contd.

of including in the system equations any known continuous disturbance.

The study carried out using the Elliott computer gave quantitative results for a comparison in performance between standard schemes for optimum digital control and the scheme developed in the thesis. The proposed scheme achieved significantly better control in a case when the number of state variables appearing in the index did not exceed the number of control inputs. Of the two other schemes, N-stage control has the ability to look ahead and to organise its control resources to best effect, but even then will improve on single-stage optimal control only in cases when control is difficult or impossible. This type of case should not arise often in industrial situations provided that the choice of performance index is carried out with care. N-stage control is also limited by the necessity of choosing a set of "cost of control" multipliers, $\underline{\lambda}$, which will ensure that the bounds are satisfied during each and every sampling interval. The $\underline{\lambda}$ chosen corresponds to the most difficult of the intervals, and for the remaining intervals this $\underline{\lambda}$ is likely to constitute a harder constraint, or greater

4) contd.

4.6) contd.

cost, than is necessary. Conventional single-stage control cannot improve on the proposed scheme, and is likely to give significantly worse results.

On the practical side, the limitations of standard schemes became apparent early in the investigation. The difficulty in finding a suitable set of multipliers, $\underline{\lambda}$, was demonstrated. It is not easy to see how this difficulty could be surmounted. A conventional iterative routine may not even converge, since the values of the components of $\underline{\lambda}$ reflect on the values of the control inputs not directly but through the system response. An example of the effect this can have was given in the results section. p.90 Bearing in mind that these tests were carried out with one level of disturbance only, at one particular operating condition, with a constant set of bounds and a constant performance index, it is considered that the practical limitations of control schemes based on the Kalman-Tou index have been shown to be significant.

The optimal single-stage control scheme was designed to avoid the difficulties inherent in the Kalman-Tou approach to bounded-input systems. Changes in any of the features of a test, whether model coefficients, disturbance, bounds or index, can be dealt with on-line by

4) contd.

4.6) contd.

straightforward changes in input data to the control program. State minimisation time is within acceptable limits. Tests also show that under certain conditions, quite likely to be fulfilled in practice, the optimal single-stage scheme will probably yield a system performance improving on the best which can be achieved by either of the schemes with which it has been compared.

5) CONCLUSIONS

The problem which has been examined in the thesis is that of the control of a linear multi-variable system whose performance is judged by the value of a quadratic index. The popularity of this formulation of the problem is due to the existence of an analytical solution in feedback form, provided that control input constraints are absent. In particular, the solution for optimal on-line control by a digital computer reduces to a set of matrix recurrence relations between the discrete-time values of the control inputs.

Further investigation of this problem in the thesis has yielded new results in the theory of N-stage optimum digital control systems. The equivalence under certain conditions between N-stage and single-stage control policies has been demonstrated,

5) contd.

and may account for a conclusion by Nicholson[10] that N-stage control of his boiler model did not appear to be superior to single-stage control[22].

Existing methods for dealing with the addition of control input bounds to the formulation of the problem have been shown to be non-optimal. The practical limitations of existing bounded-input control schemes have been discussed, with particular regard to the simplifications made when using a linearised mathematical model. An alternative method of introducing control input bounds to the solution of the problem has been developed, based on proven geometrical properties of the performance index. The method, which is fully optimal with respect to a single-stage index, has been designed to yield the maximum degree of optimality compatible with ease of implementation in practice. Using the proposed optimal single-stage scheme, changes in the properties of the control problem, such as operating level, performance index or input bound values, can be made during on-line control of the process. Further investigation might lead to an improved minimisation algorithm reducing the time taken to compute the optimal control input vector and therefore increasing the order of model to which it would be feasible to apply the technique. At present the technique might be applied to systems with up to 5 or 6 control inputs, since

5) contd.

the minimisation time depends mainly on the number of inputs rather than on the number of state variables. Optimal control of systems with larger numbers of inputs would create problems using any valid optimisation technique.

Numerical results have been obtained from a simulation study of the application of optimisation techniques to the dynamic control of a power station boiler under changing load conditions. The performance of the boiler model has been in general satisfactory. A feature of the tests was the stability of all state variables. The practical difficulties associated with existing techniques for optimal control of bounded-input systems have been shown to be considerable. The proposed optimal single-stage control policy has been shown to yield a system performance which at worst is comparable with and at best is significantly superior to conventional N-stage control.

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APPENDIX 1.

Mathematical model of a natural circulation boiler.

The equations given below were obtained by Harris, Leigh, Mudge and Sutton during a study of boiler dynamics[16]. They are similar to equations given in the well-known paper of Chien, Ergin, Ling and Lee[12], but use enthalpy throughout rather than a mixture of enthalpy and pressure. The method of representation of steam flow is also different in that steam flow is treated as a dependent variable rather than as an independent variable. The following assumptions have been made

(i) that the fluid in the downcomer leg is incompressible

(ii) that there is no evaporation to and from the surface of the drum

(iii) that the feed to the drum from the economiser is substantially at saturation temperature.

