TEMPORAL FINITE ELEMENTS IN THE DYNAMIC ANALYSIS OF STRUCTURES

Geoffrey Frank Howard

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SUMMARY

This thesis demonstrates that the use of finite elements need not be confined to space alone, but that they may also be used in the time domain. It is shown that finite element methods may be used successfully to obtain the response of systems to applied forces, including, for example, the accelerations in a tall structure subjected to an earthquake shock. It is further demonstrated that at least one of these methods may be considered to be a practical alternative to more usual methods of solution.

A detailed investigation of the accuracy and stability of finite element solutions is included, and methods of applications to both single- and multi-degree of freedom systems are described. Solutions using two different temporal finite elements are compared with those obtained by conventional methods, and a comparison of computation times for the different methods is given.

The application of finite element methods to distributed systems is described, using both separate discretizations in space and time, and a combined space-time discretization. The inclusion of both viscous and hysteretic damping is shown to add little to the difficulty of the solution.

Temporal finite elements are also seen to be of considerable interest when applied to non-linear systems, both when the system parameters are time-dependent and also when they are functions of displacement.

Solutions are given for many different examples, and the computer programs used for the finite element methods are included in an Appendix.

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CONTENTS

NOTATION

a b c d A B C	Coefficients
[A][B][C][D]	Matrices of coefficients
A	Amplitude
	or Cross-sectional area
c	Damping coefficient
$[C_N]$	Normalized damping matrix
D	Differential operator
E	Young's modulus
F	Force
g	Gravitational acceleration
$[G][H][G_f][H_f]$	Matrices of coefficients
$[h^\circ][h^I][h^{II}]$	Matrices of functions of τ
i j	Integers
I	Second moment of area
[I]	Unit matrix
j	$\sqrt{-1}$
k	Stiffness
K	Equivalent stiffness
[K]	Modal stiffness matrix
ℓ	Length of element
L	Length of beam
m	Mass
[M]	Modal mass matrix
M	Couple
n	Number of intervals per cycle
	or Number of degrees of freedom

cont'd

N	Total number of intervals
p	Non-linear parameter $\sqrt{K/m}$
	or Rate of loading
	or Principal coordinate
P,Q	Forces
[P][Q][R][S]	Matrices of functions of m, k and τ
q	Displacement
r	Number of cycles
s	Non-dimensional parameter t/τ
S	Sum of series
t	Time
T	Kinetic energy
	or Period of vibration
u	Displacement
$\dot{u}$ $\ddot{u}$	Velocity, acceleration
u'	Slope
v	$\dot{u}\tau$
$[v_N]$ $[u_N]$	Normalized eigenvector matrix
V	Potential energy
W	Work done by generalized forces
x	Spatial coordinate
[Z]	$[k] + D^2[m]$

cont'd

α	$\sqrt{EI/\rho A}/L^2$
γ	$c/2m$
Δ	Static deflection
ε	Error
θ	Angle
	or $\tau^2/420$
ρ	Density
τ	Length of temporal element, time interval
ϕ	Phase lag
ψ	Interpolation function
ω	Natural angular frequency
Ω	Forcing frequency
μ	Hysteretic damping parameter

CHAPTER 1

INTRODUCTION

In many engineering applications a knowledge of the dynamic response of a system to a known input force is of very great practical importance. The problem may be, for example, concerned with the comfort of the occupants of a vehicle travelling along a rough road, or the design of a building to withstand earthquake shocks. The methods of solution available for these and many other cases fall into two main categories; those giving exact solutions, which are often difficult to use, and numerical methods which are, in general, easier to apply but which give only approximate solutions.

Of the numerical methods, probably the most widely used is that using finite differences. This method has the advantage of relative simplicity, although it may require special starting procedures to obtain the required solution in the time domain. Since the same finite difference techniques may be used to obtain, for example, a solution in space giving the deflected shape of a beam, and a solution in time to determine the displacement of a vibrating mass, it seems logical to consider what other spatial techniques may be applied to time-dependent phenomena. Probably the most generally successful of the methods now used to obtain solutions to many different spatial problems is that using finite elements, and this thesis is principally concerned with the derivation and applications of temporal finite elements, i.e. finite elements in the time domain.

This method, like the finite difference method, gives values of displacements at selected values of time, but the finite element method also gives values of velocities, which form one set of components of the generalized displacements. A complete history of displacements and velocities is therefore obtained.

In describing the use of the temporal finite element method, a number of examples concerning free vibrations is included, and it should be emphasized that these are for checking purposes only. It is not suggested that this method is an improvement on the conventional method of obtaining a time history using natural frequencies and normalized eigenvectors when no damping is present.

Similarly, for undamped forced vibrations, problems may be solved by conventional methods provided that Duhamel's integral can be evaluated, and finite element solutions are included for comparison only.

For arbitrary forces, however, Duhamel's integral is evaluated numerically, and it is shown in Chapter 5 that the use of a refined temporal element provides a serious alternative method of solution. If damping is also present, then conventional methods become considerably more complicated, while finite element solutions are still readily obtainable.

Temporal finite element methods may also be applied with considerable success to non-linear systems, and are particularly useful where the parameters are time-dependent.

The concept of a finite element in time is perhaps not as obvious as a spatial element, and so a brief description of a spatial finite element is first given, after which the change of dimension from space to time is not difficult.

Many sophisticated finite element techniques are now available for the solution of spatial problems, see for example Zienkiewiecz [1], Dugdale and Ruiz [2], Martin and Carey [3], Robinson [4], Desai and Abel [5], but the basic concept of a spatial finite element is relatively simple, at least in the one-dimensional case about to be described.

A long, thin beam may be considered to have only one dimension, that of length, and deflections perpendicular to the length. The beam may be considered to be divided into a number of finite elements, each of which satisfies the conditions for compatibility with adjacent elements. The deflected shape of each element may then be described by an assumed "shape function", usually a polynomial function, which, in general, only approximately represents the true shape. By suitable means, each coefficient in the function may be expressed in terms of the generalized nodal displacements, or nodal parameters, which may, for instance, be the deflections and slopes at the points where elements are connected. It should be obvious that any nodal parameter has the same value for adjacent elements, and that there must therefore be continuity of this parameter between the two elements. If the generalized displacements are deflection and slope, there can be no discontinuities in these quantities anywhere along the beam. This point is of great importance for temporal elements.

When the appropriate equations are used to find the shape of the beam, instead of an algebraic expression, which may be used to calculate any deflection, the finite element method gives a number of linear equations, the unknown variables of which are the nodal parameters. By coupling adjacent elements together a number of equations are obtained from which the values of all the nodal parameters may be found simultaneously.

This method may also be applied to temporal elements, and its use is described in Chapter 2, but the method of simultaneous solution of the equations is found to require a great deal of data to be stored.

An alternative step-by-step method of solution for temporal elements solves the equations for only one element at a time. Since the initial conditions must be known, the values of the variables at the end of the first element may be found. These values then become the initial conditions for the next element, and the process of equation solution is repeated. The values of all the nodal parameters are thus found in sequence. The advantage of this method is that only a small amount of data need be stored, but on the other hand the method of simultaneous solution of equations for coupled elements gives rather more accurate results. The explanation for this is given in Chapter 2.

The simplest possible beam element which can be used to describe the shape of a beam is one which ensures continuity of deflection and slope at the nodes. The generalized nodal displacements must therefore be u and $\frac{du}{dx}$ at each node, where u is the deflection at a distance x

along the beam. Any number of space derivatives may be included in the same way, so that, in general, the generalized displacements are u, $\frac{du}{dx}$,..... $\frac{d^{n-1}u}{dx^{n-1}}$ where the shape function has 2n terms. The functions are actually Hermitian polynomials and the coefficients may be found by using the properties of these polynomials as described, for example, by Ralston [6]. Pestel uses values 2n = 6 and 2n = 8 for refined elements [7].

It is not difficult to see that if the space dimension x of the spatial beam element is replaced by time t, the nodal displacements become u, $\frac{du}{dt}$ (velocity), $\frac{d^2u}{dt^2}$ (acceleration) etc. By this simple changing of variables a temporal finite element has been produced, which will have exactly the same form as the corresponding beam element. The shape of the temporal finite element will, of course, now be a function of the displacements, velocities etc at the beginning and end of each element. When the coefficients of the shape function have been found, Hamilton's principle may be used to generate equations having, as the unknown variables, the displacements, velocities etc.at the nodes. These equations may then be solved by either the step-by-step method or simultaneously as described above in connection with spatial elements. The derivation and methods of use of two temporal finite elements are described in detail in Chapter 2.

Since the use of temporal finite elements gives approximate solutions it is essential to know the likely magnitude of the errors, and an investigation of the accuracy obtained for a number of single degree of freedom systems is described in Chapter 3. It is found, as

expected, that in all cases accuracy is improved by increasing the number of elements in a given time, and it is indeed shown that for the simplest (basic) element the errors vanish when an infinite number of elements is used.

In Chapters 4 to 8 multi-degree of freedom discrete and distributed systems are shown to be capable of solution by using the same temporal elements as for a single-degree of freedom system. For distributed systems the generalized displacements are functions of both time and space and methods of solution using a finite element discretization in space, together with a solution in time using temporal finite elements are described. In cases where there is any possibility of ambiguity between the two types of element a temporal finite element is referred to as an 'interval'. The possibility of combining space and time to give a space-time finite element is also considered in Chapter 6.

It is shown in the appropriate chapters that temporal finite element solutions may be obtained for free, damped and forced vibrations of both discrete and distributed systems, and various types of damping of distributed systems are examined in Chapter 8.

Since the temporal finite element automatically gives the complete solution, including any transient, for any system, it may seem rather odd that many of the examples in Chapters 2 to 8 are concerned with steady state solutions. The reasons for this choice are that an exact steady state solution is usually rather easier to obtain than a transient solution, and also that a comparison of finite element errors

for different systems is more logical for steady state solutions. It should be made quite clear that there is no difficulty in obtaining a transient solution by finite element methods, and indeed a number of transient solutions are so obtained.

Perhaps the most interesting part of this work is that concerned with the non-linear vibrations of single-degree of freedom systems. It is shown in Chapter 9 that, by the further use of Hermitian interpolation, finite element formulations are possible for various types of non-linearity. The degree of accuracy obtained in a number of examples is indeed extremely high, and these methods could well have important applications in more complicated systems.

CHAPTER 2

SINGLE DEGREE OF FREEDOM SYSTEMS

2.1 Introduction

The application of finite element methods to time-dependent phenomena has been suggested by several authors; Zienkiewicz [1], for example, points out that Hermitian interpolation may be used in time as well as in space, while Wilson and Clough [8] suggest using a step-by-step method of solution.

Argyris and Sharpf [9] and Fried [10] show how a temporal finite element may be used to investigate the response of a mechanical system to applied forces, and derive the necessary equations by using Hamilton's principle. Since this temporal element is the basis upon which all later applications depend, a detailed description is given in 2.2 of the basic element and its use in describing a dynamic problem in finite element terms.

The method of use of the temporal finite element is then demonstrated in 2.3 where solutions are obtained for a number of simple problems. An alternative method of dealing with an arrangement of temporal elements is examined in 2.4, and in 2.5 a refined element is used to give increased accuracy.

Although, initially, to emphasize the nature of the method, the term "element" is used to describe a discrete part of time, the alternative "interval" appears later, as this term, it is hoped, will indicate the temporal nature of the discretization.

2.2 The basic element

The "shape" of a temporal finite element may be described in terms of a number of nodal generalized displacements or nodal parameters. It is obviously necessary to have no discontinuity of either displacement or velocity between adjacent elements, and this condition can, in general, be satisfied only if these two quantities are taken as nodal parameters at the beginning and end of each element. The minimum number of nodal parameters for one element is therefore 4 denoted by u_0, $\dot{u}_0$, u_1, $\dot{u}_1$ where the suffix $_0$ or $_1$ refers to the beginning and end respectively of an element occupying a time interval τ, as shown in Fig. 2.1.

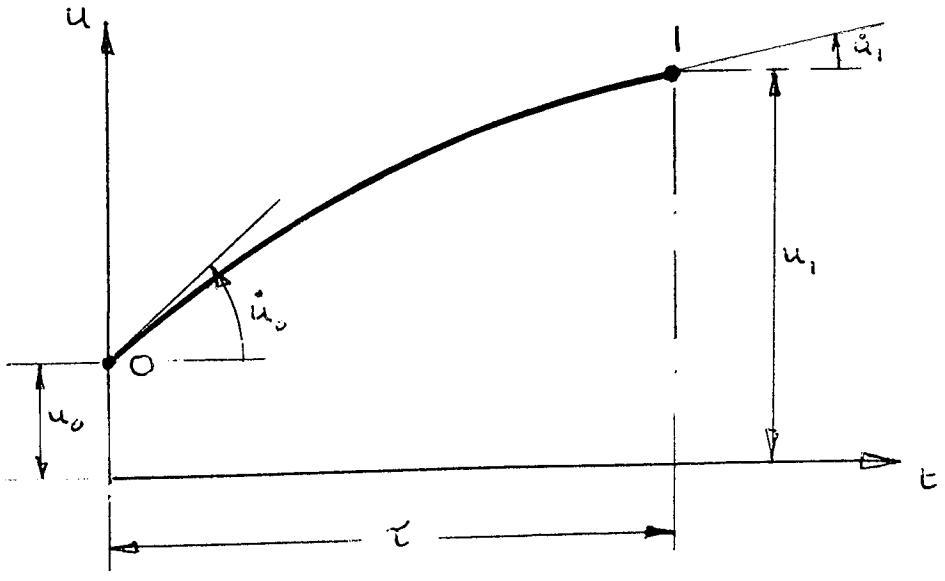


Fig. 2.1

Since 4 nodal parameters are used, the "shape" of the element may be described in terms of a 4-term polynomial, which will in general, give only approximate values for the displacement in the actual system. Subsequent calculations are simplified by the introduction of two more variables, i.e. $s = t/\tau$ and $v = \dot{u}\tau$ and hence $v_0 = \dot{u}_0\tau$ and $v_1 = \dot{u}_1\tau$. The required polynomial is thus

$$u = a_0 + a_1 s + a_2 s^2 + a_3 s^3 \qquad (2.2.1)$$

and by differentiating with respect to t

$$\dot{u} = (a_1 + 2a_2 s + 3a_3 s^2)/\tau \qquad (2.2.2)$$

and $$v = a_1 + 2a_2 s + 3a_3 s^2 \qquad (2.2.3)$$

Hence, by substituting the nodal parameters into (2.2.1) and (2.2.3) where $s = 0$ and 1 for initial and final values respectively

$$\begin{Bmatrix} u_0 \\ v_0 \\ u_1 \\ v_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad (2.2.4)$$

or $\{u\} = [A]\{a\}$

and hence $\{a\} = [A]^{-1}\{u\}$ (2.2.5)

where $$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & 1 \\ 2 & 1 & -2 & 1 \end{bmatrix}$$

By substituting (2.2.5) into (2.2.1) and (2.2.2), the displacement u and velocity $\dot{u}$ may be found for all values of t between 0 and τ in terms of the 4 nodal parameters, i.e.

$$u = (1-3s^2+2s^3)u_0 + (s-2s^2+s^3)v_0 + (3s^2-2s^3)u_1 + (-s^2+s^3)v_1$$

$$\dot{u} = [(-6s+6s^2)u_0 + (1-4s+3s^2)v_0 + (6s-6s^2)u_1 + (-2s+3s^2)v_1]/\tau$$

or $$u = \psi_1 u_0 + \psi_2 v_0 + \psi_3 u_1 + \psi_4 v_1 \qquad (2.2.6)$$

$$\dot{u} = \dot{\psi}_1 u_0 + \dot{\psi}_2 v_0 + \dot{\psi}_3 u_1 + \dot{\psi}_4 v_1 \qquad (2.2.7)$$

where $\psi_1 = (1-3s^2+2s^3)$ etc.

Equations (2.2.6) and (2.2.7), in fact, merely use Hermitian interpolation to find the values of a variable and its derivative, and could be used in any system where these two quantities are required, e.g. in heat transfer problems where the variable is temperature.

In the present case the behaviour of systems having displacements and velocities is studied, and a suitable method of presenting such a problem in finite element terms is required. The most convenient method of doing this appears to be to use Hamilton's principle, which states that

$$I = \int_0^\tau (T+W)dt \text{ is a minimum}$$

$$\text{or } \delta\int_0^\tau (T+W)dt = 0 \qquad (2.2.8)$$

where T and W are respectively the kinetic energy of the system and the work done by the forces.

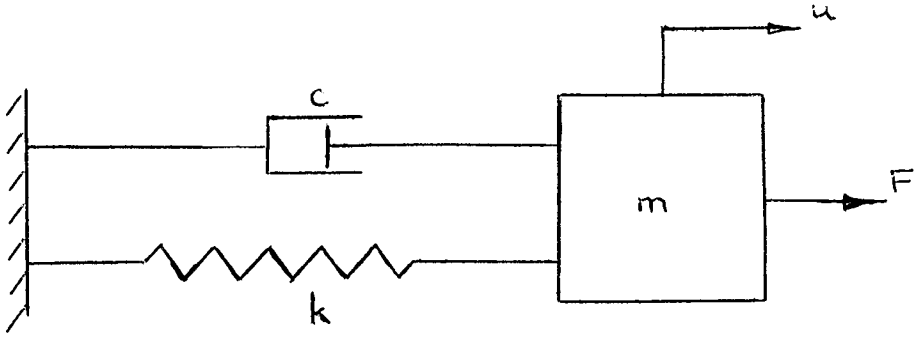


Fig. 2.2

In the single degree of freedom system of Fig. 2.2, the kinetic energy of the mass is $T = \frac{1}{2} m\dot{u}^2$, and the net force on the mass is $F - c\dot{u} - ku$ so that the work done, $W = (F-c\dot{u}-ku)u$. Equation (2.2.8) may therefore be written as

$$\delta\int_0^T (\tfrac{1}{2} m\dot{u}^2 + \{F-c\underline{\dot{u}}-k\underline{u}\}\, u)du = 0 \qquad (2.2.9)$$

It should be noted that the quantities underlined in (2.2.9) represent parts of the force and must be treated as constants during the partial differentiations with respect to u_0 etc. performed below.

If the force F varies with time, the same method of interpolation as that used for the displacement in (2.2.6) may be employed so that

$$F = \psi_1 F_0 + \psi_2 \dot{F}_0\tau + \psi_3 F_1 + \psi_4 \dot{F}_1\tau \qquad (2.2.10)$$

where F_0, F_1 are the values of F at $t = o$ and $t = \tau$ respectively and $\dot{F}_0$ $\dot{F}_1$ are the corresponding rates of change of F.

When (2.2.6), (2.2.7) and (2.2.10) are substituted into (2.2.9), each term in the integral becomes a function of the nodal parameters u_0 v_0 u_1 v_1, and in order to satisfy the condition that $\delta I = 0$, the partial differential coefficient of I with respect to each of these quantities must be zero. Four equations are thus obtained which may be given in the form

$$(m[h^0] - c[h^{II}] - k[h^I])\{u\} + [h^I]\{F\} = \{0\} \qquad (2.2.11)$$

where $\{u\} = [u_0\ v_0\ u_1\ v_1]^t$

and $\{F\} = [F_0\ \dot{F}_0\tau\ F_1\ \dot{F}_1\tau]^t$

The three time-dependent matrices are found from

$$h^{O}_{ij} = \tau \int_{O}^{1} \dot{\psi}_i \dot{\psi}_j ds$$
$$h^{I}_{ij} = \tau \int_{O}^{1} \psi_i \psi_j ds \qquad (2.2.12)$$
$$h^{II}_{ij} = \tau \int_{O}^{1} \psi_i \dot{\psi}_j ds$$

The $[h^O]$ and $[h^I]$ matrices are symmetrical while the $[h^{II}]$ matrix is unsymmetrical, as shown in (2.2.13).

$$[h^O] = \frac{1}{30\tau} \begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ \text{Symm} & & & 4 \end{bmatrix}$$

$$[h^I] = \frac{\tau}{420} \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ \text{Symm} & & & 4 \end{bmatrix} \qquad (2.2.13)$$

$$[h^{II}] = \frac{1}{60} \begin{bmatrix} -30 & 6 & 30 & -6 \\ -6 & 0 & 6 & -1 \\ -30 & -6 & 30 & 6 \\ 6 & 1 & -6 & 0 \end{bmatrix}$$

Now if the initial values u_0 and v_0 are known, only two equations are required to find the final values u_0 and v_1 in equations (2.2.11). The displacements u_0 and u_1 must, however, be taken to have prescribed values and so the first and third equations of (2.2.11) obtained from $\frac{\partial I}{\partial u_0}$ and $\frac{\partial I}{\partial u_1}$ must be discarded. This requirement is easily verified by

considering, for example, the case of the free undamped vibrations of a single degree of freedom system and substituting known exact values of u_0 v_0 u_1 v_1 into equations (2.2.11). It is found that, in general, the first and third equations are not even approximately satisfied, while the second and fourth equations are, for all conditions, almost exactly satisfied. (2.2.11) must therefore be reduced to only two equations which may be written as

$$[G]\begin{Bmatrix}u_0\\v_0\end{Bmatrix}+[H]\begin{Bmatrix}u_1\\v_1\end{Bmatrix}+[G_f]\begin{Bmatrix}F_0\\F_0\tau\end{Bmatrix}+[H_f]\begin{Bmatrix}F_1\\F_1\tau\end{Bmatrix}=\{0\} \qquad (2.2.14)$$

where $$[G]=\frac{m}{30\tau}\begin{bmatrix}3&4\\3&-1\end{bmatrix}-\frac{c}{60}\begin{bmatrix}-6&0\\6&1\end{bmatrix}-\frac{k\tau}{420}\begin{bmatrix}22&4\\-13&-3\end{bmatrix}$$

$$[H]=\frac{m}{30\tau}\begin{bmatrix}-3&-1\\-3&4\end{bmatrix}-\frac{c}{60}\begin{bmatrix}6&-1\\-6&0\end{bmatrix}-\frac{k\tau}{420}\begin{bmatrix}13&-3\\-22&4\end{bmatrix}$$

$$[G_f]=\frac{\tau}{420}\begin{bmatrix}22&4\\-13&-3\end{bmatrix} \text{ and } [H_f]=\frac{\tau}{420}\begin{bmatrix}13&-3\\-22&4\end{bmatrix}$$

It will be noted that, in equations (2.2.14), not only are the initial and final values of the force F_0 and F_1, required but also their respective rates of change $\dot{F}_0$ and $\dot{F}_1$. In the examples solved in this chapter, easy forcing functions are chosen so that values of the derivatives may be obtained exactly by differentiation. An alternative, approximate, method for evaluating F is given in Appendix A for use in cases where the forcing function is not easily differentiated, or is perhaps available only as a list of numerical values.

2.3 Method of use of the basic element

It has been noted in 2.2 that, if the values of the variables at the beginning of an element, u_0 and v_0, are known, then, if values of F_0 $\dot{F}_0$ F_1 and $\dot{F}_1$ are also known the values of the variables at the end of the element, u_1 and v_1, may be calculated from the two equations of (2.2.14). Then, by stepping forward one element, these values of u_1 and v_1 become the initial values for the next element. By further use of equations (2.2.14), values of u and v at the end of the second element may be calculated, and by repeated use of this step-by-step method all the required values of u and v may eventually be found. This is an obvious, but not the only, way of determining the response of a single degree of freedom system. (See 2.4 for a description of an alternative method.)

It is possible for a solution of this type to become unstable (see, for example, Crandall[11]), and the conditions for stability are examined in 6.4. There are, however, no stability problems with the examples of this chapter, and the reason for this becomes obvious when the condition for stability is established.

In this section, temporal finite elements are used to obtain solutions for a number of simple examples having known exact solutions. It is thus possible to show that the use of finite elements does provide a practical method of solution in these cases, and also enables a preliminary estimate of the accuracy of the method to be obtained.

Since the method used involves the solution of many equations, the use of a digital computer is almost essential. Equations (2.2.14) must

therefore be re-arranged into a suitable form for computer programming. Initially every term in equations (2.2.14) is multiplied by τ/m giving

$$\left(\frac{1}{30}\begin{bmatrix}3 & 4\\ 3 & -1\end{bmatrix} - \frac{c\tau}{60m}\begin{bmatrix}-6 & 0\\ 6 & 1\end{bmatrix} - \frac{k\tau^2}{420m}\begin{bmatrix}22 & 4\\ -13 & -3\end{bmatrix}\right)\begin{Bmatrix}u_0\\ v_0\end{Bmatrix}$$

$$+\left(\frac{1}{30}\begin{bmatrix}-3 & -1\\ -3 & 4\end{bmatrix} - \frac{c\tau}{60m}\begin{bmatrix}6 & -1\\ -6 & 0\end{bmatrix} - \frac{k\tau^2}{420m}\begin{bmatrix}13 & -3\\ -22 & 4\end{bmatrix}\right)\begin{Bmatrix}u_1\\ v_1\end{Bmatrix}$$

$$+\frac{\tau^2}{420m}\left(\begin{bmatrix}22 & 4\\ -13 & -3\end{bmatrix}\begin{Bmatrix}F_0\\ \dot{F}_0\tau\end{Bmatrix} + \begin{bmatrix}13 & -3\\ -22 & 4\end{bmatrix}\begin{Bmatrix}F_1\\ \dot{F}_1\tau\end{Bmatrix}\right) = \begin{Bmatrix}0\\ 0\end{Bmatrix} \qquad (2.3.1)$$

It may also be noted for future reference that for a free undamped vibration the natural angular frequency $\omega = \sqrt{k/m}$ so that $k\tau^2/m = (\omega\tau)^2$

Equations (2.3.1) may then be re-written in the form

$$[A]\begin{Bmatrix}u_1\\ v_1\end{Bmatrix} = [B]\begin{Bmatrix}u_0\\ v_0\end{Bmatrix} + [C]\begin{Bmatrix}F_0\\ \dot{F}_0\tau\end{Bmatrix} + [D]\begin{Bmatrix}F_1\\ \dot{F}_1\tau\end{Bmatrix}$$

and since all quantities on the right-hand side of the equations are known, values of u_1 and v_1 are easily calculated. These values then replace the original u_0 and v_0 and the whole process is repeated as many times as required. The computer program required to solve these two equations repeatedly may be regarded as a very simple form of the program used for the general solution, details of which are given in Appendix B.

In order to compare errors for different systems, solutions in all except the last of the examples in this section are obtained for one complete cycle. In Examples 2.1, 2.3 and 2.4, steady state solutions are obtained, the correct initial conditions being found from the known exact solution. This is, of course, only necessary to ensure that there is no transient component of the solution and is not necessary when a complete solution is required. Examples 2.3 and 2.5 have transient solutions, and no reference to the exact solutions is necessary to obtain finite element solutions, although the solutions can be compared.

Frequent references to errors occur in the present and later chapters. The term "error" may mean many different things, and the following definitions are used throughout this thesis:-

Actual error = Calculated value - Correct value

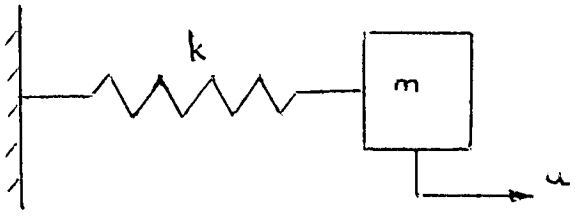
$$\text{Percentage true error} = \left| \frac{\text{Actual error}}{\text{Correct value}} \right| \times 100$$

$$\text{Percentage range error} = \left| \frac{\text{Actual error}}{\text{Amplitude}} \right| \times 100 \quad \text{for displacement}$$

$$\text{or} \quad \left| \frac{\text{Actual error}}{\omega \times \text{Amplitude}} \right| \times 100 \quad \text{for velocity}$$

Maximum range error = Maximum value of the percentage range error.

Example 2.1



For a free undamped vibration $\omega = \sqrt{k/m}$ and therefore $k\tau^2/m = (\omega\tau)^2$. Also the period $T = 2\pi/\omega$, so that if the period is divided into n equal intervals $\omega\tau = 2\pi/n$. If n is given a particular value, the coefficients of equations (2.3.1) will all have numerical values, and since c, F_0, $\dot{F}_0$, F_1, $\dot{F}_1$ are all zero in this case equations (2.3.1) may be simplified to

$$\left(\frac{1}{30}\begin{bmatrix}-3 & -1\\ -3 & 4\end{bmatrix} - \frac{(\omega\tau)^2}{420}\begin{bmatrix}13 & -3\\ -22 & 4\end{bmatrix}\right)\begin{Bmatrix}u_1\\ v_1\end{Bmatrix} = -\left(\frac{1}{30}\begin{bmatrix}3 & 4\\ 3 & -1\end{bmatrix} - \frac{(\omega\tau)^2}{420}\begin{bmatrix}22 & 4\\ -13 & -3\end{bmatrix}\right)\begin{Bmatrix}u_0\\ v_0\end{Bmatrix}$$

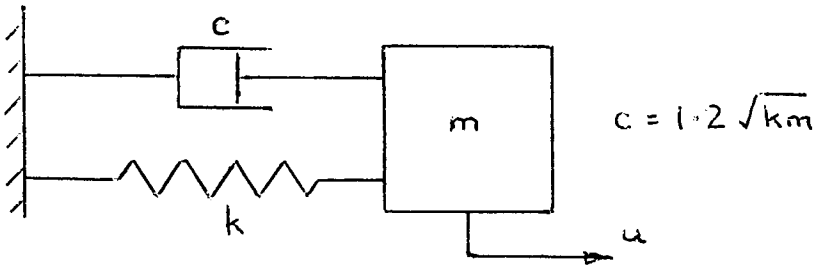
Choosing initial conditions $u_0 = A$, $v_0 = 0$ and using 10 intervals per cycle ($n = 10$), these equations may be solved 10 times to find the 10 values of displacement and velocity ($= v/\tau$). These results are shown in Table 2.1 together with those for the same system but using the value $n = 20$. Since the exact solution is $u = A \cos \omega t$, $\dot{u} = -\omega A \sin \omega t$, errors in the finite element solutions are easily evaluated, and percentage true errors are shown in brackets in Table 2.1.

It may be seen that at the end of one cycle the finite element solutions give values of displacement and velocity very close to the correct values. Even with only 10 intervals per cycle, errors are not large, and are considerably reduced when the number of intervals per cycle is doubled.

$\frac{\omega t}{2\pi}$	10 intervals/cycle		20 intervals/cycle	
	u/A	$\dot{u}/\omega A$	u/A	$\dot{u}/\omega A$
0	1.000	0.0000	1.000	0.0000
0.1	0.8079 (0.14%)	-0.5913 (0.60)	0.8087 (0.04%)	-0.5887 (0.16)
0.2	0.3053 (1.20)	-0.9554 (0.46)	0.3080 (0.33)	-0.9522 (0.12)
0.3	-0.3146 (1.81)	-0.9523 (0.13)	-0.3105 (0.48)	-0.9514 (0.04)
0.4	-0.8136 (0.57)	-0.5833 (0.76)	-0.8102 (0.15)	-0.5866 (0.20)
0.5	-0.9999 (0.01)	0.0099	-1.0000 (0)	0.0026
0.6	-0.8020 (0.87)	0.5993 (1.96)	-0.8072 (0.22)	0.5907 (0.50)
0.7	-0.2959 (4.24)	0.9583 (0.76)	-0.3056 (1.10)	0.9529 (0.19)
0.8	0.3240 (4.85)	0.9491 (0.21)	0.3129 (1.26)	0.9506 (0.05)
0.9	0.8193 (1.27)	0.5752 (2.14)	0.8117 (0.33)	0.5845 (0.56)
1.0	0.9998 (0.02)	-0.0198	1.0000 (0)	-0.0051

Table 2.1

Example 2.2



For the free vibrations of a damped system the angular frequency ω_d is known to be $\sqrt{\omega^2 - \gamma^2}$ where $\omega = \sqrt{k/m}$ and $\gamma = c/2m$. Hence for this particular system $\gamma = 0.6\omega$ and $\omega_d = 0.8\omega$, and for a finite element solution using n intervals per cycle $\omega\tau = 2.5\pi/n$. Since $c = 1.2\sqrt{km}$, $c\tau/m = 1.2\omega\tau$, so that for a particular value of n, numerical values may be assigned to the coefficients of equations (2.3.1). (For a free vibration F_0, $\dot{F}_0$, F_1, $\dot{F}_1$ are all zero.)

Choosing initial conditions $u_0 = A$, $\dot{u}_0 = 0$, finite element solutions are easily obtained for $n = 10$ and $n = 20$. The calculated values are shown in Table 2.2.

In this case it may be shown that the exact solution is

$$u = Ae^{-0.6\omega t}(\cos 0.8\omega t + 0.75 \sin 0.8\omega t)$$

$$\dot{u} = -1.25\omega Ae^{-0.6\omega t} \sin 0.8\omega t$$

so that percentage true errors in the finite element solutions may be calculated. These also are shown in Table 2.2.

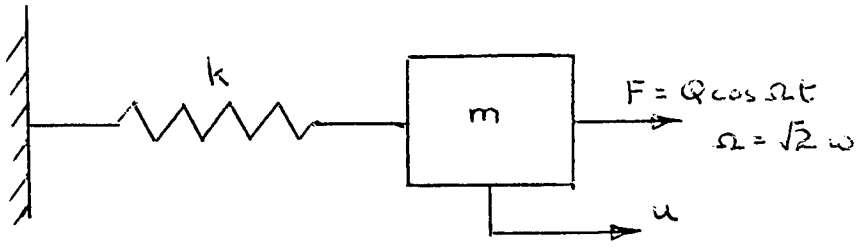
$\frac{\omega_d t}{2\pi}$	10 intervals/cycle		20 intervals/cycle	
	u/A	$\dot{u}/\omega$ A	u/A	$\dot{u}/\omega$ A
0	1.0000	0.0000	1.0000	0.0000
0.1	0.7812 (0.13)	-0.4555 (0.69)	0.7805 (0.04)	-0.4579 (0.16)
0.2	0.4017 (0.83)	-0.4600 (0.71)	0.3993 (0.23)	-0.4625 (0.16)
0.3	0.1031 (4.82)	-0.2881 (0.37)	0.0996 (1.26)	-0.2890 (0.06)
0.4	-0.0515 (7.95)	-0.1128 (1.13)	-0.0548 (2.02)	-0.1119 (0.34)
0.5	-0.0919 (3.06)	-0.0024	-0.0941 (0.75)	-0.0006
0.6	-0.0729 (1.47)	0.0413 (4.97)	-0.0737 (0.32)	0.0429 (1.23)
0.7	-0.0380 (0.64)	0.0426 (2.90)	-0.0378 (0.24)	0.0436 (0.68)
0.8	-0.0102 (8.96)	0.0271 (1.29)	-0.0095 (2.43)	0.0273 (0.24)
0.9	0.0045 (15.83)	0.0108 (2.25)	0.0051 (4.07)	0.0106 (0.71)
1.0	0.0084 (6.08)	0.0004	0.0089 (1.48)	0.0001

Table 2.2

In this example also, it is clear that an increase in the number of intervals per cycle results in a reduction in the magnitude of the errors. A comparison of the results in Tables 2.1 and 2.2 shows that, for the same number of intervals per cycle, percentage true errors are rather higher for the damped vibration than for the undamped vibration, although the order of magnitude is the same for both systems. It should be noted, however, that, for the damped system, the greatest errors occur at values of t for which the displacement or velocity is small, and these errors are actually very small compared with the maximum displacement or velocity. Excepting these errors at values of $\omega_d t/2\pi$ of 0.4, 0.8, 0.9 and 1.0, the remaining errors are very similar to those obtained in the undamped systems.

In the two previous examples no external force is applied, but in the case of a forced vibration, values of the force and its rate of change must be known at the beginning and end of each interval. Nothing is gained at this stage by introducing complicated forcing functions, so in the next two examples harmonic forces are used, as their rates of change are easily found by differentiation.

Example 2.3



For the forcing function shown, $F = Q \cos \Omega t$, $\dot{F} = -\Omega Q \sin \Omega t$. Values of F and $\dot{F}$ may therefore be calculated for any value of Ωt and used in equations (2.3.1). Also, in this case, since $\Omega = \sqrt{2}\omega$, $\omega\tau = \sqrt{2}\pi/n$ where n is the number of intervals per cycle of the force. For an undamped system $c = 0$ and again equations (2.3.1) may be used to obtain finite element solutions for $n = 10$ and $n = 20$. The exact solution is $u = \Delta \cos \Omega t$, $\dot{u} = -\Omega\Delta \sin \Omega t$ where $\Delta = -Q/k$.

The two finite element solutions, with percentage true errors in brackets are shown in Table 2.3.

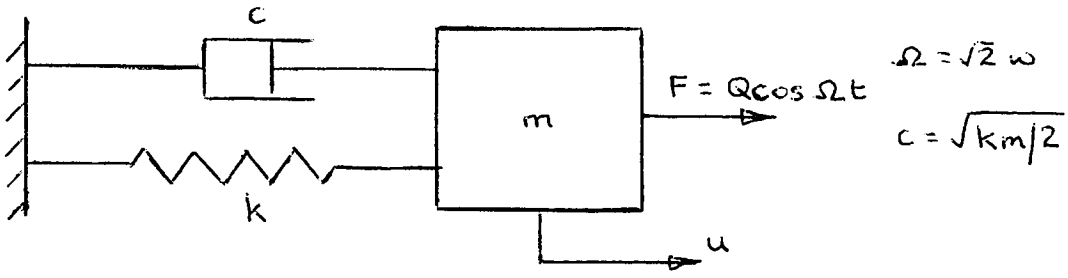
Errors in this case are very similar to those in the free undamped vibration of Example 2.1 for both 10 and 20 intervals per cycle.

The addition of a damper to the system of Example 2.3 does not make a finite element solution more difficult; the only requirement is that in equations (2.3.1) the appropriate value of c must be used.