A schematic diagram of the process is given in Fig.A.1.

The equations are in linearised form suitable for the study of small perturbations about a given operating point. This form was achieved by replacement of derivatives with respect to variables other than time by finite difference approximations, and by reduction of the order of the system using physical and engineering considerations. Other methods of systematic reduction

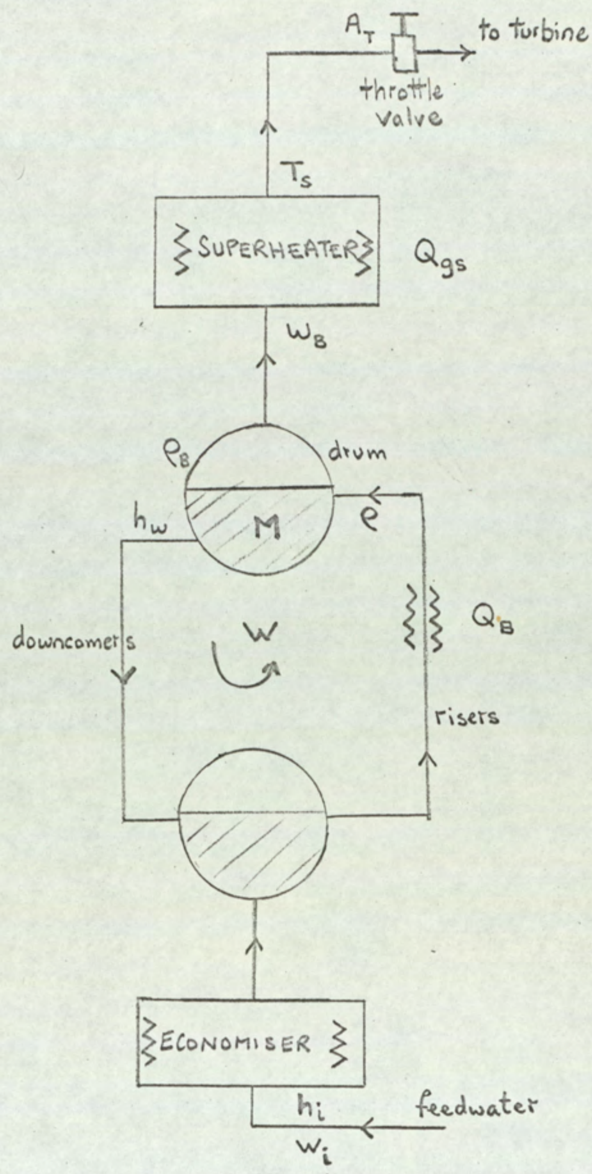

FIG. A.1 - SCHEMATIC DIAGRAM OF BOILER

of order are now available[35,10].

List of symbols used in boiler model

p_B	=	drum pressure
p_s	=	superheater outlet pressure
p_T	=	turbine inlet pressure
ρ_B	=	saturated vapour density corresponding to p_B
ρ_s	=	density at superheater outlet
ρ_w	=	liquid density in downcomer-riser loop
x	=	quality of mixture at riser outlet
ρ	=	density of liquid-vapour mixture at riser outlet
T_s	=	superheater outlet temperature
h_i	=	enthalpy of feedwater
h_B	=	enthalpy of saturated vapour corresponding to p_B
h_{wB}	=	enthalpy of saturated liquid " " "
h_w	=	enthalpy of drum and downcomer liquid
h_s	=	enthalpy of steam leaving superheater
h	=	enthalpy of mixture leaving riser
h_{fg}	=	enthalpy of evaporation corresponding to p_B
Q_{gs}	=	heat-input rate from hot gases into superheater tube walls
Q_B	=	heat-input rate from riser tube walls into boiling liquid
w_i	=	feedwater mass-flow rate
w	=	riser mass-flow rate
w_B	=	steam mass-flow rate from drum into superheater

M	=	mass of liquid in drum
M_s	=	mass of superheater tubes
V_B	=	volume of vapour phase in drum
L_s, L_r	=	superheater and riser tube lengths
A_s, A_r, A_d	=	superheater, riser and downcomer tube cross-sectional areas
A_T	=	area of opening of throttle valve
C_s, C_r	=	heat capacitance of superheater, and riser tubes
W	=	sum of mass-flow rates in riser and downcomer tubes.