$\frac{\Omega t}{2\pi}$	10 intervals/cycle		20 intervals/cycle	
	u/Δ	$\dot{u}/\Omega\Delta$	u/Δ	$\dot{u}/\Omega\Delta$
0	1.0000	0.0000	1.0000	0.0000
0.1	0.8078 (0.15)	-0.5915 (0.63)	0.8087 (0.04)	-0.5887 (0.16)
0.2	0.3049 (1.33)	-0.9563 (0.55)	0.3080 (0.33)	-0.9524 (0.14)
0.3	-0.3160 (2.26)	-0.9545 (0.36)	-0.3108 (0.58)	-0.9519 (0.09)
0.4	-0.8167 (0.95)	-0.5865 (0.22)	-0.8110 (0.25)	-0.5874 (0.07)
0.5	-1.0050 (0.50)	0.0072	-1.0013 (0.13)	0.0018
0.6	-0.8080 (0.13)	0.5995 (1.99)	-0.8088 (0.03)	0.5908 (0.51)
0.7	-0.3001 (2.89)	0.9636 (1.32)	-0.3068 (0.72)	0.9542 (0.33)
0.8	0.3247 (5.08)	0.9597 (0.91)	0.3130 (1.29)	0.9532 (0.23)
0.9	0.8278 (2.32)	0.5885 (0.12)	0.8138 (0.59)	0.5880 (0.04)
1.0	1.0162 (1.62)	-0.0088	1.0041 (0.41)	-0.0022

Table 2.3

Example 2.4



In this example also, the required values of F and $\dot{F}$ are easily calculated. To obtain the steady state solution, the correct initial conditions must be found from the exact solution, which may be shown to be $u = \Delta \cos(\Omega t - 3\pi/4)$ and hence $\dot{u} = -\Omega\Delta \sin(\Omega t - 3\pi/4)$ where $\Delta = Q/\sqrt{2}k$. Since, in this case, $c = \sqrt{km/2}$, the value of $c\tau/m$ required in equations (2.3.1) in $\omega\tau/\sqrt{2}$ which is equal to π/n, where n is the number of intervals per cycle, since $\tau = 2\pi/\Omega n$. The results obtained from finite element solutions using values n = 10, and n = 20, with percentage true errors in brackets, are shown in Table 2.4

$\frac{\Omega t}{2\pi}$	10 intervals/cycle		20 intervals/cycle	
	u/Δ	$\dot{u}/\Omega\Delta$	u/Δ	$\dot{u}/\Omega\Delta$
0	0.7071	0.0000	0.7071	0.0000
0.1	0.5714 (0.12)	-0.5909 (0.53)	0.5719 (0.03)	-0.5886 (0.14)
0.2	0.2163 (1.02)	-0.9547 (0.38)	0.2179 (0.27)	-0.9520 (0.10)
0.3	-0.2219 (1.55)	-0.9525 (0.15)	-0.2194 (0.41)	-0.9514 (0.04)
0.4	-0.5753 (0.56)	-0.5854 (0.40)	-0.5729 (0.15)	-0.5872 (0.10)
0.5	-0.7085 (0.20)	0.0059	-0.7075 (0.05)	0.0015
0.6	-0.5705 (0.28)	0.5953 (1.27)	-0.5717 (0.07)	0.5897 (0.33)
0.7	-0.2138 (2.13)	0.9572 (0.65)	-0.2173 (0.55)	0.9526 (0.17)
0.8	0.2250 (2.99)	0.9532 (0.23)	0.2202 (0.78)	0.9516 (0.06)
0.9	0.5784 (1.11)	0.5847 (0.52)	0.5737 (0.29)	0.5870 (0.13)
1.0	0.7111 (0.57)	-0.0075	0.7082 (0.15)	-0.0019

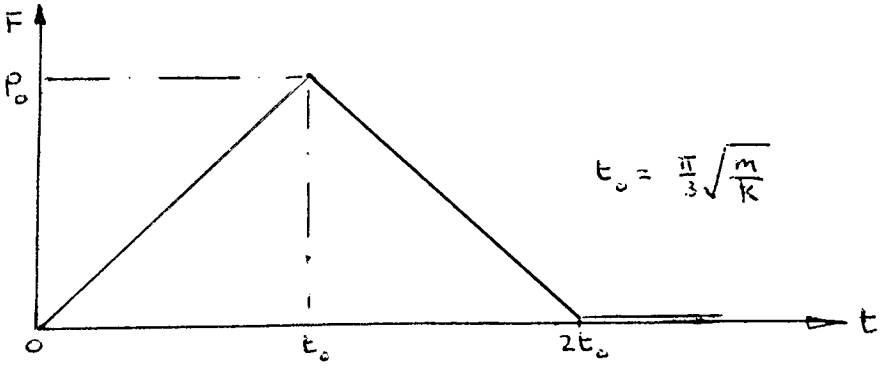
Table 2.4

It will again be seen that percentage true errors in these solutions are very similar to those for the free undamped vibration in Example 2.1.

All of the previous examples, except Example 2.2, are to some extent artificial, as steady state solutions, requiring known initial conditions from exact solutions, are obtained. It should not, of course, be necessary to have an exact solution to obtain a finite element solution, and these examples are included to show that the finite element method does give quite accurate steady-state solutions, which are easily compared with exact solutions.

Example 2.2 gives a transient solution, and finite element solutions may be obtained without any knowledge of the exact solution. A very common practical problem is that of a system initially at rest suddenly subjected to a force. Solutions for a limited number of forcing functions are given by Przmieniecki [12], and one of these is used in Example 2.5 below. For more difficult forcing functions, the evaluation of Duhamel's integral used to obtain an exact solution may become very difficult, but a finite element solution is possible in all cases where values of F and $\dot{F}$ are available.

Example 2.5



If an undamped single degree of freedom system, which is initially at rest, is subjected to the force pulse shown above, values of F and $\dot{F}$ are required to give a finite element solution by using equations (2.3.1). These values may be calculated from:-

$0 < t < t_0$	$F = P_0 t/t_0$	$\dot{F} = P_0/t_0$
$t_0 < t < 2t_0$	$F = P_0(2-t/t_0)$	$\dot{F} = -P_0/t_0$
$t > 2t_0$	$F = 0$	$\dot{F} = 0$

It should be noted that there are discontinuities in $\dot{F}$ at t_0 and $2t_0$, and these must be allowed for in the finite element solution by changing the value of $\dot{F}$ at the node common to the two elements. Since in this example, $2t_0$ is equal to one-third of the period of a free undamped vibration, if an interval of length $t_0/5$ is used this is equivalent to 30 intervals per cycle of a free vibration. A finite element solution may then be obtained from equations (2.3.1) by using the values of F and $\dot{F}$ as described above, and, since the system is initially at rest, making $u_0 = v_0 = 0$. Since, in this solution, a considerable number of values are obtained, only values of displacement at the end of every fifth interval are shown in Table 2.5. Also shown in this table are the exact values calculated from the solution given by Przmieniecki [12]

$0 < t < t_0$	$u = P/kt_0(t - \sin \omega t/\omega)$
$t_0 < t < 2t_0$	$u = P_0/kt_0[2t_0-t+2 \sin \omega(t-t_0)/\omega - \sin \omega t/\omega]$
$t > 2t_0$	$u = P_0/k\omega t_0[2 \sin \omega(t-t_0) - \sin \omega(t-2t_0) - \sin \omega t]$

$\frac{t}{t_0}$	$u/(P_0/k)$		Percentage true error
	Finite element	Exact	
0	0	0	
1	0.1731	0.1730	0.069
2	0.8274	0.8270	0.052
3	0.8269	0.8270	0.015
4	-0.0011	0	
5	-0.8280	-0.8270	0.117
6	-0.8263	-0.8270	0.081
7	0.0022	0	
8	0.8285	0.8270	0.183
9	0.8258	0.8270	0.147
10	-0.0033	0	
11	-0.8291	-0.8270	0.249
12	-0.8252	-0.8270	0.213
13	0.0044	0	

Table 2.5

Errors in this example are rather less than those of previous examples using 20 intervals per cycle. This is to be expected as the present example uses 30 intervals per cycle.

It has therefore been shown that the basic temporal element may be used to obtain solutions in a number of different single degree of freedom systems. Examination of the solutions in Examples 2.1 to 2.4

shows that, for the same number of intervals per cycle, errors of similar magnitudes appear for free, damped and forced vibrations, and that in each case there is a fluctuating error within each cycle. This is most clearly shown by observing how the actual error (i.e. calculated value - correct value) varies during one cycle. This variation is shown for a free undamped vibration, using 20 intervals per cycle, in Fig. 2.3. It will be noted that although the final displacement error is almost zero, there is a considerable velocity error at the end of the cycle. It therefore appears that, for more than one cycle, there is a steadily increasing error (particularly in the case of velocity), on which is superimposed the fluctuation in error during each cycle. Table 2.5 shows that the percentage true error in the displacement does indeed vary in this way.

The results of a more detailed examination of errors appear in the next chapter.

2.4 Solution with elements coupled

It was noted in 2.3 that the step-by-step method is not the only method of solution, and Fried [10] advocates coupling the elements together end to end. If there are n elements, the initial values u_0 and v_0 must be known, but u_1, v_1, u_2, v_2,.....u_n, v_n (i.e. 2n values) are all unknown, and these values are to be found simultaneously.

By combining the matrices of (2.2.13) for each element, in exactly the same way as that used to assemble the stiffness matrix for a beam, a (2n+2) x (2n+2) matrix is obtained.

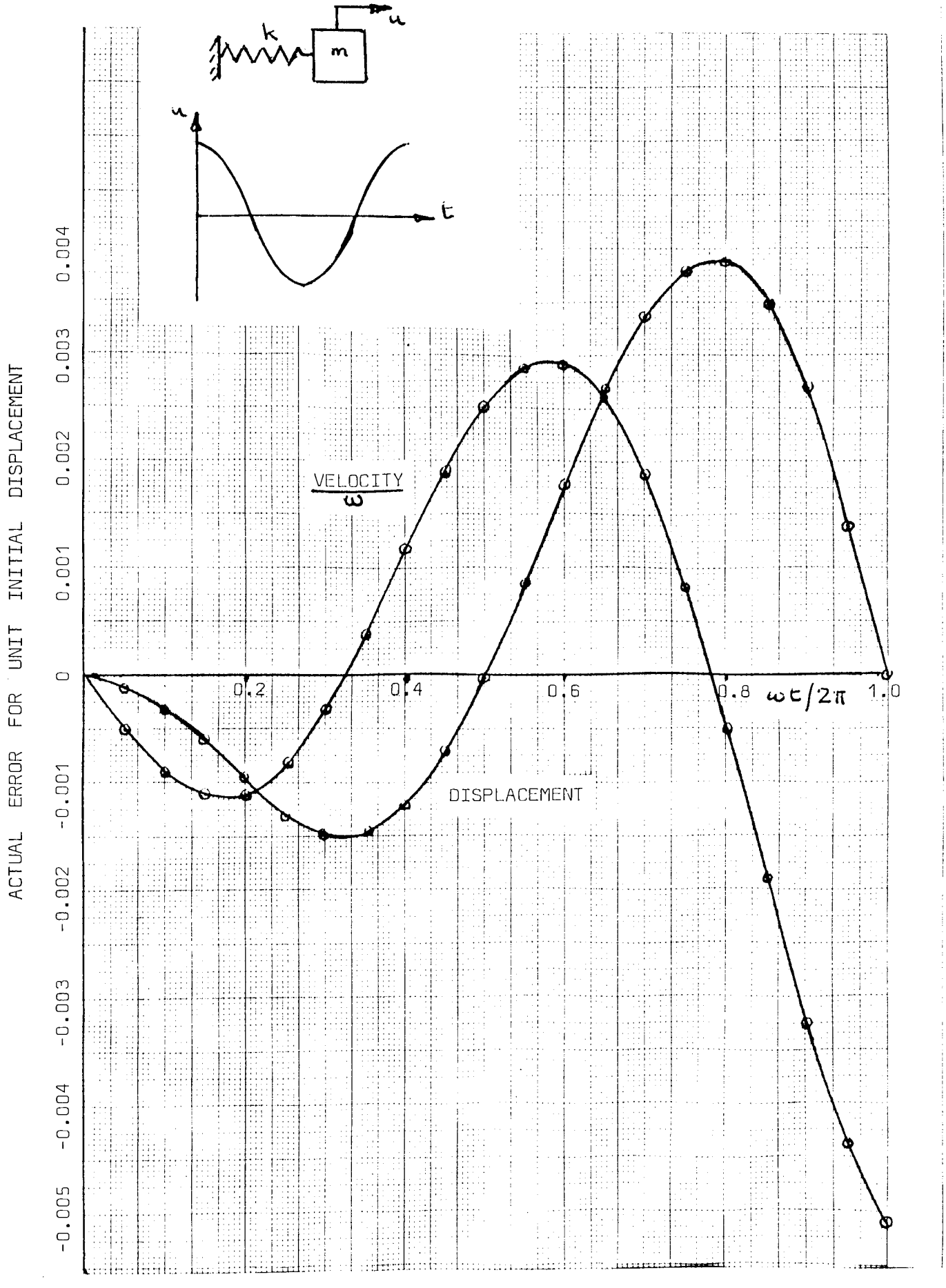


FIG.2.3 VARIATION OF ACTUAL ERROR DURING ONE CYCLE.

It has been explained in 2.2 that the initial and final displacements must be treated as prescribed quantities, and when n elements are coupled together these two displacements are u_0 and u_n. The two equations obtained from $\frac{\partial I}{\partial u_0}$ and $\frac{\partial I}{\partial u_n}$ must therefore be discarded, but all other equations found from $\frac{\partial I}{\partial u_i} = 0$, $(i \neq 0, i \neq n)$, are valid and must be used to give the necessary 2n equations to evaluate the 2n variables.

For all "nodes" except the first and last, the equations obtained are functions of the displacements and velocities for two adjacent elements, and, for a free undamped vibration, each pair of equations is of the type:-

$$\begin{aligned} fu_{i-1} + cv_{i-1} + 2eu_i + fu_{i+1} - cv_{i+1} &= 0 \\ -cu_{i-1} - dv_{i-1} + 2bv_i + cu_{i+1} + dv_{i+1} &= 0 \end{aligned} \qquad (2.4.1)$$

where the coefficients are functions of m, k and τ.

For the single element used in the step-by-step method the two equations are of the form:-

$$\begin{aligned} au_i + bv_i + cu_{i+1} + dv_{i+1} &= 0 \\ -cu_i + dv_i - au_{i+1} + bv_{i+1} &= 0 \end{aligned} \qquad (2.4.2)$$

Equations (2.4.2) thus use only 4 different coefficients, but equations (2.4.1) require two additional coefficients e and f due to the necessary use of the equations $\frac{\partial I}{\partial u_i} = 0$. The introduction of these additional coefficients means that extra conditions are imposed

on the values of the generalized displacements, and this has the effect of increasing the accuracy of the solution. This may be illustrated by dividing one complete cycle of a free undamped vibration into two elements only, as in Fig. 2.4, and comparing the results for the two methods of solution.

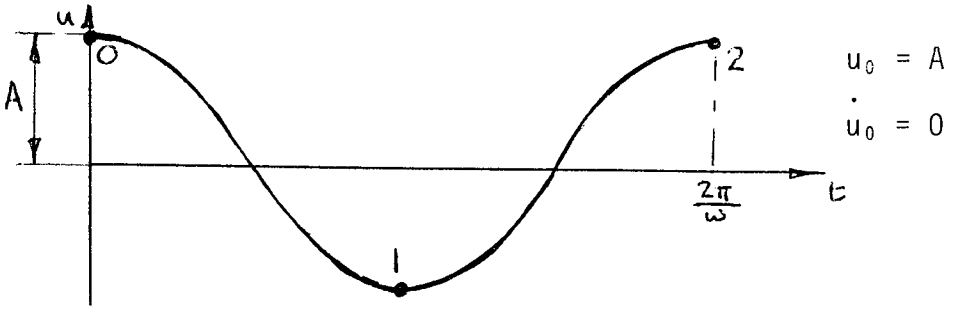


Fig. 2.4

With the two elements coupled, 4 equations are obtained, i.e.

$$\begin{aligned} au_0 + bv_0 + cu_1 + dv_1 &= 0 \\ fu_0 + cv_0 + 2eu_1 + fu_2 - cv_2 &= 0 \\ -cu_0 + dv_0 + 2bv_1 + cu_2 + dv_2 &= 0 \\ -cu_1 + dv_1 - au_2 + bv_2 &= 0 \end{aligned} \qquad (2.4.3)$$

where

$$a = \frac{3}{30} - \frac{22}{420}(\omega\tau)^2, \quad b = \frac{4}{30} - \frac{4}{420}(\omega\tau)^2, \quad c = -\frac{3}{30} - \frac{13}{420}(\omega\tau)^2$$

$$d = -\frac{1}{30} + \frac{3}{420}(\omega\tau)^2, \quad e = \frac{36}{30} - \frac{156}{420}(\omega\tau)^2, \quad f = -\frac{36}{30} - \frac{54}{420}(\omega\tau)^2$$

For 2 elements per cycle, $\omega\tau = \pi$, and since $u_0 = A$, $\dot{u}_0 = 0$, equations (2.4.3.) become

$$\begin{bmatrix} -0.405488 & 0.037164 & 0 & 0 \\ -4.931706 & 0 & -2.468949 & 0.405488 \\ 0 & 0.078674 & -0.405488 & 0.037164 \\ 0.405488 & 0.037164 & 0.416979 & 0.039337 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = A \begin{Bmatrix} 0.416979 \\ 2.468949 \\ -0.405488 \\ 0 \end{Bmatrix}$$

From these equations values of u_1, v_1, u_2 and v_2 may be found, and since $\dot{u} = v/\tau$ the velocities may be calculated. The values obtained are given in Table 2.6 below.

For the step-by-step method, values of a, b, c and d from (2.4.3) are calculated for $\omega\tau = \pi$ and substituted into (2.4.2). The two equations so obtained must then be solved twice:-

$$\begin{bmatrix} -0.405488 & 0.037164 \\ 0.416979 & 0.039337 \end{bmatrix} \begin{Bmatrix} u_{i+1} \\ v_{i+1} \end{Bmatrix} = \begin{bmatrix} 0.416979 & -0.039337 \\ -0.405488 & -0.037164 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

The first solution of these equations gives u_1 and v_1 from the known initial values u_0 and v_0 and the second solution then gives u_2 and v_2 . Values of displacement and velocity are again given in Table 2.6.

	u_1/A	$\dot{u}_1/\omega A$	u_2/A	$\dot{u}_2/\omega A$
Elements coupled	-1.0346	0.0329	1.0011	-0.0660
Step-by-step	-1.0008	0.0957	1.0032	-0.1915
Correct value	-1	0	1	0

Table 2.6

It is clear that the solution with elements coupled does give more accurate results, particularly for the velocities, but the increase in accuracy is not outstanding.

By increasing the number of elements per cycle to 6, the accuracy of both methods is increased, the final values of displacement and velocity being:-

	u_6/A	$\dot{u}_6/\omega A$
Elements coupled	1.0000	-0.0189
Step-by-step	0.9987	-0.0505
Correct value	1	0

The assembly and solution of the equations used in the coupled-element method is considerably more difficult than when the step-by-step method is used. In addition, the coupled-element method requires much more data to be stored than that needed by the step-by-step method. For n elements coupled together $2n(2n+1)$ values must be stored compared with only 8 values required by the step-by-step method. There may thus be computer storage problems for large values of n when the elements are coupled.

Although the results quoted above confirm Fried's statement that coupling of the elements does increase accuracy, this improvement is obtained at the expense of increased complexity of the calculations, and more computing time is required to solve $2n$ equations once than to solve 2 equations n times.

An alternative method of obtaining increased accuracy is therefore described in the next section, and, since this gives very accurate

results with the step-by-step method of solution, the solution by coupling of elements was discarded as being an interesting, but not the best method of solution.

2.5 The refined element

Argyris and Sharpf [9] suggest that any number of nodes may be used in one temporal element, and that the nodal parameters may include any number of derivatives at each end of these nodes. If, as for the basic element, only two nodes are used, but the second derivative $\ddot{u}$ is introduced as a nodal parameter, a 6-term polynomial may be used to describe the "shape" of the element. This should then give more accurate results than the 4-term polynomial of 2.2.1. The assumed "shape" of the refined element is therefore given by

$$u = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 \qquad (2.5.1)$$

and hence by using the same method as in 2.2

$$u = \begin{bmatrix} 1-10s^3 & s-6s^3 & \frac{1}{2}s^2-\frac{3}{2}s^3 & 10s^3-15s^4 & -4s^3+7s^4 & \frac{1}{2}s^3-s^4 \\ +15s^4-6s^5, & +8s^4-3s^5, & +\frac{3}{2}s^4-\frac{1}{2}s^5, & +6s^5, & -3s^5, & +\frac{1}{2}s^5 \end{bmatrix} \begin{Bmatrix} u_0 \\ \dot{u}_0\tau \\ \ddot{u}_0\tau^2 \\ u_1 \\ \dot{u}_1\tau \\ \ddot{u}_1\tau^2 \end{Bmatrix} \qquad (2.5.2)$$

By using exactly the same procedure as for the basic element it may be shown that equations (2.2.11) are again valid, the only difference being that for the refined element:

$$\{u\} = [u_0 \ \dot{u}_0\tau \ \ddot{u}_0\tau^2 \ u_1 \ \dot{u}_1\tau \ \ddot{u}_1\tau^2]^t$$

$$\{F\} = [F_0 \ \dot{F}_0\tau \ \ddot{F}_0\tau^2 \ F_1 \ \dot{F}_1\tau \ \ddot{F}_1\tau^2]^t$$

and $[h^O]$, $[h^I]$ and $[h^{II}]$ are 6 x 6 matrices, details of which are given in Appendix A.

Since the force vector now includes both the first and second derivatives at each node, these must be obtained by differentiation, or if this is not possible by using approximations such as those given in Appendix A.

Of the 6 equations now given by (2.2.11), the first and fourth must be discarded since values of u_0 and u_1 prescribed. The remaining 4 equations are all admissible and give similar but not identical results. By using any 3 of these equations simultaneously, 4 slightly different element "shapes" are obtained. Differences are however not large, and any 3 of these 4 equations should give satisfactory results. The second, fifth and sixth equations are found to give most accurate values at the end of a cycle of a free undamped vibration, and these equations are used in all subsequent calculations with the refined element.

Because acceleration, $\ddot{u}$, is now a nodal parameter there can be no discontinuity of acceleration between adjacent elements. While this is not an essential requirement (it is not, in general, satisfied by the basic element), it should improve considerably the accuracy of the solution. One minor problem is that the initial acceleration $\ddot{u}_0$ is now a required initial condition, but if this value is not known it may be easily calculated for a single degree of freedom system from

$$\ddot{u}_0 = (F_0 - c\dot{u}_0 - ku_0)/m \qquad (2.5.3)$$

where u_0 and $\dot{u}_0$ are the known initial displacement and velocity respectively.

To verify that the refined element does indeed give increased accuracy a comparison with solutions obtained with the basic element is required. The values in Table 2.7 show the results obtained with the refined element using 10 intervals per cycle for the free undamped vibration of Example 2.1, and the forced damped vibration of Example 2.4 in which $F = Q \cos \sqrt{2}\,\omega t$ and $c = \sqrt{km/2}$.

Percentage true errors are shown in brackets in Table 2.7, values of less than 0.001% being shown as zero. It will be seen that the maximum percentage true errors for the free and forced vibrations are 0.008% and 0.005% respectively. The corresponding figures for the solutions obtained with the basic element taken from Tables 2.1 and 2.4 are 4.85% and 2.99% for 10 intervals per cycle. It is obvious that the refined element gives accuracy of a much higher order then the basic element; in the two examples compared here the replacement of the basic element by the refined element reducing the maximum true error by a factor of approximately 600. A more detailed investigation of the accuracy of the two elements is given in the next chapter.

2.6 Conclusion

It has been shown that, by using finite element methods, it is possible to describe the displacement-time "shape" of a moving mass, and that the effects of damping and external forces may be included. The equations subsequently derived by using Hamilton's principle are found to give acceptable results in a number of cases, and that accuracy is increased by increasing the number of elements in a given time.

$\frac{t}{\text{Period}}$	10 intervals/cycle		20 intervals/cycle	
	u/A	$\dot{u}/\omega A$	u/Δ	$\dot{u}/\Omega\Delta$
0	1.000000	0.000000	0.7077107	0.000000
0.1	0.809015 (0)	-0.587782 (0.001)	0.572060 (0)	-0.587781 (0.001)
0.2	0.309022 (0.002)	-0.951047 (0.001)	0.218511 (0.001)	-0.951049 (0.001)
0.3	-0.309008 (0.002)	-0.951052 (0)	-0.218502 (0.003)	-0.951053 (0)
0.4	-0.809006 (0.001)	-0.587789 (0.001)	-0.572054 (0.001)	-0.587787 (0)
0.5	-0.999995 (0)	-0.000015	-0.707102 (0.001)	-0.000009
0.6	-0.809024 (0.001)	0.587765 (0.003)	-0.572062 (0)	0.587772 (0.002)
0.7	-0.309037 (0.006)	0.951040 (0.002)	-0.218515 (0.003)	0.951044 (0.001)
0.8	0.308990 (0.008)	0.951054 (0)	0.218497 (0.005)	0.951051 (0.001)
0.9	0.808993 (0.002)	0.587802 (0.003)	0.572049 (0.002)	0.587788 (0)
1.0	0.999992 (0.001)	0.000033	0.707097 (0.001)	0.000012

Table 2.7

Determination of the nodal displacements and velocities in a period of time spanned by several elements may be found by solving the two equations for each element in sequence, or by solving all the equations simultaneously. The second of these methods does give increased accuracy but at the cost of an increase in complexity of the solution. It is likely that any required accuracy can be achieved by the step-by-step method by simply increasing the number of elements.

Finally, the use of a refined element using acceleration as a nodal parameter results in much improved accuracy for only a small increase in the difficulty of the solution.

Since accuracy is mentioned on many occasions in this chapter, it now seems that a thorough investigation of this topic is essential, and the whole of the next chapter is devoted to this study.

CHAPTER 3

ACCURACY OF FINITE ELEMENT SOLUTIONS

3.1 Introduction

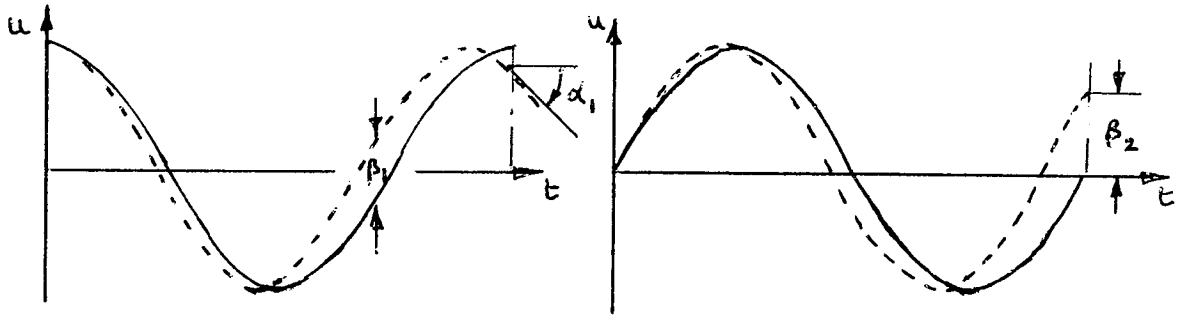
Because the assumed displacement-time relationships of (2.2.1) and (2.5.1) are, in general, only approximations, it follows that any displacements and velocities calculated from these assumed "shapes" must also be approximate. To use an approximate method successfully a knowledge of the magnitude of the errors is required and in the finite element solutions it appears that this magnitude depends on the size of the element. In this chapter it is shown that the errors do indeed become smaller as the element size is reduced, and it is proved that, in the case of a free undamped vibration, the errors approach zero as the number of elements per cycle becomes very large.

Solutions for more than one cycle are also investigated, and it is shown that, for a number of cycles, the maximum error is likely to be one of phase, and that this error is proportional to the number of cycles.

Errors for the basic and refined elements are compared, and it is shown that, for a given number of intervals per cycle, both of these give more accurate results than the finite difference method.

3.2 Errors within a single cycle

If the results of Examples 2.1 to 2.4 are examined it will be seen that, for a given number of elements per cycle, errors for free, damped and forced vibrations are very similar. In order to keep the number of variables to a minimum, the effect of variation of the number of elements per cycle of a free undamped vibration is examined. For both the basic and refined finite elements the displacement varies as shown in Fig. 3.1(a) or 3.1(b), depending on whether the cycle starts at a position of maximum displacement or maximum velocity.



(a) (b)

Fig. 3.1

_____ Exact solution

---------- Finite element solution

α = maximum velocity error

β = maximum displacement error

If these two diagrams are examined it will be seen that in both cases the finite element solution gives a vibration period which is too small, and the results of Example 2.1 show that the maximum actual error is the velocity error α_1 in (a). It is also found that the maximum actual error in (b) is the displacement error β_2 and that $\alpha_1 = \omega\beta_2$ exactly.

It will therefore be noted that in (a) the maximum actual error is a velocity error at a position where the correct velocity is zero. Similarly in (b) the maximum actual error is in the displacement where this quantity should be zero. It would therefore be meaningless in either case to express the error as a percentage of the correct value of the variable in the appropriate position.

If, however, the error is expressed as a percentage of the maximum value of the variable then the maximum range errors are:-

in (a), $\alpha_1/\omega A \times 100\%$ of the maximum velocity (ωA)

in (b), $\beta_2/A \times 100\%$ of the maximum displacement (A)

It has previously been noted that $\alpha_1 = \omega\beta_2$, and therefore these two expressions for the percentage error give identical values. It may therefore be seen that the maximum error may be one of velocity or displacement, depending on whether the velocity or displacement is zero at the beginning of the cycle. In general, neither the velocity nor displacement may be zero and the maximum error may then be one of either velocity or displacement. The maximum range error given in Fig. 3.2 is therefore defined as:-

$$\text{Maximum range error } \varepsilon\,\% = \text{Whichever is the greater of } \left\{ \left| \frac{\text{Maximum actual velocity error}}{\omega A} \right| \times 100 \quad \left| \frac{\text{Maximum actual displacement error}}{A} \right| \times 100 \right\} \tag{3.2.1}$$

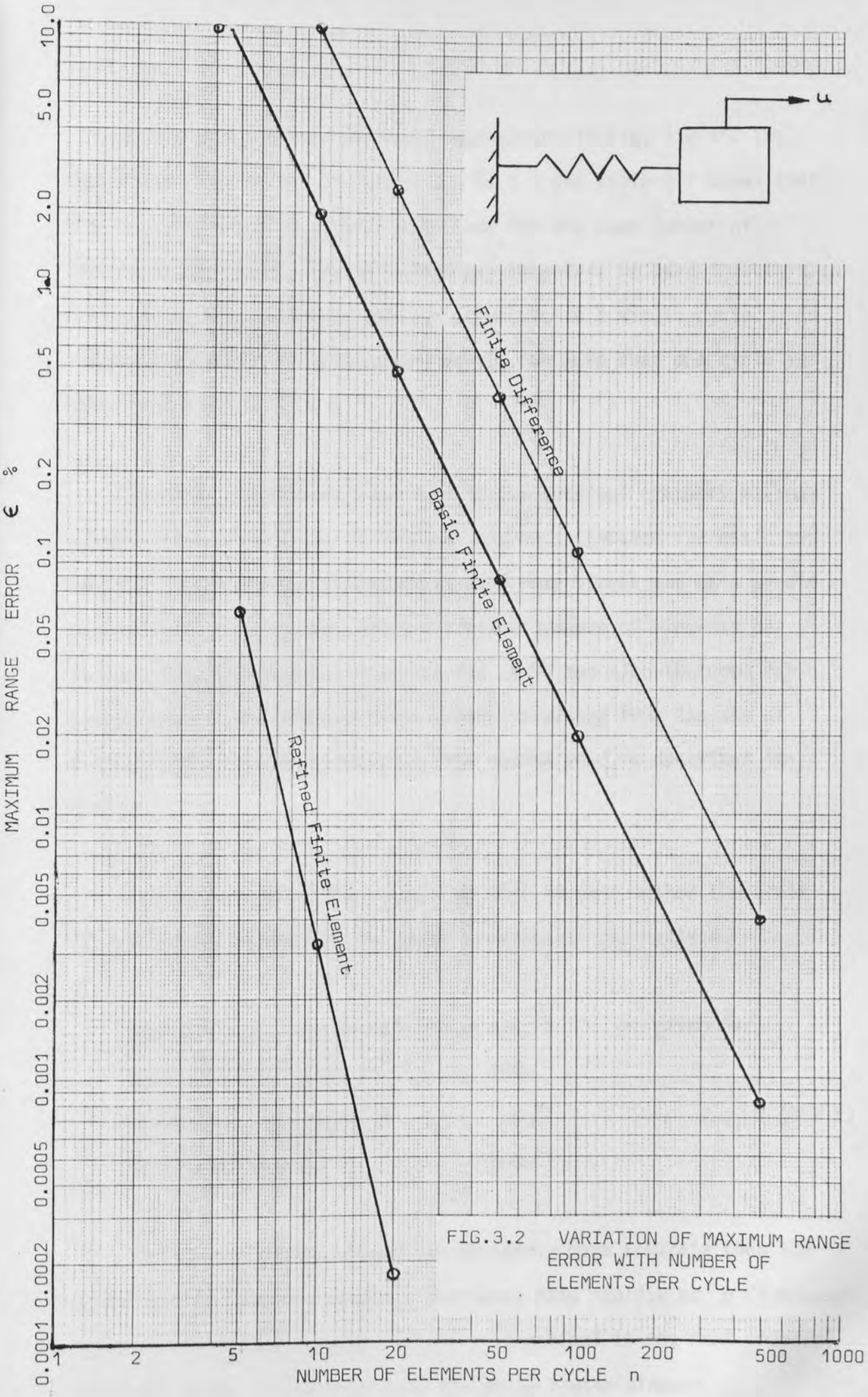


FIG.3.2 VARIATION OF MAXIMUM RANGE ERROR WITH NUMBER OF ELEMENTS PER CYCLE

A comparison of the maximum range errors in Fig. 3.2 and the percentage true errors in Tables 2.1 to 2.4 and Table 2.7 shows that the two errors are of similar magnitude for the same number of intervals per cycle. Either method may therefore be used to obtain an estimate of the maximum percentage error within a single cycle. An alternative method of calculating errors for more than one cycle is used in 3.3.

By using the methods described in the previous chapter, maximum range errors, as defined in (3.2.1), may be calculated for basic and refined finite element solutions for the free vibrations of a single degree of freedom system, using different numbers of elements per cycle. These results are shown in Fig. 3.2, and also included for comparison are the maximum range errors resulting from the use of finite difference approximations (The method used is described in Chapter 5.).

An examination of Fig. 3.2 shows that in each method the error is reduced by increasing the number of elements per cycle n.

More precisely the maximum range errors (%) are given by:-

Basic finite element	$\varepsilon = 206n^{-2}$	
Refined finite element	$\varepsilon = 50n^{-4}$	(3.2.2)
Finite difference	$\varepsilon = 1000n^{-2}$	

The refined finite element is obviously more accurate than the basic element, and its accuracy increases more rapidly as n increases. The finite difference method (which is described in the next chapter) compares rather poorly with even the basic finite element.

It should be noted that the results given in Fig. 3.2 are obtained for a free undamped vibration, but very similar values have been shown to occur in other types of vibration. The results of Fig. 3.2 can therefore be used to give quite a good estimate of the maximum range error in any single degree of freedom system.

An examination of the results presented in Fig. 3.2 suggests that the maximum errors in one cycle continue to diminish indefinitely as the number of elements per cycle is increased, and therefore that the finite element solutions converge on the correct solution. It is possible to show that this is indeed true for a free undamped vibration using the basic element. For this case, equations (2.3.1) reduce to

$$\left(\frac{1}{30}\begin{bmatrix}3 & 4\\ 3 & -1\end{bmatrix} - \frac{k\tau^2}{420m}\begin{bmatrix}22 & 4\\ -13 & -3\end{bmatrix}\right)\begin{Bmatrix}u_0\\ v_0\end{Bmatrix} + \left(\frac{1}{30}\begin{bmatrix}-3 & -1\\ -3 & 4\end{bmatrix} - \frac{k\tau^2}{420m}\begin{bmatrix}13 & -3\\ -22 & 4\end{bmatrix}\right)\begin{Bmatrix}u_1\\ v_1\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{3.2.3}$$

These two equations may be written in the form

$$\begin{aligned} au_0 + bv_0 + cu_1 + dv_1 &= 0\\ -cu_0 + dv_0 - au_1 + bv_1 &= 0 \end{aligned} \tag{3.2.4}$$

where

$$\begin{aligned} a &= \frac{3}{30} - \frac{22}{420}\frac{k\tau^2}{m}\\ b &= \frac{4}{30} - \frac{4}{420}\frac{k\tau^2}{m}\\ c &= -\frac{3}{30} - \frac{13}{420}\frac{k\tau^2}{m}\\ d &= -\frac{1}{30} + \frac{3}{420}\frac{k\tau^2}{m} \end{aligned} \tag{3.2.5}$$

and also $k/m = \omega^2$

Equations (3.2.4) apply to the first element, but can be made to apply to any element by using the appropriate values of u and v, so that for the i'th element after the start of the vibration

$$\begin{aligned} au_{i-1} + bv_{i-1} + cu_i + dv_i &= 0 \\ -cu_{i-1} + dv_{i-1} - au_i + bv_i &= 0 \end{aligned} \qquad (3.2.6)$$

Since the initial values u_{i-1} and v_{i-1} are known when the step-by-step method is used, the unknown variables are u_i and v_i and by eliminating each in turn from equations (3.2.6), separate expressions for u_i and v_i are obtained, i.e.

$$\begin{aligned} u_i &= Au_{i-1} + Bv_{i-1} \\ v_i &= Cu_{i-1} + A_{vi-1} \end{aligned} \qquad (3.2.7)$$

$$\text{where } A = -\frac{ab + cd}{ad + bc}, \quad B = \frac{d^2 - b^2}{ad + bc}, \quad C = \frac{c^2 - a^2}{ad + bc} \qquad (3.2.8)$$

and a, b, c and d are as given in (3.2.5)

Hence equations (3.2.7) may be used with known initial values of u_0 and v_0 to find u_1 and v_1. These new values are then used in (3.2.7) to find u_2 and v_2 and so on for any number of elements. If, instead of using numerical values, the algebraic expressions for u and v are re-inserted in equations (3.2.7) to obtain expressions for the next pair of variables, it may be shown that at the end of the n'th element

$$\begin{aligned} u_n &= S_1 u_0 + S_2 v_0 \\ v_n &= \frac{C}{B} S_2 u_0 + S_1 v_0 \end{aligned} \qquad (3.2.9)$$

where $S_1 = A^n + \frac{n!}{(n-2)!2!} A^{n-2}BC + \frac{n!}{(n-4)!4!} (BC)^2 + \ldots$ (3.2.10)

and

$$S_2 = \frac{n!}{(n-1)!1!} A^{n-1}C + \frac{n!}{(n-3)!3!} A^{n-3}BC^2 + \ldots$$

The last term in each series depends on whether n is odd or even as follows

	n Odd	n Even
Last term of S_1	$n\ A(BC)^{(n-1)/2}$	$(BC)^{n/2}$
Last term of S_2	$B^{(n-1)/2}C^{(n+1)/2}$	$(n-1)AB^{(n/2-1)}C^{n/2}$

Since one complete cycle is being considered, $\tau = \frac{2\pi}{\omega n}$ where n is the number of elements per cycle, and as n is increased the value of τ becomes smaller. By substituting $\omega\tau = \frac{2\pi}{n}$ into equations (3.2.5) and then substituting these expressions into equations (3.2.8) it may be shown that as $n \to \infty$, $A \to 1$, $BC \to -\frac{2\pi}{n}$ and $\frac{C}{B} \to -\frac{3}{4}$.