The equations are:

$$\frac{\delta W}{W}\left\{\left[AX - \left(1 - \frac{\rho_B}{\rho_w}\right)\right] + s\,\frac{\rho L_r A_r}{wC_r}\;C\right\}\left(1 + \frac{A_r}{A_d}\right)K$$

$$= \frac{\delta\rho}{\rho}\left\{\frac{1}{2}\left[\frac{1 - \frac{w}{w_i}\,\frac{\rho_B}{\rho_w}\;X}{\left(1 - \frac{\rho_B}{\rho_w}\right)}\right] C + A\left[X - \left(1 + \frac{A_r}{A_d}\right)\right]\right\}K \quad (1)$$

$$+\left[\left(1+\frac{A_r}{A_d}\right)-X\right]\left[\frac{\delta\rho_B}{\rho_B}\left\{\left(1+\frac{\rho L_r A_r}{V_B\rho_B}\ \frac{w_i}{w}\right)B\right\}-\frac{\delta h_w}{h_w}\left\{\frac{h_w}{h}\right\}+\right.$$

$$+\frac{\delta w_i}{w_i}\left\{\frac{\rho_B}{\rho_w}\ \frac{\rho L_r A_r}{V_B\ \rho_B}\ \frac{w_i}{w}\ B\right\}$$

$$\left.-\frac{\delta w_B}{w_B}\left\{\frac{\rho L_r A_r}{V_B\ \rho_B}\ \frac{w_i}{w}\ B\right\}-\frac{\delta Q_B}{Q_B}\left\{\frac{xh_{fg}}{h}\right\}\right]$$

$$\frac{\delta\rho}{\rho}\left\{A+s\,\frac{\rho L_r A_r}{w\left(1+\frac{A_r}{A_d}\right)}\ C\right\}K=\frac{\delta W}{W}\left\{\left(1-\frac{\rho_B}{\rho_w}\right)K\right\}+\frac{\delta\rho_B}{\rho_B}\left\{\left(1+\frac{\rho L_r A_r}{V_B\ \rho_B}\ \frac{w_i}{w}\right)B\right\}-$$

$$-\frac{\delta h_w}{h_w}\left\{\frac{h_w}{h}\right\}+\frac{\delta w_i}{w_i}\left\{\frac{\rho_B}{\rho_w}\ \frac{\rho L_r A_r}{V_B\ \rho_B}\ \frac{w_i}{w}\ B\right\}$$

$$-\frac{\delta w_B}{w_B}\left\{\frac{\rho L_r A_r}{V_B\ \rho_B}\ \frac{w_i}{w}\ B\right\}-\frac{\delta Q_B}{Q_B}\left\{\frac{xh_{fg}}{h}\right\} \qquad (2)$$

$$\frac{\delta M}{M}\left\{s\ \frac{M}{w_i}\ K\ C\right\}=-\frac{\delta W}{W}\left\{\left(1+\frac{A_r}{A_d}\right)\frac{w}{w_i}\ K\right\}+\frac{\delta\rho}{\rho}\left\{\frac{\left(1+\frac{A_r}{A_d}\right)\frac{w}{w_i}\ KA}{\left(1+\frac{\rho_B}{\rho_w}\right)}\right\}-$$

$$-\frac{\delta\rho_B}{\rho_B}\left\{\frac{KC}{\left(1-\frac{\rho_B}{\rho_w}\right)}+\left(1+\frac{\rho L_r A_r}{V_B\ \rho_B}\ \frac{w_i}{w}\right)B\right\}$$

$$+ \frac{w}{w_i}\left[1+(1-x)\frac{A_r}{A_d}\right]\left[\frac{\delta h_w}{h_w}\left\{\frac{h_w}{h}\right\} + \frac{\delta w_B}{w_B}\left\{\frac{\rho L_r A_r}{V_B\,\rho_B}\;\frac{w_i}{w}\;B\right\} + \frac{\delta Q_B}{Q_B}\left\{\frac{xh_{fg}}{h}\right\}\right] \qquad (3)$$

$$+ \frac{\delta w_i}{w_i}\left\{\frac{xh_{fg}}{h}\,\frac{C}{(1-\frac{\rho_B}{\rho_w})} + \frac{A_r}{A_d}\;\frac{\rho L_r A_r}{V_B\,\rho_B}\;\frac{w_i}{w}\;B\right\}$$

$$\frac{\delta\rho_B}{\rho_B}\left\{\left[B - \frac{xh_{fg}}{h(1-\frac{\rho_B}{\rho_w})}\right] + s\,\frac{V_B\rho_B}{w_i}\right\} = \frac{\delta h_w}{h_w}\left\{\frac{h_w}{h}\right\} + \frac{xh_{fg}}{h}\left[\frac{\delta w_i}{w_i}\left\{\frac{\rho_B/\rho_w}{(1-\frac{\rho_B}{\rho_w})}\right\}\right.$$

$$\left. - \frac{\delta w_B}{w_B}\left(\frac{1}{1-\frac{\rho_B}{\rho_w}}\right) + \frac{\delta Q_B}{Q_B}\right] \qquad (4)$$

$$\frac{\delta h_w}{h_w}\left\{\frac{w}{w_i} + s\,\frac{M}{w_i}\right\}\frac{h_w}{h} = \frac{\delta\rho_B}{\rho_B}\left\{\frac{1-x}{x}\,\frac{\rho_B}{h}\,\left.\frac{\partial h_{wB}}{\partial\rho_B}\right|_{\rho_B}\right\} + \frac{\delta h_i}{h_i}\left\{\frac{h_w}{h}\right\} \qquad (5)$$

$$\frac{\delta T_s}{T_s}\left\{Z\,\frac{\partial p_s}{\partial\rho_s} + s\,\frac{M_s C_s T_s}{w_i\;h_s}\right\} = \frac{\delta\rho_B}{\rho_B}\left\{\left(\frac{p_B-p_s}{p_s} + \frac{\rho_B}{p_s}\,\frac{\partial p_B}{\partial\rho_B}\right)\frac{p_s Z}{T_s}\cdot\frac{\partial T_s}{\partial\rho_s}\right\} \qquad (6)$$

$$- \frac{\delta w_B}{w_B}\left\{\left(1 - \frac{h_B}{h_s}\right) + \frac{2(p_B-p_s)}{p_s}\,\frac{p_s Z}{T_s}\cdot\frac{\partial T_s}{\partial\rho_s}\right\} + \frac{\delta Q_{gs}}{Q_{gs}}\left(1 - \frac{h_B}{h_s}\right)$$

where $s \equiv \frac{d}{dt}$

$$A = \frac{\rho_B}{\rho_w}\,\frac{w}{w_i} + \left(1 - \frac{\rho_B}{\rho_w}\right)$$

$$B = \frac{xh_{fg}}{h}\left(\frac{1}{1-\frac{\rho_B}{\rho_w}}\right) + (1-x)\frac{\rho_B}{h}\left.\frac{\partial h_{wB}}{\partial \rho_B}\right|_{\rho_B}$$

$$C = \frac{\rho_B}{\rho_w}\frac{w}{w_i}\left(1 + \frac{A_r}{A_d}\right) + \frac{A_r}{A_d}\left(1 - \frac{\rho_B}{\rho_w}\right)$$

$$K = \frac{xh}{h}fg\,\frac{1}{\left(1 - \frac{\rho_B}{\rho_w}\right)} + \frac{\rho L_r A_r}{V_B\,\rho_B}\,\frac{w_i}{w}\,B$$

$$X = 1 - \frac{2}{C_r}\frac{\rho}{\rho_w} + \frac{\rho}{\rho_w}\frac{C_d}{C_r}\left(\frac{A_r}{A_d}\right)^2$$

$$Z = \frac{T_s}{h_s\left(\frac{\partial p_s}{\partial \rho_s}\frac{\partial T_s}{\partial h_s} - \frac{\partial p_s}{\partial h_s}\frac{\partial T_s}{\partial \rho_s}\right)}$$

Define new variables as follows:-

$$x_1 = \frac{K\frac{L_r A_r}{V_w}\left(1 + \frac{A_r}{A_d}\right)\frac{C}{A}\frac{\rho_B}{\rho_w}\frac{w}{w_i}}{C_r\left[1 + \frac{A_r}{A_d} - X\right]}\frac{\delta W}{W} \qquad 7(a)$$

$$x_2 = K\frac{L_r A_r}{V_w}\,\frac{C}{A\left(1 + \frac{A_r}{A_d}\right)}\,\frac{\rho_B}{\rho_w}\,\frac{w}{w_i}\,\frac{\delta\rho}{\rho} \qquad 7(b)$$

$$x_3 = \frac{KC}{\left(1+\frac{A_r}{A_d}\right) - \frac{w_i}{w}\frac{A_r}{A_d}} \frac{\delta M}{M} \qquad 7(c)$$

$$x_4 = K \frac{V_B}{V_w} \frac{\rho_B}{\rho_w} \frac{w}{w_i} \frac{\delta\rho_B}{\rho_B} \qquad 7(d)$$

$$x_5 = \frac{h_w}{h} \frac{\delta h_w}{h_w} \qquad 7(e)$$

$$x_6 = \frac{M_s C_s T_s}{M h_s} \frac{w}{w_i} \frac{1}{Z \frac{\partial p_s}{\partial \rho_s}} \frac{\delta T_s}{T_s} \qquad 7(f)$$

with $$u_1 = \frac{1}{\frac{w}{w_i} \frac{h_{wB}}{h_{fg}}} \frac{\delta Q_B}{Q_B} \qquad 7(g)$$

$$u_2 = \frac{\rho_B/\rho_w}{\left(1 - \frac{\rho_B}{\rho_w}\right)\left(\frac{w}{w_i} \frac{h_{wB}}{h_{fg}}\right)} \frac{\delta w_i}{w_i} \qquad 7(h)$$