For an infinite number of elements, the expressions for S_1 and S_2 become infinite series, and by substituting the limiting values of A, BC and $\frac{C}{B}$ into (3.2.10) it may be seen that when $n = \infty$

$$S_1 = 1 - \frac{(2\pi)^2}{2!} + \frac{(2\pi)^4}{4!} - \frac{(2\pi)^6}{6!} + \ldots = \cos 2\pi = 1$$

$$S_2 = \sqrt{\frac{3}{4}} \left[-2\pi + \frac{(2\pi)^3}{3!} - \frac{(2\pi)^5}{5!} + \ldots \right] = \sqrt{\frac{3}{4}} \sin 2\pi = 0$$

Thus, from (3.2.9)

$$u_\infty = 1u_0 + 0v_0 = u_0$$

and

$$v_\infty = -\frac{3}{4}.0u_0 + 1v_0 = v_0$$

i.e. for an infinite number of elements per cycle the values of u and v at the end of the cycle are exactly the same as those at the beginning of the cycle for any initial conditions. Since this is an exact solution for a free undamped vibration, the basic finite element does converge to the correct solution.

Although this proof is for a free undamped vibration, because very similar errors arise in damped and forced vibrations when the same number of elements per cycle is used, it seems very likely that in these systems also the finite element solutions approach the correct solutions as the number of elements per cycle becomes very large. In Examples 2.1 to 2.4 an increase in the number of elements per cycle certainly does increase the accuracy of the solution.

It is unlikely that, in general, only one cycle of a vibration needs to be studied, but before the errors in more than one cycle are investigated, the errors for a single cycle are given in an alternative form. It will be remembered that Fig. 3.1 shows that the maximum error may be in either the velocity or the displacement depending on whether the velocity or displacement is zero at the beginning of the cycle. In both cases the displacement-time curves for finite element and exact solutions are close together i.e. in the first case the finite element solution is approximately a cosine curve and in the second case a sine curve.

For an undamped free vibration in which $u_0 = A_0$ and $\dot{u}_0 = 0$ the exact solution is

$$u = A_0 \cos \omega t \qquad (3.2.11)$$

Near the end of the cycle the finite element solution is given almost exactly by

$$u = A \cos (\omega t - \phi) \qquad (3.2.12)$$

and hence $\dot{u} = - \omega A \sin (\omega t - \phi)$

where A = amplitude and ϕ = Phase lag

Fig. 3.3 shows typical exact and assumed curves given by (3.2.11) and (3.2.12) respectively.

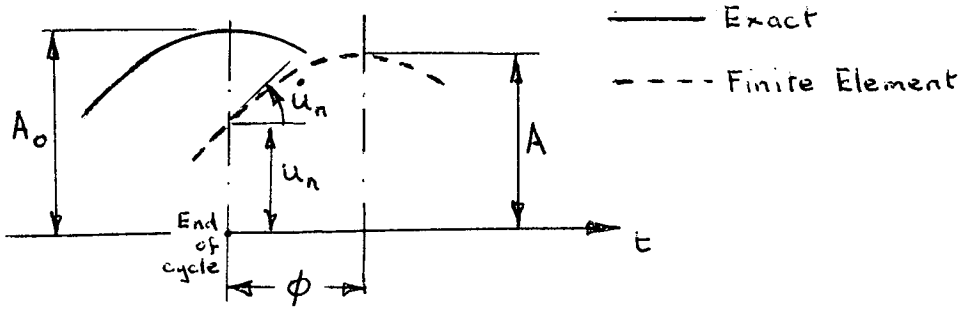


Fig. 3.3

Now if the finite element and exact solutions agreed precisely, A would be equal to A_0 and ϕ would be zero. Thus by comparing the values of A and A_0 and calculating the value of ϕ, amplitude and phase errors may be used instead of the displacement and velocity errors previously calculated. If the cycle is divided into n elements, the values of u_n and v_n $(= \dot{u}_n \tau)$ may be calculated, either by the original step-by-step finite element method as used in 2.3 or by summation of the series S_1 and S_2 of (3.2.10), and the substitution of these values into equations (3.2.9). In either case values of u_n and $\dot{u}_n$ at the end of the cycle may be found.

Now, at the end of the cycle, $\omega t = 2\pi$ and by substituting this value into equations (3.2.12) two more equations are obtained, i.e.

$$\begin{aligned} u_n &= A \cos \phi \\ \frac{\dot{u}_n}{\omega} &= A \sin \phi \end{aligned} \tag{3.2.13}$$

and hence $A = \sqrt{u_n^2 + (\dot{u}_n/\omega)^2}$

$$\text{and } \phi = \tan^{-1}\left(\frac{\dot{u}_n}{\omega u_n}\right) \tag{3.2.14}$$

Values of A/A_0 and ϕ for various values of n are given in Table 3.1. These values apply when the basic element is used, but a similar pattern appears for the refined element.

n	u_n/A_0	$\dot{u}_n/\omega A_0$	A/A_0	ϕ rad
1	0.882821	2.006572	2.192192	1.156319
2	1.003178	-0.191486	1.021290	-0.188611
5	0.997710	-0.068483	1.000058	-0.068533
10	0.999806	-0.019774	1.000001	-0.019775
20	0.999987	-0.005111	1.000000	-0.005111
50	1.000000	-0.000825	1.000000	-0.000825
100	1.000000	-0.000207	1.000000	-0.000207
200	1.000000	-0.000052	1.000000	-0.000052
500	1.000000	-0.000008	1.000000	-0.000008

Table 3.1

It is immediately obvious that the amplitude ratio converges very rapidly on the correct value of 1, but although the phase error is reduced by increasing the number of intervals per cycle it is measurable when the error in the amplitude ratio is less than 10^{-6}. It may therefore be deduced that for one cycle of a free undamped vibration, when more then 10 intervals per cycle are used, the error in amplitude is negligible while the phase error is still appreciable. It may be further noted that, since ϕ is negative, maximum displacement occurs before the correct time. The frequency of the vibration obtained by this finite element method is therefore too high.

3.3 Errors in more than one cycle

Although, with the exception of Example 2.5, all previous calculations have been for one cycle only, the finite element method described in 2.3 may be applied to any number of cycles. Alternatively the series S_1 and S_2 of (3.2.10) may be evaluated for any number of elements n, which will now be the total number and not the number per cycle. Values of u_n and v_n calculated from equations (3.2.9) will then be the values of these variables at the end of the n'th element. When values of u_n and v_n have been found, the method described at the end of the previous section may then be used to find the amplitude and phase errors, provided that the total number of elements is an exact multiple of the number of elements per cycle. For example, if 10 elements per cycle are to be used, then values of u_n and v_n must be calculated only at the end of 10, 20, 30 . . . elements.

Both the step-by-step method of 2.3 and the series summations of 3.2 may be used to find these values of u_n and v_n, and in a number of cases for which comparisons are available are found to differ by very small amounts which are due to rounding off errors in the computer calculations. Values of amplitude and phase errors are then easily found, and these errors for various numbers of elements per cycle are shown in Fig. 3.4. It should again be noted that these results are obtained by using the basic element.

It is again obvious that both amplitude and phase errors for a given number of cycles are reduced by increasing the number of intervals per cycle. The amplitude error decreases rapidly as the number of intervals per cycle increases, and appears to be approximately proportional to (Number of intervals/cycle)$^{-6}$ up to about 50 intervals per cycle. It is thus so small (about 10^{-9}) that errors shown for higher numbers of intervals per cycle are probably mainly rounding off in the computer. A further reference to Fig. 3.4 shows that, for more than about 20 intervals per cycle, the amplitude errors are very small, even for a number of cycles.

The phase error, however, is still measurable at up to 100 intervals per cycle, and because of these greater errors it is possible to be fairly precise about the size of this error. It is found that the phase error ϕ is proportional to (Number of intervals/cycle)$^{-2}$, and directly proportional to the number of cycles.

More precisely, for n intervals per cycle, after r cycles, the

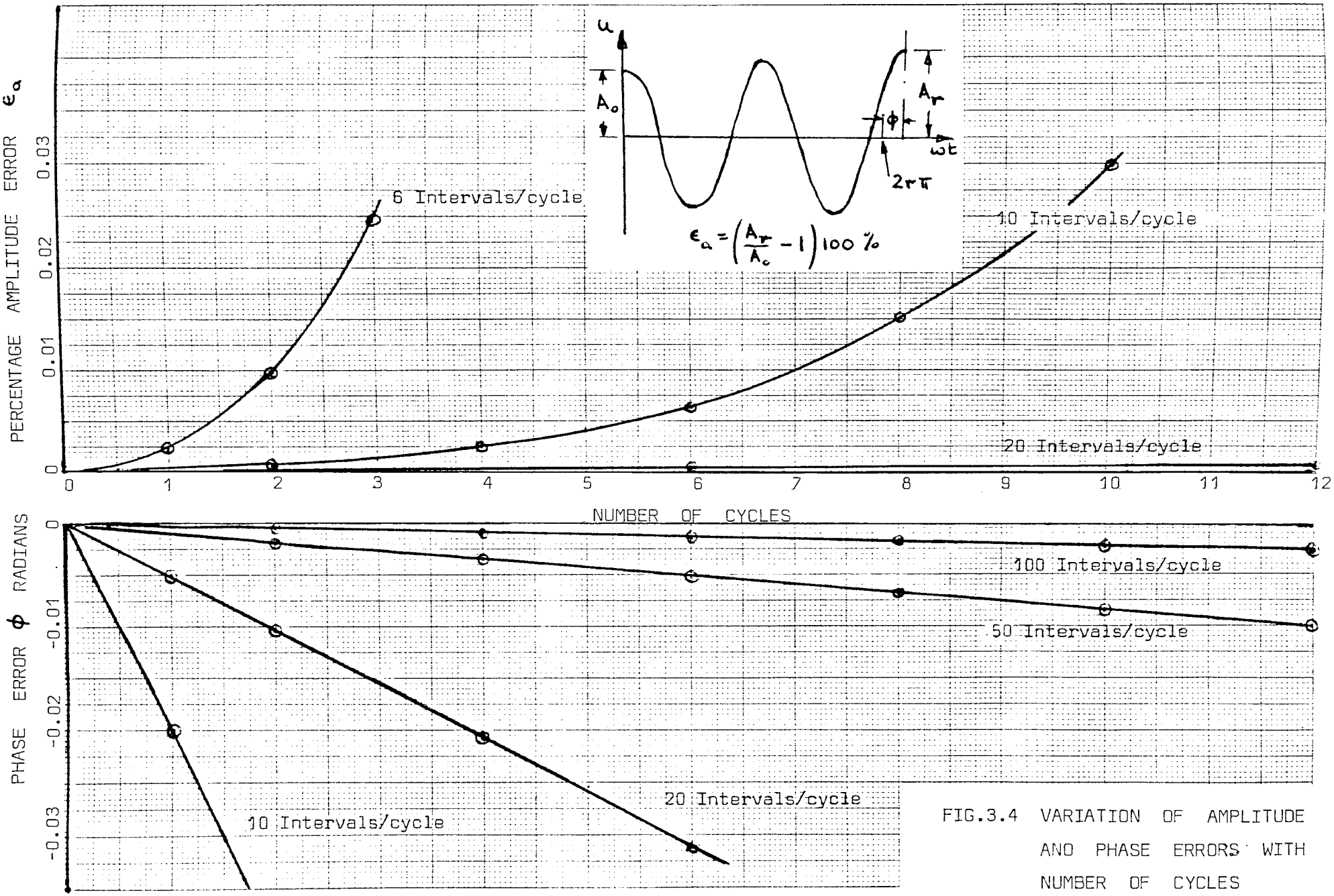


FIG.3.4 VARIATION OF AMPLITUDE ANO PHASE ERRORS WITH NUMBER OF CYCLES

phase error ϕ in radians is given by

$$\phi = 2.06\ r/n^2 \text{ approximately if } n \geq 20 \qquad (3.2.15)$$

Alternatively, inspection of the results in Table 3.1 for a single cycle shows that except for small values of n, the figures for $\dot{u}_n/\omega A_0$ and ϕ are almost exactly equal (in some cases agreeing to 6 figures), and the same agreement is found for more than one cycle. Since $\dot{u}_n/\omega A_0$ is the maximum velocity error expressed as a fraction of the maximum velocity, equation (3.2.15) may also be written as

$$\varepsilon\ (\%) = 206\ r/n^2 \qquad (3.2.16)$$

where ε is the maximum range error as defined in (3.2.1).

Alternatively if the maximum range error ε is to have a prescribed value at the end of r cycles, equation (3.2.16) may be re-arranged to give the required number of intervals per cycle n, i.e.

$$n = 14.4\ \sqrt{r/\varepsilon} \qquad (3.2.17)$$

The total number of intervals for the r cycles, N, is then obtained from N = rn or

$$N = 14.4\ r^{1.5}\varepsilon^{0.5} \qquad (3.2.18)$$

It has previously been noted on several occasions that the patterns of results are similar for the basic and refined finite elements. It may therefore be deduced that for more than one cycle an expression similar to (3.2.16) may be obtained for the refined element, and an examination of the results given in Fig. 3.2 suggests that, for the refined element

$$\varepsilon = 50r/n^4 \qquad (3.2.19)$$

and (3.2.19) does indeed give an accurate indication of the maximum range error in a free undamped vibration extending over several cycles when the refined finite element is used.

3.4 Conclusion

The basic finite element gives results for which the maximum range error in one cycle is proportional to (Number of intervals/cycle)$^{-2}$, and in the limiting case of an infinite number of intervals per cycle gives an exact solution for a free undamped vibration, and almost certainly for damped and forced vibrations also.

The refined element gives even more accurate results, and in this case the error is proportioned to (Number of intervals/cycle)$^{-4}$, so that fewer intervals per cycle will give the same accuracy as the basic element.

For more than one cycle the amplitude error is likely to be very small for the basic element, and for more than 20 intervals per cycle is negligible. The phase error, on the other hand, is appreciable, even with as many as 100 intervals per cycle, and this error is directly proportional to the number of cycles. The phase lag ϕ is shown to be negative, indicating that the frequency of a free vibration obtained from a finite solution is too high. This is also the case for the refined element although errors here are smaller.

CHAPTER 4

MULTI-DEGREE OF FREEDOM SYSTEMS

4.1 Introduction

It has been shown in Chapters 2 and 3 that a finite element solution may be obtained for single degree of freedom systems, and that any desired accuracy may be obtained by a suitable choice of the number of temporal elements to be used. Since exact solutions for single degree of freedom systems are usually not difficult to obtain it might be suggested that the finite element method achieves nothing new. The situation is, however, very different for multi-degree of freedom systems, where exact solutions are often very tedious (see Chapter 5 where these methods are described). The present chapter shows that the finite element method may be used for multi-degree of freedom systems in exactly the same way as for a single degree of freedom system. This method then provides a practical way of dealing with complicated arrangements.

4.2 Solution using the basic element

Although a system of masses and springs may be connected together in many ways, it is perhaps most convenient to consider, first, an arrangement of the type shown in Fig. 4.1. For the sake of simplicity, dampers are not included at this stage, but it is shown later than damping effects are easily allowed for.

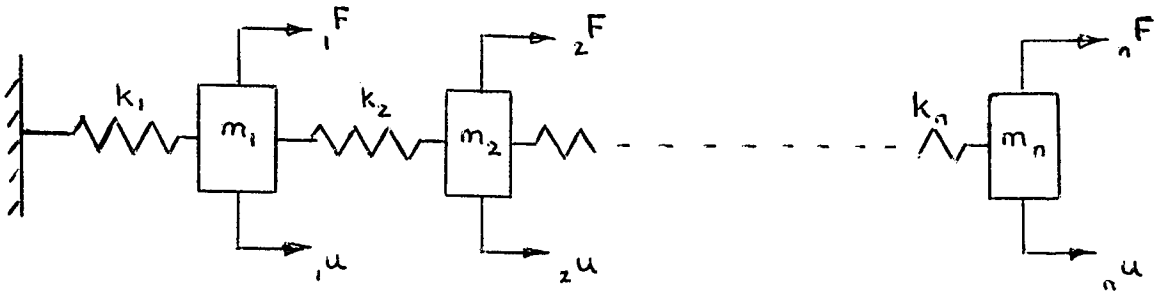


Fig. 4.1

In this system, the displacement of n masses must be considered, and the finite element method requires that values of u and $\dot{u}$ for each mass shall be used when $t = 0$ and $t = \tau$. The vector of the generalized displacements $\{u\}$ therefore contains $4n$ components which are most simply arranged as

$$[{}_1u_0\,{}_2u_0 \ldots {}_nu_0\ {}_1v_0\ {}_2v_0 \ldots {}_nv_0\ {}_1u_1\ {}_2u_1 \ldots {}_nu_1\ {}_1v_1\ {}_2v_1 \ldots {}_nv_1]^t$$

where the prefix identifies the mass (i.e.refers to spatial position) and the suffix ^{0} or ^{1} refers to time giving either the initial or final value of u or v. As in previous chapters $v = \dot{u}_\tau$.

The methods used for a multi-degree of freedom system are very similar to those previously applied to a single degree of freedom system. The modifications necessary may be shown by considering the kinetic energy of a multi-degree of freedom system.

Using the basic element of 2.2, the interpolation function of (2.2.6) is again used, i.e.

$$[\psi] = [1-3s^2 + 2s^3 \quad s-2s^2 + s^3 \quad 3s^2-2s^3 \quad -s^2 + s^3]$$

or $[\psi] = [\psi_1\ \psi_2\ \psi_3\ \psi_4]$

and $[\dot{\psi}] = [\dot{\psi}_1\ \dot{\psi}_2\ \dot{\psi}_3\ \dot{\psi}_4]$

Then if $[\Psi] = [\psi_1[I_n] \quad \psi_2[I_n] \quad \psi_3[I_n] \quad \psi_4[I_n]]$

and $[\dot{\Psi}] = [\dot{\psi}_1[I_n] \quad \dot{\psi}_2[I_n] \quad \dot{\psi}_3[I_n] \quad \dot{\psi}_4[I_n]]$ (4.2.1)

where $[I_n]$ is the n x n unit matrix

the kinetic energy of the system is given by

$$T = \frac{1}{2}[u]^t[\Psi]^t[m][\dot{\Psi}]\{u\} \tag{4.2.2}$$

where [m] is the mass matrix for the system.

Again using Hamilton's principle and considering for the present only the kimetic energy, the partial derivatives with respect to the ${}_iv_j$ ($i = 1,2,.....n$, $j = 0,1$) are required. It should be remembered that the derivatives with respect to the ${}_iu_j$ are not required, since the initial and final displacements must not be treated as variables. Thus reference to 2.2 shows that

$$\frac{\partial}{\partial {}_iv_j}\int_0^\tau T\,dt = \begin{bmatrix} h^\circ_{21}[m] & h^\circ_{22}[m] & h^\circ_{23}[m] & h^\circ_{24}[m] \\ h^\circ_{41}[m] & h^\circ_{42}[m] & h^\circ_{43}[m] & h^\circ_{44}[m] \end{bmatrix}\{u\} \tag{4.2.3}$$

A comparison with (2.2.11) shows that the single mass m for a single degree of freedom system is replaced by the mass matrix [m] in a multi-degree of freedom system.

When the processes of integration and partial differentiation are similarly applied to the generalized forces $[\{F\} - [c]\{\dot{u}\} - [k]\{u\}]^t$, it is found that the matrices [c] and [k] replace the single values c and k of (2.2.11), and the force vector for a multi-degree of freedom system is

$$[{}_1F_0 \;\; {}_2F_0 \ldots {}_1\dot{F}_{0\tau} \;\; {}_2\dot{F}_{0\tau} \ldots {}_1F_1 \;\; {}_2F_1 \ldots {}_1\dot{F}_{1\tau} \;\; {}_2\dot{F}_{1\tau} \ldots]^t$$

The equations now obtained by the use of Hamilton's principle are of exactly the same form as those of (2.2.14) and may be expressed as

$$[G]\begin{Bmatrix}\{u_0\}\\ \{v_0\}\end{Bmatrix}+[H]\begin{Bmatrix}\{u_1\}\\ \{v_1\}\end{Bmatrix}+[G_f]\begin{Bmatrix}\{F_0\}\\ \{F_{0\tau}\}\end{Bmatrix}+[H_f]\begin{Bmatrix}\{F_1\}\\ \{F_{1\tau}\}\end{Bmatrix}=\{0\} \qquad (4.2.4)$$

where $[G]=\frac{1}{30\tau}\begin{bmatrix}3[m] & 4[m]\\ 3[m] & -1[m]\end{bmatrix}-\frac{1}{60}\begin{bmatrix}-6[c] & 0\\ 6[c] & 1[c]\end{bmatrix}-\frac{\tau}{420}\begin{bmatrix}22[k] & 4[k]\\ -13[k] & -3[k]\end{bmatrix}$

$$[G_f]=\frac{\tau}{420}\begin{bmatrix}22[I] & 4[I]\\ -13[I] & -3[I]\end{bmatrix}$$

and similar expressions may be obtained for $[H]$ and $[H_f]$ by comparison with (2.2.14).

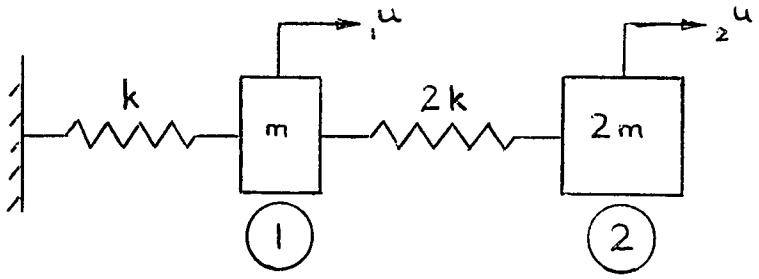
For an n-degree of freedom system $[m]$, $[c]$ and $[k]$ are $n \times n$ matrices, so that $[G]$, $[H]$, $[G_f]$ and $[H_f]$ are $2n \times 2n$ matrices. By solving the $2n$ equations of (4.2.4) the n values of displacement and n values of velocity at the end of the interval are obtained.

By employing the usual step-by-step method, a solution may thus be obtained for any period of time. It is shown in the following examples that percentage true errors are of the same order for single and multi-degree of freedom systems for the same number of intervals per cycle.

Because of the large number of calculations required to solve these examples, the use of a digital computer is essential. Details of the computer programsused are given in Appendix B.

In order to check the accuracy of the results, exact solutions must be known, and these are most easily obtained when the masses move with harmonic motion. This is only possible if the initial conditions are those appropriate to a particular mode of vibration, which may be found by conventional methods. It should be noted that the calculation of eigenvalues and eigenvectors is required for checking purposes only and is not necessary for the normal use of the finite element method. In the last two examples harmonic forces are applied, since the calculation of amplitudes by conventional methods is then relatively easy, but a finite element solution could be obtained with any type of force.

Example 4.1



This example illustrates the application of finite element methods to the free vibrations of a 2-degree of freedom system. The steady state solution is required so that the accuracy of the finite element solution may be checked easily. An exact solution must therefore be obtained so that the two masses may be given their correct initial positions to vibrate in one mode only.

Conventional methods show that the two natural frequencies are given by

$$\omega^2 = (2\pm\sqrt{3})k/m$$

and that the corresponding ratio of displacements is

$$\frac{{}_1u}{{}_2u} = -1 \mp \sqrt{3}$$

Thus for the higher frequency $\omega_2 = 1.931852\sqrt{k/m}$ and ${}_1u_0/{}_2u_0 = -2.732051$.

A finite element solution may be obtained for one complete cycle by dividing the period of $2\pi/\omega_2 = 3.252416\sqrt{m/k}$ into a convenient number of intervals n, and in the present case n = 20.

Mass and stiffness matrices are also required, and these are:-

$$[m] = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad [k] = k\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

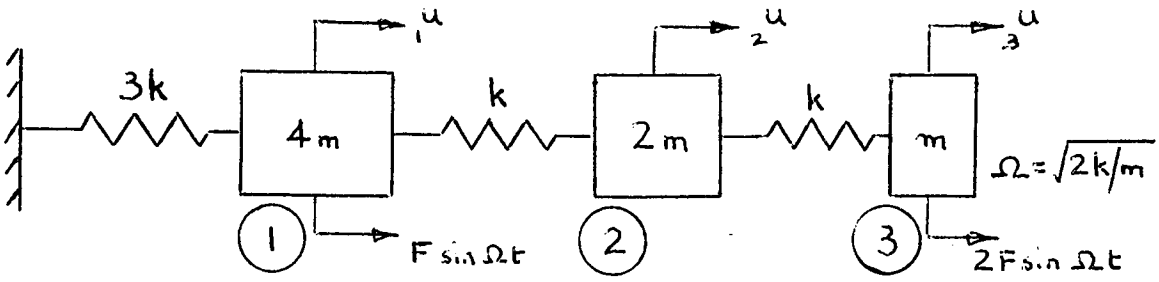
Initial and final values of displacement and velocity and percentage range errors are given in Table 4.1.

	${}_1u$	${}_1\dot{u}/\sqrt{k/m}$	${}_2u$	${}_2\dot{u}/\sqrt{k/m}$
Initial value	-2.732051	0	1	0
Value after 1 cycle	-2.732015	0.026978	0.999987	-0.009875
Range error %	0.001	0.511	0.001	0.511

Table 4.1

Not only are the two displacement errors and the two velocity errors respectively equal, but these figures are identical to the range errors previously obtained for the free vibration of a single degree of freedom system, also using 20 intervals per cycle.

Example 4.2



Conventional methods give an exact steady state solution for the forced vibrations of this system as

$$\begin{Bmatrix} {}_1u \\ {}_2u \\ {}_3u \end{Bmatrix} = \begin{Bmatrix} -1 \\ 3 \\ -5 \end{Bmatrix} \frac{F}{k} \sin \Omega t$$

The mass and stiffness matrices required for the finite element solution are:-

$$[m] = m \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [k] = k \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

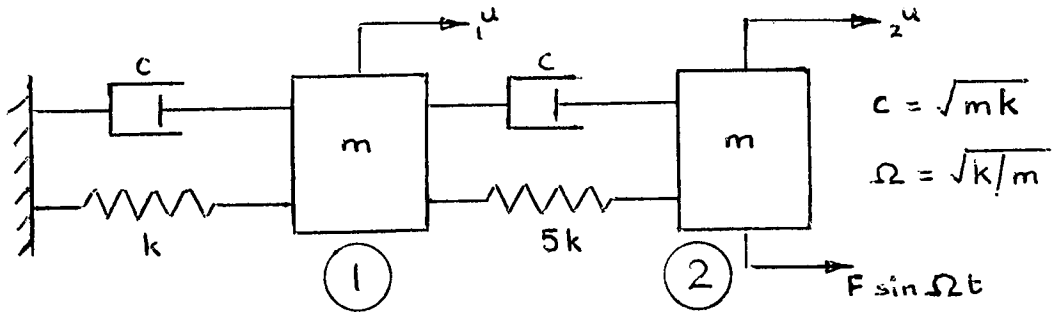
As for a single degree of freedom system the time derivatives of the forces are required, and these are easily found from $_1\dot{F} = \Omega F \cos\Omega t$, $_3\dot{F} = 2\Omega F \cos \Omega t$. Initial and final values of displacement and velocity and percentage range errors in the final values are given in Table 4.2, the calculations being for one complete cycle divided into 10 intervals.

	$_1u/(F/k)$	$_1\dot{u}/(F/\sqrt{km})$	$_2u/(F/k)$	$_2\dot{u}/(F/\sqrt{km})$	$_3u/(F/k)$	$_3\dot{u}/(F/\sqrt{km})$
Initial value	-1	0	3	0	-5	0
Final value	-1.001153	0.026181	3.008146	-0.092360	-5.020918	0.117251
Range error%	0.115	1.851	0.271	2.177	0.418	1.658

Table 4.2

In this case the displacement errors are not all equal and neither are the corresponding velocity errors. The velocity errors are, however, not very different from each other and their average value is very close to the 2% maximum range error for a single degree of freedom system obtained from Fig. 3.2.

Example 4.3



The steady state solution for the forced damped vibrations of the system in this example may be given in complex form as

$$\begin{Bmatrix} {}_1u \\ {}_2u \end{Bmatrix} = [-9 \quad -7j, \quad -8 \quad -9j] \frac{F}{15k} \sin \Omega t$$

Initial displacements and velocities may thus be calculated, and these values are given in Table 4.3.

The matrices required for a finite element solution are

$$[m] = m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad [k] = k\begin{bmatrix}6 & -5\\ -5 & 5\end{bmatrix} \qquad [c] = c\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$$

Using 10 intervals per cycle, the values of displacement and velocity at the end of the cycle may be calculated. These are shown, together with percentage range errors in Table 4.3.

	$_1u/(F/k)$	$_1\dot{u}/(F/\sqrt{km})$	$_2u/(F/k)$	$_2\dot{u}/(F/\sqrt{km})$
Initial value	-0.466667	-0.6	-0.6	-0.533333
Final value	-0.474753	-0.599765	-0.608909	-0.532253
Range error %	1.064	0.031	1.110	0.135

Table 4.3

Once again the percentage range errors for the two masses are not equal, but are of the same order of magnitude as those previously obtained for a single degree of freedom system.

4.3 Series solution for free vibrations

In 3.2 it was shown that displacements and velocities could be obtained for a single degree of freedom system by series summation. It was further shown that, by reducing the size of the element to zero, two infinite series gave an exact solution. In the present chapter, Example 4.1 for a 2-degree of freedom system gives identical errors to those previously obtained for a single degree of freedom system. It therefore seems probable that a finite element solution for a multi-degree of freedom system behaves in exactly the same way as for a single degree of

freedom system and, in the limit when $\tau = 0$, an exact solution will be obtained. This may be verified for the 2-degree of freedom system of Fig. 4.2

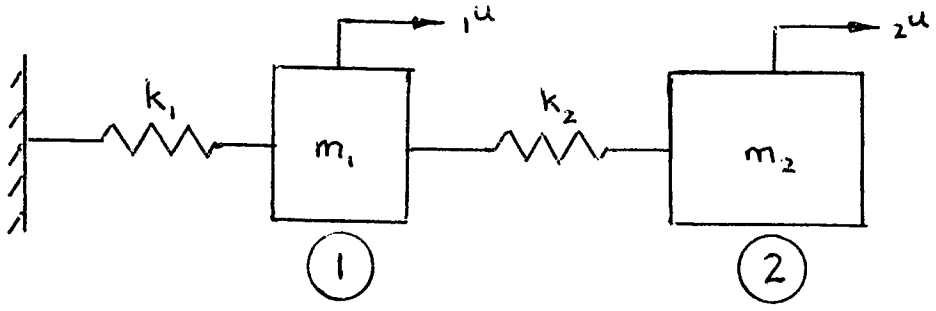


Fig. 4.2

For this system the mass and stiffness matrices are:-

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

When these matrices are substituted into (4.2.4) the [G] matrix may be obtained, and by comparison with (2.2.14) it is possible to write down the [H] matrix. For a free vibration the $[G_f]$ and $[H_f]$ matrices of (4.2.4) need not be included, so that, since n = 2, the 2n = 4 equations may be written in the form

$$\begin{bmatrix} [P] & [Q] \\ [R] & [S] \end{bmatrix} \begin{Bmatrix} {}_1u_1 \\ {}_2u_1 \\ {}_1v_1 \\ {}_2v_1 \end{Bmatrix} = \begin{bmatrix} -[R] & -[S] \\ [P] & [Q] \end{bmatrix} \begin{Bmatrix} {}_1u_0 \\ {}_2u_0 \\ {}_1v_0 \\ {}_2v_0 \end{Bmatrix} \qquad (4.3.1)$$

where $[P] = \begin{bmatrix} -\frac{1}{10}m_1 - 13(k_1+k_2)\theta & 13k_2\theta \\ 13k_2\theta & -\frac{1}{10}m_2 - 13k_2\theta \end{bmatrix}$

$$[Q] = \begin{bmatrix} -\frac{1}{30}m_1 + 3(k_1+k_2)\theta & -3k_2\theta \\ -3k_2\theta & -\frac{1}{30}m_2 + 3k_2\theta \end{bmatrix}$$

$$[R] = \begin{bmatrix} \frac{1}{10} m_1 - 22(k_1+k_2)\theta & 22k_2\theta \\ 22k_2\theta & \frac{1}{10} m_2 - 22k_2\theta \end{bmatrix}$$

$$[S] = \begin{bmatrix} \frac{4}{30} m_1 - 4(k_1+k_2)\theta & 4k_2\theta \\ 4k_2\theta & \frac{4}{30} m_2 - 4k_2\theta \end{bmatrix}$$

and $\theta = \tau^2/420$

It may then be shown that

$$\{u_1\} = [[Q]^{-1}[P]+[S]^{-1}[R]^{-1}]^{-1}\{(-[Q]^{-1}[R]+[S]^{-1}[P])\{u_0\} - ([S]^{-1}[Q]+[Q]^{-1}[S])\{v_0\}\}$$

$$\{v_1\} = [[P]^{-1}[Q]+[R]^{-1}[S]^{-1}]^{-1}\{(-[P]^{-1}[R]+[R]^{-1}[P])\{u_0\} - ([P]^{-1}[S]+[R]^{-1}[Q])\{v_0\}\} \qquad (4.3.2)$$

where $\{u_1\} = \begin{Bmatrix} {}_1u_1 \\ {}_2u_1 \end{Bmatrix}$ etc.

The matrix operations of (4.3.2) are very difficult to carry out algebraically, but it is possible to carry out the necessary calculations using a computer if these can be performed numerically. In order to do this it is necessary to assign values to the masses and stiffnesses and also to choose an interval τ. Since a solution has previously been obtained for a system of this type in Example 4.1, values are chosen here to give the same results, i.e. $m_1 = 1$, $m_2 = 2$, $k_1 = 1$, $k_2 = 2$. The higher frequency is then $\omega_2 = 1.931852$ and for 10 intervals per cycles $\tau = 0.325242$.

The two equations of (4.3.2) are then found to be of the form

$$\{u_1\} = [A]\{u_0\} + [B]\{v_0\}$$
$$\{v_1\} = [C]\{u_0\} + [A]\{v_0\} \qquad (4.3.3)$$

where in this case

$$[A] = \begin{bmatrix} 0.845472 & 0.102767 \\ 0.051384 & 0.948239 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0.947723 & 0.034814 \\ 0.017407 & 0.982537 \end{bmatrix}$$

$$[C] = \begin{bmatrix} -0.298976 & 0.198204 \\ 0.099102 & -0.100772 \end{bmatrix}$$

Because of the similarity between equations (4.3.3) and (3.2.7) for 2 and 1 degree-of-freedom systems respectively, it follows that the displacements and velocities after several intervals may be expressed in the form of a series for the 2 degree-of-freedom system of exactly the same form as that of (3.2.9) for a 1 degree-of-freedom system.

In the present case where n = 10 and $\{v_0\} = 0$

$$\{u_{10}\} = ([A]^{10}+45[A]^{8}[B][C]+210[A]^{6}([B][C])^{2}+210[A]^{4}([B][C])^{3} +45[A]^{2}([B][C])^{4}+ ([B][C])^{5})\{u_0\} \qquad (4.3.4)$$

The necessary operations may again be carried out by computer, and (4.3.4) then reduces to

$$\begin{Bmatrix} {}_1u_{10} \\ {}_2u_{10} \end{Bmatrix} = \begin{bmatrix} 0.764656 & -0.642440 \\ -0.321220 & 0.122216 \end{bmatrix} \begin{Bmatrix} {}_1u_0 \\ {}_2u_0 \end{Bmatrix} \qquad (4.3.5)$$

Since, to vibrate in this particular mode $_1u/_2u$ must have the value $-(1+\sqrt{3})/1$, initial values $_1u_0 = -2.732051$ $_2u_0 = 1$ are taken. The values then obtained from (4.3.5) are $_1u_{10} = -2.731520$ $_2u_{10} = 0.999806$. It is found that the value $_2u_{10}$ is exactly the same (to 6 figures) as the value of u_{10} for a single degree-of-freedom system, thus confirming the results of Example 4.1. It may be further deduced that since the series (3.2.9) for a single degree-of-freedom system gives a ratio u_n/u_o approaching the limit 1 as the number of intervals per cycle $n \to \infty$, the same result will be obtained for the series (4.3.4) for a 2 degree of freedom system, and that this solution also converges on the correct solution as $n \to \infty$. There seems no reason to doubt that similar results would be obtained for any desired number of degrees of freedom. Although it would thus be possible to evaluate the matrices [A], [B] and [C] and to find the required displacements from the series (4.3.4), this method requires a number of matrix products and inversions and would therefore be rather slow. Since exactly the same degree of accuracy is obtained from the step-by-step method described in 4.2 with fewer calculations, this step-by-step method is to be preferred.

4.4 Solution using the refined element

It has been shown in 4.2 that a solution may be obtained for a multi-degree of freedom system, using basic temporal elements, by replacing the single values m, k and c of a single degree of freedom system by the mass, stiffness and damping matrices. By using exactly the same processes with the refined temporal element it is again found that the same methods may be used for multi-degree of freedom systems,

with the appropriate matrices replacing the single values of the single degree of freedom systems. For an n degree of freedom system there will now be 3n equations giving the values of displacement, velocity and acceleration at the end of the interval.

One minor difficulty encountered in the use of the refined element is that initial values of displacement velocity and acceleration are required. Since the [m], [k] and [c] matrices are known, it is however possible to specify initial values of displacement and velocity only and to calculate the accelerations from

$$\{\ddot{u}\} = [m]^{-1}(\{F\} - [k]\{u\} - [c]\{\dot{u}\}) \qquad (4.4.1)$$

This requires the inversion of the mass matrix which increases the computing time, but does reduce the amount of data to be supplied.