$$u_3 = \frac{w_i}{w} \frac{h_w}{h} \frac{\delta h_i}{h_i} \qquad 7(i)$$

$$u_4 = \frac{\left(1 - \frac{h_B}{h_s}\right)}{Z\, \partial p_s/\partial \rho_s} \frac{\delta Q_{gs}}{Q_{gs}} \qquad 7(j)$$

$$n = \frac{1}{\left(1 - \frac{\rho_B}{\rho_W}\right)\left(\frac{w}{w_i}\,\frac{h_{wB}}{h_{fg}} + 1\right)} \quad \frac{\delta w_B}{w_B} \qquad 7(k)$$

and $$t' = \frac{w_i}{M}\;\frac{w}{w_i}\;t \qquad 7(\ell)$$

The equations (1 - 6) may be written in matrix form as

$$\frac{d\underline{x}}{dt'} = A\underline{x}(t') + B\underline{u}(t') + \underline{n}(t') \qquad (8)$$

where $\underline{x}(t')$ is the state vector of the process

$\underline{u}(t')$ is the control vector

$\underline{n}(t')$ is the disturbance vector for changes in steam flow

A is the coefficient matrix of the process

B is the driving matrix

The equations (1 - 6) assume that the steam flow from the boiler, w_B, may be freely chosen. In practice this is not so, since flow will be controlled by a valve and will depend on both the valve opening and the pressure drop across the valve. Assuming that the flow is dependent on the area of the opening and on the square root of the pressure drop, we have

$$\frac{\delta w_B}{w_B} = \left\{\frac{p_s - p_T}{p_B - p_T}\right\} \frac{\delta A_T}{A_T} + \frac{1}{2}\left\{\frac{(p_B - p_s) + \rho_B \frac{\partial p_B}{\partial \rho_B}}{p_B - p_T}\right\} \frac{\delta \rho_B}{\rho_B} - \frac{1}{2}\left\{\frac{p_T}{p_B - p_T}\right\} \frac{\delta p_T}{p_T} \quad (9)$$

If constant pressure on the outlet side of the throttle valve is maintained, this equation reduces to

$$n = N + \xi x_4$$

where

$$N = \frac{1}{\left(1 - \frac{\rho_B}{\rho_w}\right)\left(\frac{w}{w_i}\frac{h_{wB}}{h_{fg}} + 1\right)} \left(\frac{p_s - p_T}{p_B - p_T}\right) \frac{\delta A_T}{A_T} \qquad 7(m)$$

$$\xi = \frac{\frac{V_w \rho_w}{V_B \rho_B} \frac{w_i}{w}}{K\left(1 - \frac{\rho_B}{\rho_w}\right)\left(\frac{w}{w_i}\frac{h_{wB}}{h_{fg}} + 1\right)} \frac{1}{2}\left\{\frac{p_B - p_s + \rho_B \frac{\partial p_B}{\partial \rho_B}}{p_B - p_T}\right\} \qquad 10(a)$$

The coefficient matrix A and the driving matrix B now become

$$A = \begin{bmatrix} -a_{11}\alpha & -a_{12}\beta & 0 & \gamma-\xi\mu & -1 & 0 \\ a_{21}\alpha & -\beta & 0 & \gamma-\xi\mu & -1 & 0 \\ -a_{31}\alpha & a_{32}\beta & 0 & -a_{34}\gamma-\xi\mu & 1 & 0 \\ 0 & 0 & 0 & -\delta-\xi & 1 & 0 \\ 0 & 0 & 0 & \delta & -1 & 0 \\ 0 & 0 & 0 & a_{64}\gamma+\eta\xi & 0 & -\epsilon \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & \mu & 0 & 0 \\ -1 & \mu & 0 & 0 \\ 1 & b_{32}\mu & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with

$$\alpha = \frac{\left[1 + \frac{A_r}{A_d} - X \right] C_r}{\frac{L_r A_r}{V_w} \; \frac{C}{A} \; \frac{\rho_B}{\rho_w} \; \frac{w}{w_i}} \qquad 10(b)$$

$$\beta = \frac{V_w}{L_r\ A_r} \; \frac{A^2}{C} \left(1 + \frac{A_r}{A_d}\right) \frac{\rho_w}{\rho_B} \; \frac{w_i}{w} \qquad 10(c)$$

$$\gamma = \frac{\left(1 + \frac{L_r A_r}{A V_B}\right) B}{K \frac{V_B}{V_w} \; \frac{\rho_B}{\rho_w} \; \frac{w}{w_i}} \qquad 10(d)$$

$$\delta = \frac{\left(1 - \frac{w_i}{w}\right) \frac{\rho_w}{h_{fg}} \left.\frac{\partial h_{wB}}{\partial \rho_B}\right|_{\rho_B}}{K \frac{V_B}{V_w} \left(\frac{w}{w_i} \frac{h_{wB}}{h_{fg}} + 1\right)} \qquad 10(e)$$

$$\epsilon = \frac{M}{M_s C_s} \frac{h_s Z}{T_s} \frac{w_i}{w} \frac{\partial p_s}{\partial \rho_s} \qquad 10(f)$$

$$\eta = -\left(1 - \frac{\rho_B}{\rho_w}\right)\left(\frac{w}{w_i} \frac{h_{wB}}{h_{fg}} + 1\right)\left\{ \left(1 - \frac{h_B}{h_s}\right) \frac{1}{Z \frac{\partial p_s}{\partial \rho_s}} + \frac{2(p_B - p_s)}{T_s} \frac{\partial T_s/\partial p_s}{\partial p_s/\partial \rho_s} \right\} \qquad 10(g)$$

$$\mu = \frac{L_r A_r}{V_B} \frac{B}{A} \left(1 - \frac{\rho_B}{\rho_w}\right)\left(\frac{w}{w_i} \frac{h_{wB}}{h_{fg}} + 1\right) \qquad 10(h)$$

$$a_{11} = \frac{AX - \left(1 - \frac{\rho_B}{\rho_w}\right)}{1 + \frac{A_r}{A_d} - X} \qquad 10(i)$$

$$a_{12} = 1 - \frac{C}{2A} \frac{1 - \frac{\rho_B}{\rho_w}\left(1 + \frac{w}{w_i} X\right)}{\left(1 + \frac{A_r}{A_d} - X\right)\left(1 - \frac{\rho_B}{\rho_w}\right)} \qquad 10(j)$$

$$a_{21} = \frac{1 - \frac{\rho_B}{\rho_w}}{1 + \frac{A_r}{A_d}} \qquad 10(k)$$

$$a_{31} = \frac{1}{1 + \frac{A_r}{A_d} - \frac{w_i}{w}\frac{A_r}{A_d}} \qquad 10(\ell)$$

$$a_{32} = \frac{a_{31}}{a_{21}} \qquad 10(m)$$

$$a_{34} = 1 + \frac{KC}{B\left(1 - \frac{\rho_B}{\rho_w}\right)\left(1 + \frac{L_r A_r}{A V_B}\right)} \quad \frac{1}{\frac{w}{w_i}\left(1 + \frac{A_r}{A_d} - \frac{w_i}{w}\frac{A_r}{A_d}\right)} \qquad 10(n)$$

$$a_{64} = \frac{\frac{p_s}{T_s}\frac{\partial T_s}{\partial p_s}\left(\frac{p_B - p_s}{p_s} + \frac{\rho_B}{p_s}\frac{\partial p_B}{\partial \rho_B}\right)}{\frac{\partial p_s}{\partial \rho_s}\left(1 + \frac{L_r A_r}{A V_B}\right)B} \qquad 10(p)$$

$$b_{32} = \frac{\frac{\rho_w}{\rho_B}\left(\frac{C}{\mu} + \frac{A_r}{A_d}\right)}{\frac{w}{w_i}\left(1 + \frac{A_r}{A_d} - \frac{w}{w_i}\frac{A_r}{A_d}\right)} \qquad 10(q)$$

The disturbance vector, $\underline{n}$ becomes

$$\underline{n}(t)' = [-\mu N \quad -\mu N \quad \mu N \quad -N \quad 0 \quad \eta N]' \qquad (11)$$

APPENDIX 2.

Minimisation procedure and computer results.

The control program for tests of the optimal single-stage policy developed in this thesis involves a procedure which finds the values of the elements of the input vector, $\underline{u}$, such that the performance index is minimised:

$$C = \min_{\underline{a} \leqslant \underline{u} \leqslant \underline{b}} \left[\left\{ \phi\underline{x} + G\underline{u} + \underline{d} \right\}' Q \left\{ \phi\underline{x} + G\underline{u} + \underline{d} \right\} \right]$$

The procedure CNTRL, which is reproduced here, is written in Elliott Algol, and uses standard Elliott matrix procedures. CNTRL is a parameterless procedure and therefore requires the prior declaration of ϕ, G, Q, $\underline{d}$, $\underline{u}$, $\underline{a}$, $\underline{b}$ and $\underline{x}$ (here X1) as two-dimensional arrays.

In the given form CNTRL is written for systems with exactly 6 state variables and 4 control inputs. To generalise to any number of state variables, N, would involve only changes of the dimensional bounds of the arrays and of the elements of some of the for - lists, both to become dependent on N. The addition of an extra control input would involve the nesting of the procedure body within an extra loop, with some modifications to the counting variables.

This version of the minimisation procedure allows some redundant calculations to be carried out in order that the detailed programming should stay relatively simple. The redundant calculations may be removed by the addition of <u>Boolean arrays</u> to keep track of input value combinations which have already been considered.