To confirm that a solution can be obtained for a multi-degree of freedom system this method may be used to solve the forced damped system of Example 4.3. The data input now required is the same as for the basic element solution except that values of $\ddot{F}$ are required, and these are of course easily calculated for a harmonic force. As with the previous basic element solution 10 intervals per cycle are used, and the values of displacement and velocity at the beginning and end of one cycle are given in Table 4.4. The percentage range errors are also shown.

	$_1u/(F/k)$	$_1\dot{u}/(F/\sqrt{km})$	$_2u/(F/k)$	$_2\dot{u}/(F/\sqrt{km})$
Initial value	-0.466667	-0.6	-0.6	-0.533333
Final value	-0.466645	-0.599992	-0.599972	-0.533313
Range error %	0.003	0.001	0.003	0.003

Table 4.4

It is again obvious that, as for single degree of freedom systems, the use of the refined finite element gives greatly increased accuracy, as the errors for the basic element solution of this problem are of the order of 1%. A more detailed comparison of basic and refined element solutions is given in Chapter 5.

4.5 Conclusion

The use of the finite element method is not confined to single degree of freedom systems, but it may also be used in multi-degree of freedom systems. Both the basic and refined finite elements may be used, and in both cases the accuracy obtained in multi-degree of freedom systems is similar to that for single degree of freedom systems. By using finite elements, solutions may be obtained for free, damped and forced vibrations, and even in cases where the forces are known only as a list of values the required values of their derivatives may be obtained very accurately by the approximations given in Appendix A.

CHAPTER 5

EXACT AND NUMERICAL SOLUTIONS FOR DISCRETE SYSTEMS

5.1 Introduction

Problems requiring the response of a discrete system to applied forces can, at least in principle, be solved exactly by conventional methods. Unfortunately, except for certain forcing functions, these methods may become very tedious, and alternative numerical methods may offer more easily applied procedures. While it is true that these numerical methods give only approximate solutions, it has been shown in Chapter 3 that one of these methods, i.e. that using temporal finite elements, can be made to give almost any required degree of accuracy by increasing the number of elements. It should, for example, be possible to obtain a finite element solution to at least 6-figure accuracy by merely using a sufficient number of elements.

Nevertheless, it seems appropriate at this point to compare finite element methods with both exact and other numerical methods of solution. The methods used in this chapter are:-

A Exact methods

A1 Direct solution of the equations of motion

A2 Modal analysis with Duhamel's integral evaluated exactly

(If this is not possible B1 may be used)

B Numerical methods

B1 Modal analysis with Duhamel's integral evaluated numerically

B2 Finite difference method

B3 Basic finite element method

B4 Refined finite element method

The examples solved in this chapter are chosen so that solutions may be obtained without difficulty by each of the methods. In practice, as noted earlier, all except the simplest type of forcing function may make the exact methods of solution very difficult, but the numerical methods may be used for any type of forcing function.

The inclusion of damping makes the exact methods and the numerical integration method much more difficult, but presents no problem when finite differences and finite element methods are used. Information on the various methods is given by Przmieniecki [12], Salvadori and Baron [13] and Meirovitch [14].

A description of the methods of solution is given in the next two sections of this chapter, including solutions obtained by the exact methods. A comparison of computing times for varying numbers of degrees of freedom and standards of accuracy is then given.

5.2 Exact Methods

A1. Direct solution

For an undamped system, the equations of motion may be expressed in matrix form as

$$[m]\{\ddot{u}\} + [k]\{u\} = \{F\} \tag{5.2.1}$$

where $\{u\}$ and $\{F\}$ are the displacement and force vectors respectively.

It is theoretically possible to obtain the complete solution of these equations directly by using Laplace transforms. After transformation, equations (5.2.1) become

$$[m]\{s^2\bar{u} - su_0 - \dot{u}_0\} + [k]\{\bar{u}\} = \{\bar{F}\} \tag{5.2.2}$$

where $\{u_0\}$ and $\{\dot{u}_0\}$ are respectively the initial displacement and velocity vectors, and hence

$$\{\bar{u}\} = ([k] + s^2[m])^{-1}([m]\{su_0 + \dot{u}_0\} + \{\bar{F}\}) \tag{5.2.3}$$

or $\{\bar{u}\} = [z]^{-1}\{\bar{U}\}$

where $[z] = [k] + s^2[m]$ and $\{\bar{U}\} = [m]\{su_0 + \dot{u}_0\} + \{\bar{F}\}$

Now although it is possible, in principle, to obtain the inverse transform of equations (5.2.3), in practice the functions become so complicated that the inversion becomes almost impossible except for systems with only a few degrees of freedom, and for certain forcing functions. The task does become slightly less difficult in the special case where the system is initially at rest in its equilibrium position so that $\{u_0\} = \{\dot{u}_0\} = \{0\}$. In this case equations (5.2.3) have the simplified form

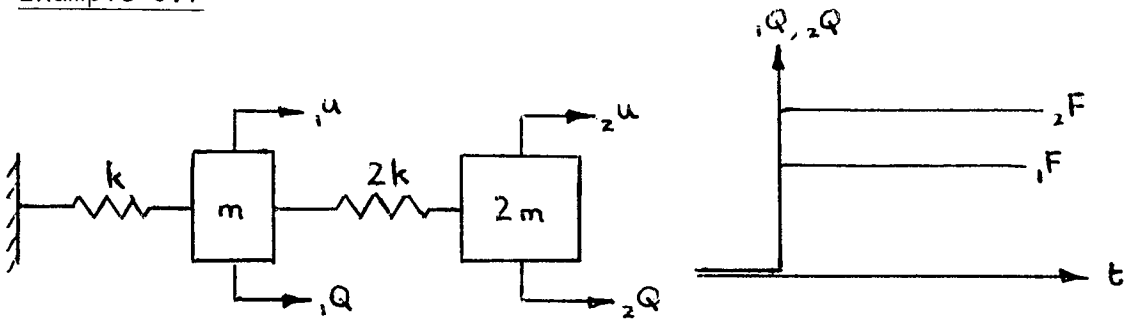
$$\{\bar{u}\} = [z]^{-1}\{\bar{F}\} \tag{5.2.4}$$

where as before $[z] = [k] + s^2[m]$.

The task of finding the inverse transform of $[z]^{-1}$ is, however, still formidable in most cases. Further details of the use of Laplace transforms are given by, among others, Spiegel [15] and Thompson [16].

The use of this method for a 2 degree of freedom system is demonstrated in Example 5.1 below.

Example 5.1



For this system

$$[m] = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad [k] = k\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\text{so that } [z] = k\begin{bmatrix} 3 + s^2m/k & -2 \\ -2 & 2 + 2s^2m/k \end{bmatrix}$$

$$\text{and hence } [z]^{-1} = \frac{1}{2k\left(s^4\left(\frac{m}{k}\right)^2+4s^2\frac{m}{k}+1\right)}\begin{bmatrix} 2 + 2s^2m/k & 2 \\ 2 & 3 + s^2m/k \end{bmatrix}$$

Now for initial conditions $\{u_0\} = \{\dot{u}_0\} = \{0\}$

and for step forces $t < 0$, $_1Q = {}_2Q = 0$; $t > 0$, $_1Q = {}_1F$, $_2Q = {}_2F$;

equations (5.2.4) become

$$\begin{Bmatrix} {}_1\bar{u} \\ {}_2\bar{u} \end{Bmatrix} = \frac{1}{2ks[s^2+(2-\sqrt{3})k/m][s^2+(2+\sqrt{3})k/m](\frac{m}{k})^2} \begin{bmatrix} 2 + 2s^2m/k & 2 \\ 2 & 3 + s^2m/k \end{bmatrix} \begin{Bmatrix} {}_1F \\ {}_2F \end{Bmatrix}$$

The inverse transformation requires the two forms

$$\mathcal{L}^{-1} \frac{1}{s(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{1}{a^2}(1-\cos a\,t) - \frac{1}{b^2}(1-\cos b\,t)\right]$$

$$\mathcal{L}^{-1} \frac{s^2+c}{s(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{c-a^2}{a^2}(1-\cos a\,t) - \frac{c-b^2}{b^2}(1-\cos b\,t)\right]$$

where, in order to find ${}_1u$, $a^2 = (2-\sqrt{3})k/m$, $b^2 = (2+\sqrt{3})k/m$ and $c = k/m$. By substituting these values it may be shown that

$${}_1u = (0.788675\,{}_1F/k + 1.077350\,{}_2F/k)(1-\cos\omega_1 t) + (0.211325\,{}_1F/k - 0.077350\,{}_2F/k)(1-\cos\omega_2 t)$$

where ω_1 and ω_2 are the natural frequencies and have the values $\sqrt{(2-\sqrt{3})k/m}$ and $\sqrt{(2+\sqrt{3})k/m}$ respectively.

A similar expression may be obtained for ${}_2u$ by using the same values of a^2 and b^2 as before but in this case $c = 1.5k/m$.

A2. Modal analysis

The previous example shows that, even for a 2-degree of freedom system, the direct solution of the equations of motion is quite difficult. This is because, with this method, the equations are coupled, i.e. each equation contains functions of the displacements of each of the masses. If the equations could be uncoupled so that each equation contained functions of one variable only, a solution could be obtained much more easily. This uncoupling of the equations is indeed possible by the use

of modal analysis, and in order to use this method a preliminary solution of the eigenvalue problem is necessary to find the eigenvalues and eigenvectors.

When these values have been found the method of solution is as follows:-

If $v^{(r)}$ is the r'th eigenvector

Then if $[v^{(r)}]^t[m]\{v^{(r)}\} = \alpha_r^2$ (5.2.5)

$\{v_N^{(r)}\} = 1/\alpha_r\{v^{(r)}\}$

where $\{v_N^{(r)}\}$ is the r'th normalized eigenvector and hence $[v_N] = [\{v_N^{(1)}\}\{v_N^{(2)}\}.....\{v_N^{(n)}\}]$ is the n x n normalized modal matrix for an n-degree of freedom system. If the transform $\{u\} = [v_N]\{p\}$ is used to convert the original coordinates $\{u\}$ to principal coordinates $\{p\}$, and also $[v_N]^t\{Q\} = \{P\}$ where $\{Q\}$ are the applied forces, then the original coupled equations may be replaced by the uncoupled equations

$$\{\ddot{p}\} + [\omega^2]\{p\} = \{P\} \quad (5.2.6)$$

These equations are then solved to give values of $\{p\}$ and finally, reverting to the original coordinate system

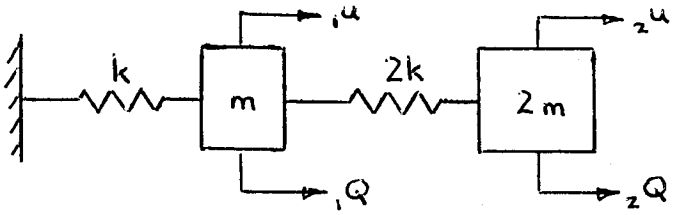
$$\{u\} = [v_N]\{p\} \quad (5.2.7)$$

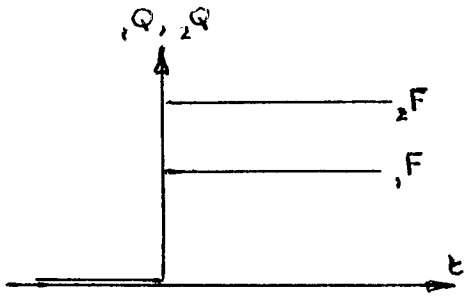
gives the values of the displacements $\{u\}$.

Certain forcing functions give easy solutions; for example, for suddenly-applied (step) forces $\{F\}$ the values of $\{P\}$ are independent of time and the solution of the r'th equation is $p_r = P_r/\omega_r^2(1- \cos \omega_r t)$. In general, however, an exact solution of the second-order differential equations may be very difficult to obtain and some form of numerical solution such as one of those described in 5.3 must be used.

So that a direct comparison of the modal analysis method can be made with the direct solution, Example 5.1, for which the equations have been solved directly, is now solved using the method of modal analysis.

Example 5.1 (Modal analysis method)





Solution of the eigenvalue problem gives

$\omega_1 = \sqrt{(2-\sqrt{3})k/m}$ and $\{v^{(1)}\} = [\sqrt{3}-1 \quad 1]^t$

$\omega_2 = \sqrt{(2+\sqrt{3})k/m}$ and $\{v^{(2)}\} = [-1-\sqrt{3} \quad 1]^t$

Hence $\alpha_1^2 = [v^{(1)}]^t[m]\{v^{(1)}\}$

$$= [\sqrt{3}-1 \quad 1]m \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \sqrt{3} & -1 \\ & 1 \end{Bmatrix}$$

$$= 2\sqrt{3}(\sqrt{3} - 1)m$$

Similarly $\alpha_2^2 = [v^{(2)}]^t[m]\{v^{(2)}\} = 2\sqrt{3}(\sqrt{3} + 1)m$

Then the normalized modal matrix is

$$[v_N] = \begin{bmatrix} \frac{\sqrt{3}-1}{\alpha_1} & \frac{-1-\sqrt{3}}{\alpha_2} \\ \\ \frac{1}{\alpha_1} & \frac{1}{\alpha_2} \end{bmatrix}$$

or $$[v_N] = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.459701 & -0.888074 \\ \\ 0.627963 & 0.325058 \end{bmatrix}$$

For step forces, ${}_1Q = {}_1F$ and ${}_2Q = {}_2F$ when $t > 0$ so that the principal forces are found from

$$\{P\} = [v_N]^t\{Q\}$$

$$\text{i.e.} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.459701 & 0.627963 \\ -0.880074 & 0.325058 \end{bmatrix} \begin{Bmatrix} {}_1F \\ {}_2F \end{Bmatrix}$$

The uncoupled equations to be solved are then

$$\{\ddot{p}\} + [\omega^2]\{p\} = \{P\}$$

where in this case P_1 and P_2 are constants.

The solution of these equations is therefore

$$p_1 = P_1/\omega_1{}^2(1 - \cos \omega_1 t)$$

$$p_2 = P_2/\omega_2{}^2(1 - \cos \omega_2 t)$$

and substituting for P_1 and P_2

$$p_1 = 1.715627\sqrt{m}/k \; {}_1F + 2.343590\sqrt{m}/k \; {}_2F$$

$$p_2 = -0.237959\sqrt{m}/k \; {}_1F + 0.087099\sqrt{m}/k \; {}_2F$$

Finally, in the original coordinate system

$$\{u\} = [v_N]\{p\}$$

$$\text{or} \begin{Bmatrix} {}_1u \\ {}_2u \end{Bmatrix} = \frac{1}{k} \begin{bmatrix} 0.459701 & -0.888074 \\ 0.627963 & 0.325058 \end{bmatrix} \begin{Bmatrix} 1.715627\,{}_1F + 2.343590\,{}_2F \\ -0.237959\,{}_1F + 0.087099\,{}_2F \end{Bmatrix}$$

so that

$${}_1u = (0.788675 \; {}_1F/k + 1.077350 \; {}_2F/k)(1 - \cos \omega_1 t)$$

$$+ (0.211325 \; {}_1F/k - 0.077350 \; {}_2F/k)(1 - \cos \omega_2 t)$$

which is exactly the same result as that obtained by direct solution.

A similar expression may be obtained for ${}_2u$.

The advantage of modal analysis is that all the calculations may be arranged so that they may be carried out by computer. This is not possible in the direct solution method where the difficulty lies in the inverse Laplace transformation.

5.3 Numerical solutions

B1. Numerical evaluation of Duhamel's integral

The exact evaluation of Duhamel's integral is possible only for a limited number of forcing functions. Apart from the obvious cases of harmonic and step forces a number of others are given by Przmieniecki[12]. In general, however, it may be necessary to evaluate this integral by some numerical method, with the result that only an approximate solution is obtained.

The numerical integration may be introduced in the following way:-

If the forces $\{Q(\tau)\}$ have known values at a number of different times, then the values of $\{P(\tau)\}$ for each value of time may be found from $\{P(\tau)\} = [v_N]^t\{Q(\tau)\}$.

Then, considering the case where all the initial displacements and velocities are zero, Duhamel's integral is used to find the values of $\{p\}$ after time t, i.e.

$$p_r = 1/\omega_r{}^2 \int_0^t P_r(\tau) \sin \omega_r(t-\tau) d\tau \qquad (5.3.1)$$

and finally the displacements are found from

$\{u\} = [v_N]\{p\}$ as in the exact method.

Various methods of numerical integration may be used, but Simpson's rule gives very accurate results with little difficulty in the examples of the next section.

B2. Finite difference method

In this method the equations of motion are solved approximately by using the central finite difference operators, so that

$$\dot{u}_i \simeq (u_{i+1} - u_{i-1})/2\tau$$
$$\ddot{u}_i \simeq (u_{i-1} - 2u_i + u_{i+1})/\tau^2 \qquad (5.3.2)$$

where u_{i-1}, u_i and u_{i+1} are successive values of u at times $t-\tau$, t and $t+\tau$.

Thus for a single degree of freedom the equation of motion is reduced to an approximate linear equation

$$m(u_{i-1}-2u_i+u_{i+1})/\tau^2 + c(u_{i+1}-u_{i-1})/2\tau + ku_i = F_i \qquad (5.3.3)$$

where $F = F(t)$.

One minor difficulty of this method is that a special starting procedure is necessary involving the introduction of a fictitious node at a negative value of t. Then, if, at time $t = 0$, the velocity is u_0, the first of equations (5.3.2) gives $\dot{u}_0 = (u_1 - u_{-1})2\tau$, from which the fictitious value u_{-1} may be replaced by a function of $\dot{u}_0$ and u_1.

For multi-degree of freedom systems the equations of motion are

$$[m]\{u\} + [c]\{\dot{u}\} + [k]\{u\} = \{F\}$$

and by using the central difference operators as in (5.2.3) the finite difference approximations for the equation of motion become

$$\frac{1}{\tau^2}[m](\{u_{i-1}\}-2\{u_i\} + \{u_{i+1}\}) + \frac{1}{2\tau}[c](\{u_{i+1}\}-\{u_{i-1}\}) + [k]\{u_i\} = \{F_i\} \qquad (5.3.4)$$

and hence

$$\{u_{i+1}\} = ([m]+\tau/2[c])^{-1}((\tau/2[c]-[m])\{u_{i-1}\} + (2[m]-\tau^2[k])\{u_i\} + \tau^2\{F_i\}) \qquad (5.3.5)$$

Velocities are readily calculated from

$$\{\dot{u}_i\} = (\{u_{i+1}\} - \{u_{i-1}\})/2\tau \qquad (5.3.6)$$

A special starting procedure is again necessary, so that when i = 0 a fictitious node i = -1 is introduced to give

$\{\dot{u}_0\} = (\{u_1\} - \{u_{-1}\})/2\tau$ where $\{\dot{u}_0\}$ are the initial velocities

and hence $\{u_{-1}\} = \{u_1\} - 2\tau\{\dot{u}_0\}$

Making this substitution in (5.3.4) when i = 0 gives

$$\frac{2}{\tau^2}[m](\{u_1\} - \{u_0\} - \tau\{\dot{u}_0\}) + [c]\{\dot{u}_0\} + [k]\{u_0\} = \{F_0\}$$

and hence

$$\{u_1\} = \{u_0\} + \tau\{\dot{u}_0\} + \tau^2/2[m]^{-1}(\{F_0\} - [c]\{\dot{u}_0\} - k\{u_0\}) \qquad (5.3.7)$$

Further details of the use of finite difference methods are given by Crandall [11], Salvadori and Baron [13] and Bajpai et al.[17].