An example is given of the form in which the results of a simulation test are output by the Elliott 803 digital computer. The example corresponds to the application of the optimal single-stage scheme to the control of the model following a 10% extra demand for steam, under an index

$$\underline{q} = [0 \quad 0.001 \quad 1 \quad 10 \quad 0 \quad 1]$$

These results also appear, as response curves, in Fig.4.5. Features of the results are the speed of convergence to a steady state condition and the absence of off-set values, in the steady state, of the principal weighted variables, x_3, x_4 and x_6.

```
   PROCEDURE CNTRL'
 BEGIN INTEGER BND,BND1,BND2,BND3,COUNT'  REAL X,FNMIN'
      ARRAY  SIG,G3,F,H(1:6,1:1),Z,STORE(1:4,1:1),
             U1(1:3,1:1),U12(1:2,1:1),S8,U123(1:1,1:1)
      ,G1(1:6,1:3),G2(1:6,1:2),R1,S1(1:4,1:6),S2(1:4,1:4),
      R3,S3(1:3,1:6),S4(1:3,1:3),R5,S5(1:2,1:6),
      S6(1:2,1:2),R7,S7(1:1,1:6)'
      REAL PROCEDURE FNVAL'
      BEGIN MXPROD(SIG,G,U)'
            MXSUM(SIG,F,SIG)'
            MXPROD(G3,Q,SIG)'
            MXTRANS(S7,SIG)'
            MXPROD(S8,S7,G3)'
            FNVAL:=S8(1,1)
      END OF FNVAL'

      MXPROD(F,PHI,X1)'   MXSUM(F,F,D)'
      MXTRANS(S1,G)' MXPROD(R1,S1,Q)' MXPROD(S2,R1,G)'
      INVMX(S2)' MXPROD(S1,S2,R1)' MXPROD(U,S1,F)'
      MXNEG(Z,U)'  BND:=0'  MXCOPY(STORE,Z)'  FNMIN:=1000000'
BEGIN INTEGER I'    SWITCH S:=L2'
   FOR I:=1 STEP 1 UNTIL 4 DO
     BEGIN X:=Z(I,1)'
           IF X LESS A(I) THEN U(I,1):=A(I) ELSE
           IF X  GR  B(I) THEN U(I,1):=B(I) ELSE
           BEGIN U(I,1):=X' GOTO L2 END'

           BND:=BND+1'
        FOR COUNT:=1 STEP 1 UNTIL 6 DO
        BEGIN H(COUNT,1):=G(COUNT,I)*U(I,1)'
           G1(COUNT,1):=IF I=1 THEN G(COUNT,2) ELSE G(COUNT,1)'
  G1(COUNT,2):=IF I=1 OR I=2 THEN G(COUNT,3) ELSE G(COUNT,2)'
 G1(COUNT,3):=IF I=4 THEN G(COUNT,3) ELSE G(COUNT,4) END'

        MXSUM(SIG,F,H)' MXTRANS(S3,G1)' MXPROD(R3,S3,Q)'
        MXPROD(S4,R3,G1)' INVMX(S4)' MXPROD(S3,S4,R3)'
        MXPROD(U1,S3,SIG)' MXNEG(U1,U1)' BND1:=0'

 BEGIN INTEGER V,J' SWITCH SS:=L4'
      FOR V:=1 STEP 1 UNTIL 3 DO
     BEGIN
          J:=IF V GREQ I THEN 1 ELSE 0'
          X:=U1(V,1)'
          IF X LESS A(V+J) THEN U(V+J,1):=A(V+J) ELSE
          IF X  GR  B(V+J) THEN U(V+J,1):=B(V+J) ELSE
          BEGIN U(V+J,1):=X' GOTO L4 END'
             BND1:=BND1+1'
   FOR COUNT:=1 STEP 1 UNTIL 6 DO
   BEGIN
       H(COUNT,1):=G(COUNT,I)*U(I,1)+G1(COUNT,V)*U(V+J,1)'
       G2(COUNT,1):=IF V=1 THEN G1(COUNT,2) ELSE G1(COUNT,1)'
       G2(COUNT,2):=IF V=3 THEN G1(COUNT,2) ELSE G1(COUNT,3)
   END'
```

```
        MXSUM(SIG,F,H)' MXTRANS(S5,G2)' MXPROD(R5,S5,Q)'
        MXPROD(S6,R5,G2)' INVMX(S6)' MXPROD(S5,S6,R5)'
        MXPROD(U12,S5,SIG)' MXNEG(U12,U12)' BND2:=0'

    BEGIN INTEGER L,W,K' SWITCH SSS:=L6,L7,L8'
         FOR W:=1,2 DO BEGIN K:=0'
            IF W GREQ V THEN K:=1'
            IF (W+K) GREQ I THEN K:=K+1'
              X:=U12(W,1)'
            IF X LESS A(W+K) THEN U(W+K,1):=A(W+K) ELSE
            IF X  GR  B(W+K) THEN U(W+K,1):=B(W+K) ELSE
            BEGIN U(W+K,1):=X' GOTO L6 END'
                BND2:=BND2+1'

     IF I=1 OR (V+J)=1 OR (W+K)=1 THEN L:=2 ELSE BEGIN L:=1' GOTO
                                                              L7 END'
     IF I=2 OR (V+J)=2 OR (W+K)=2 THEN L:=3 ELSE GOTO L7'
     IF I=3 OR (V+J)=3 OR (W+K)=3 THEN L:=4'

   L7:    FOR COUNT:=1 STEP 1 UNTIL 6 DO BEGIN
        H(COUNT,1):=G(COUNT,I)*U(I,1)+G1(COUNT,V)*U(V+J,1)
                    +G2(COUNT,W)*U(W+K,1)'
        G3(COUNT,1):=G(COUNT,L) END'
              BND3:=0'

   MXSUM(SIG,F,H)' MXTRANS(S7,G3)' MXPROD(R7,S7,Q)'
   MXPROD(S8,R7,G3)'  X:=-(S8(1,1))' MXPROD(U123,R7,SIG)'
   X:=U123(1,1)/X'
    IF X LESS A(L) THEN U(L,1):=A(L) ELSE
                      IF X  GR  B(L) THEN U(L,1):=B(L) ELSE
                      BEGIN U(L,1):=X' GOTO L8 END'
   BND3:=1'  X:=FNVAL'  IF X LESS FNMIN THEN
    BEGIN  FNMIN:=X'   MXCOPY(STORE,U)  END'

   L6: END OF K BLOCK'

   L8: END OF W LOOP'

        IF BND2=0 OR BND3=0 THEN BEGIN
              X:=FNVAL' IF X LESS FNMIN THEN
     BEGIN FNMIN:=X'  MXCOPY(STORE,U) END  END'

   L4: END OF J BLOCK'  END OF VJ LOOP'
       IF BND1=0 THEN BEGIN X:=FNVAL' IF X LESS FNMIN THEN
     BEGIN FNMIN:=X'  MXCOPY(STORE,U) END  END'


   L2: END OF X BLOCK' END OF I LOOP'
       IF BND=0 THEN BEGIN X:=FNVAL'  IF X LESS FNMIN THEN
     BEGIN FNMIN:=X'  MXCOPY(STORE,U) END  END'

PRINT £CONTROL VECTOR?'   PRINTMX(STORE)'  MXCOPY(U,STORE)'
END OF CNTRL'
```

TIME 3
STATE VECTOR

-.00203900

-.00391400

.00282800

-.00241600

-.00213200

-.00055220

XQX .00058579

CONTROL VECTOR

.00297000

.00002720

.00006220

.00043756

TIME 6
STATE VECTOR

-.00429181

-.00808360

.00609451

-.00057109

-.00089286

.00000000

XQX .00062806

CONTROL VECTOR

.00297000

-.00006445

.00006220

.00034693

TIME 9
STATE VECTOR

-.00491370

-.00926462

.00295221

-.00027664

-.00034173

.00000000

XQX .00063798

CONTROL VECTOR

.00297000

.00001719

.00006220

.00037509

TIME 12
STATE VECTOR

-.00502782

-.00948187

.00107554

-.00010078

-.00009782

.00000000

XQX .00063929

CONTROL VECTOR

.00297000

.00005645

.00006220

.00038915

TIME 15
STATE VECTOR

-.00508272

-.00958633

.00017098

-.00001602

.00001977

.00000000

XQX .00063933

CONTROL VECTOR

.00292686

.00008699

.00006220

.00039388

TIME 18
STATE VECTOR

-.00505689

-.00953901

.00000000

-.00000000

.00005455

-.00000000

XQX .00063933

CONTROL VECTOR

.00291364

.00009161

.00006220

.00039468

$\underline{q} = [0 \;\; 0.001 \;\; 1 \;\; 10 \;\; 0 \;\; 1]$ — scheme (c) responses.

TIME 21
STATE VECTOR

-.00505033

-.00952704

.00000000

-.00000000

.00005996

.00000000
XQX .00063933
CONTROL VECTOR

.00291307

.00009115

.00006220

.00039475

TIME 24
STATE VECTOR

-.00505126

-.00952881

.00000000

-.00000000

.00006061

.00000000
XQX .00063933
CONTROL VECTOR

.00291287

.00009122

.00006220

.00039476

TIME 27
STATE VECTOR

-.00505116

-.00952863

.00000000

-.00000000

.00006069

.00000000
XQX .00063933
CONTROL VECTOR

.00291287

.00009121

.00006220

.00039476

TIME 30
STATE VECTOR

-.00505117

-.00952865

.00000000

-.00000000

.00006069

.00000000
XQX .00063933

Input bounds are

$|u_1| \leq 0.00297$

$|u_2| \leq 0.0000923$

$|u_3| \leq 0.0000622$

$|u_4| \leq 0.0491$

scheme (c) responses cont.

$\underline{q} = [0 \ 0.001 \ 1 \ 10 \ 0 \ 1]$

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