B3 & B4. Finite element methods

The use of both basic and refined elements has been described fully in Chapters 2 and 4, and so it is not necessary to repeat the details here. It may, however, be noted that for the numerical integration and finite difference methods, values of forces only are required. The basic finite element method requires, in addition, the first derivatives of the forces, while the refined finite element method requires both their first and second derivatives. Both finite element methods therefore require more data input than the other two methods, although it is quite possible to obtain approximate values of these derivatives from values of forces only (See Appendix A). To compensate for this increase of

data required, and the greater difficulty of programming, the finite element methods can be expected to give more accurate results, and the comparison of the next section shows that this is indeed so.

5.4 A comparison of the methods

In cases where it is possible to obtain an exact solution of a problem, one of the methods giving an exact solution will normally be used. It has, however, been pointed out that the forces may be "difficult" and so an exact solution may not be practically possible. One of the numerical methods must then be used, and a comparison is made here of the accuracy and computing times for the different methods.

The computing times given are for an I.C.L. 1905 computer, and although actual times will be different for other computers the relative speeds of the various methods can be expected to remain unchanged.

Although the comparison is concerned with numerical methods, an exact solution is required to enable errors to be calculated, and so suitable forcing functions must be chosen. To obtain solutions without too much difficulty, damping is not included in the examples solved in this section. Harmonic forcing functions may thus give large amplitudes if the forcing frequency is close to one of the natural frequencies of the system, and so are considered to be unsuitable for a system in which the number of degrees of freedom, and hence the natural frequencies, are to be varied. Step forces, however, produce no such undesirable effects, and so appear to be very suitable for use with systems of the type shown in Fig.5.1. The number of degrees of freedom of this system is easily varied by altering the number of masses and springs.

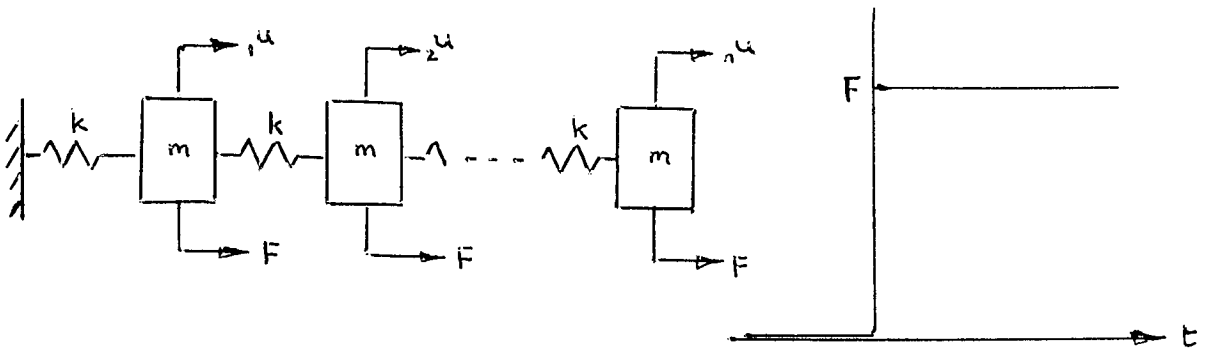


Fig. 5.1

The responses of systems of this type having 3,5,7 and 10 degrees of freedom to step forces are first found by using the exact modal analysis method described in 5.2. In each case the initial conditions are taken as $_iu_o = _i\dot{u}_o = 0$, $i = 1,2...n$. The eigenvalues and eigenvectors required for use in the solution are found by using a standard computer procedure.

For each system, displacements are calculated for a time equal to one period of the lowest natural frequency, and the general pattern is similar for each of the 4 different numbers of degrees of freedom. The variation of the displacement of each mass is shown for the 3-degree of freedom system in Fig. 5.2.

Solutions obtained by each of the 4 numerical methods are then compared with the exact solution previously obtained, the number of intervals in each of the numerical methods being adjusted to obtain the required accuracy. It should be noted that in this comparison maximum percentage true errors are quoted, and in each case this maximum error occurs when displacements are small. A maximum true error of, for example, 0.1% is therefore a very small percentage of the maximum displacement.

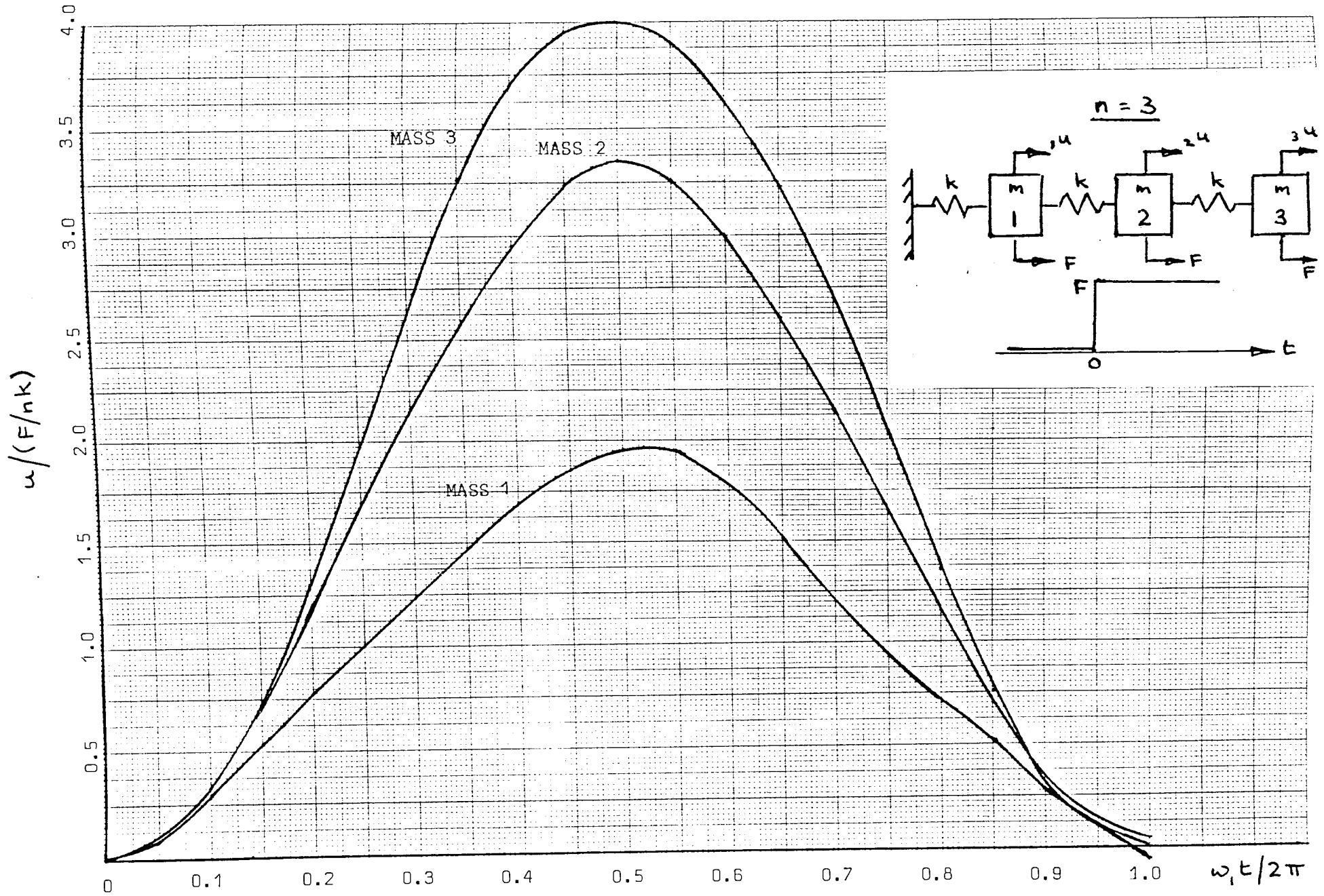


FIG.5.2 DISPLACEMENTS IN A 3 D.O.F SYSTEM DUE TO STEP FORCES

The numerical methods are compared in two ways, first by investigating the variation of computing time with different numbers of degrees of freedom, with a maximum percentage true error of 1% in each case. The effect of varying the accuracy for the 10-degree of freedom system is then investigated.

The results of these two comparisons are shown in Figs.5.3 and 5.4 respectively, and it is clear that, except for a small number of degrees of freedom, modal analysis with numerical integration is the fastest (and therefore the cheapest) method. Within the ranges used the refined finite element method, however, requires computing time comparable to that used in numerical integration, while the time requirements of the basic finite element and finite difference methods increase rapidly as either the number of degrees of freedom or accuracy required are increased. On the other hand, both finite element and the finite difference methods will allow for the inclusion of damping without difficulty, while the modal analysis method becomes considerably more complicated.

It is unlikely that identical results would be obtained for other forcing functions, but solutions of examples using harmonic and exponentially decaying forces (for which exact results are readily obtained) suggest that the relative speeds of the various methods are approximately the same as those of Fig. 5.4.

In all the previous examples in this chapter, forcing functions have been chosen so that conventional methods give exact solutions

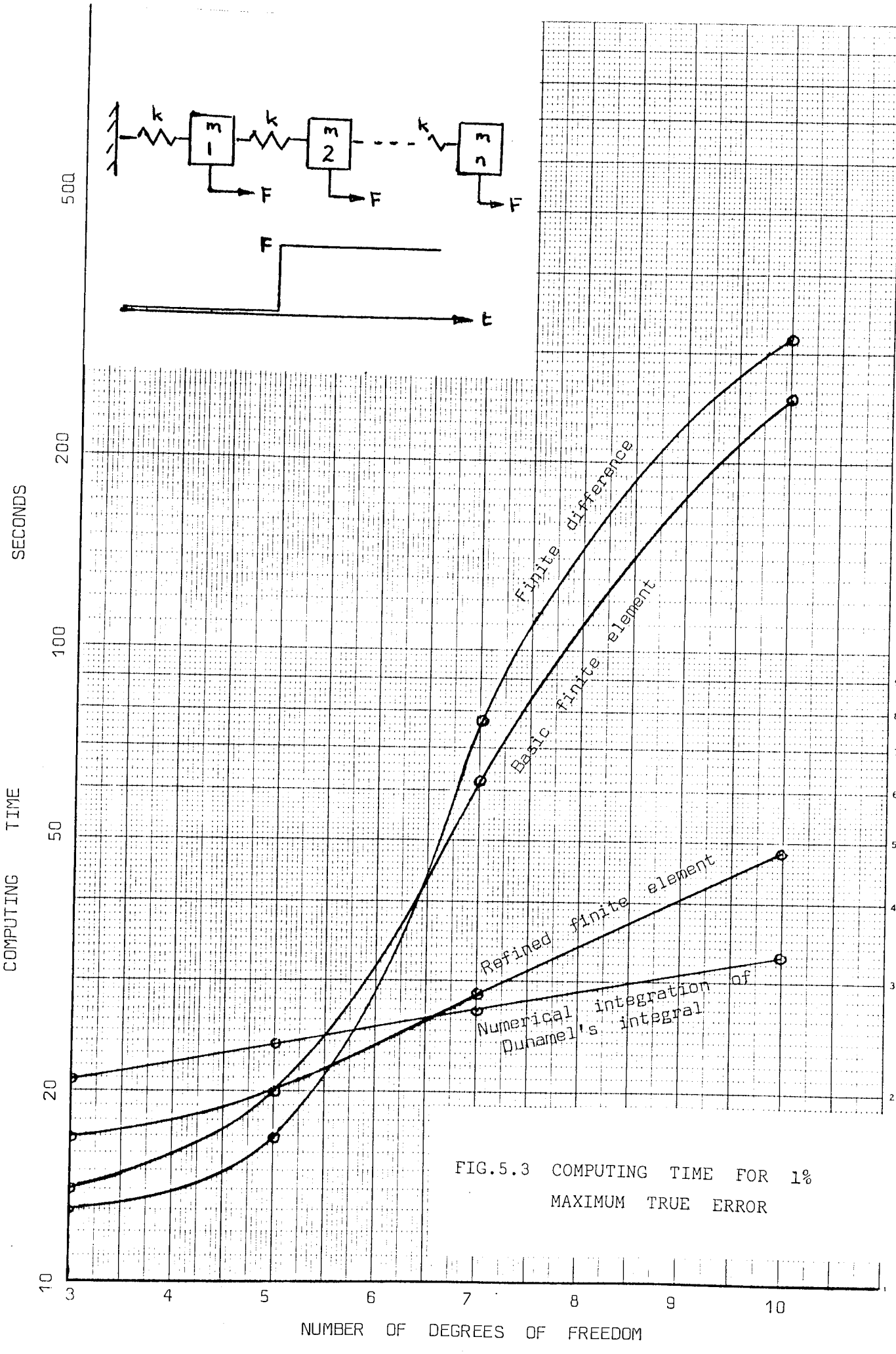


FIG.5.3 COMPUTING TIME FOR 1% MAXIMUM TRUE ERROR

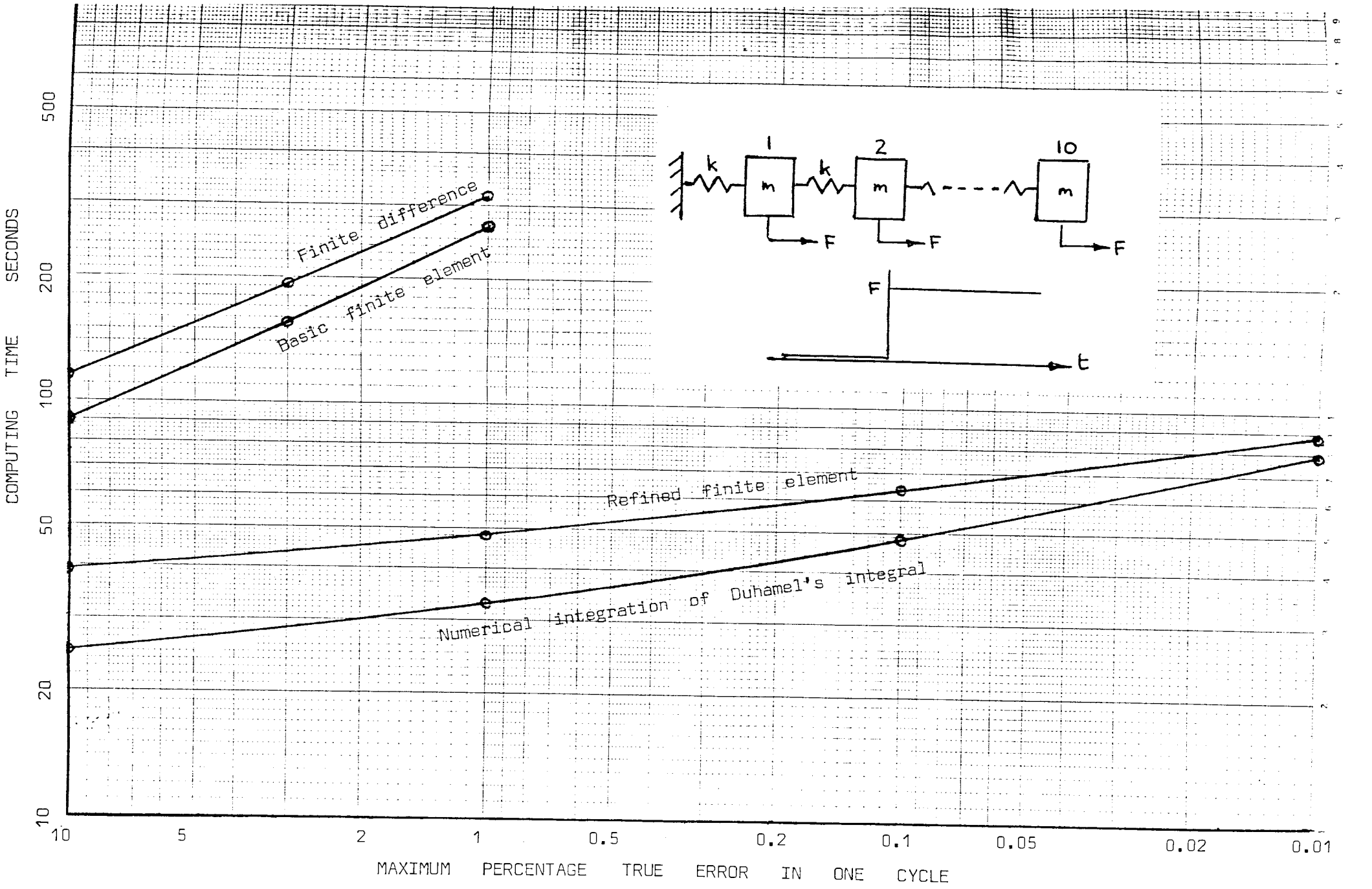


FIG.5.4 COMPUTING TIME FOR A 10 D.O.F SYSTEM

without much difficulty. This is a necessary step to check the accuracy of finite element methods, but it does not show the advantages of these methods when they are used in more difficult situations. For an arbitrary forcing function, conventional methods become very tedious, particularly when damping is present. The next two examples are therefore concerned with an arbitrary force obtained from details given by Housner [18] for an earthquake shock. In Example 5.2, damping is neglected so that a conventional solution is obtained by numerical integration of Duhamel's integral without much difficulty. This solution is compared with a refined finite element solution. In Example 5.3, damping is included, and the finite element solution, which is not made more difficult by this addition, is compared with experimental results given by Housner.

Both of these examples are concerned with a 15-storey building for which the acceleration of the basement is given. An approximate discretization of the building as suggested by Jacobsen and Ayre [19] is given in Fig. 5.5.

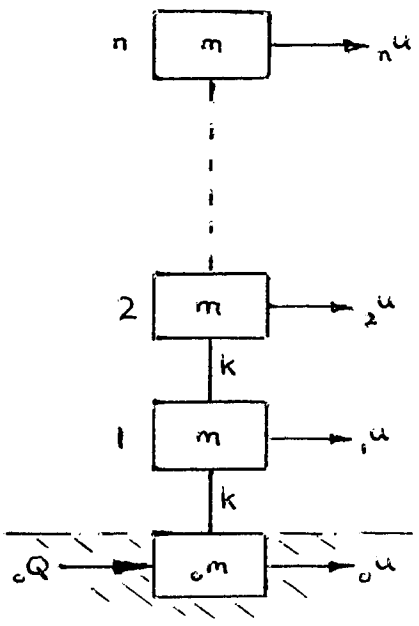


Fig.5.5

The equations of motion for this system when an unknown ground force $_0Q$ is applied are

$$\begin{bmatrix} {}_0m & & & \\ & m & & \\ & & m & \\ & & & \ddots \\ & & & & m \end{bmatrix} \begin{Bmatrix} {}_0\ddot{u} \\ \ddot{u} \\ \\ \\ {}_n\ddot{u} \end{Bmatrix} + \begin{bmatrix} k & -k & & & \\ -k & 2k & & & \\ & & & & \\ & & & & \\ & & & & k \end{bmatrix} \begin{Bmatrix} {}_0u \\ {}_1u \\ \\ \\ {}_n u \end{Bmatrix} = \begin{Bmatrix} {}_0Q \\ 0 \\ \\ \\ 0 \end{Bmatrix}$$

Discarding the first equation gives

$$[m]\{\ddot{u}\} + [k]\{u\} = \begin{Bmatrix} k_0u \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

where [m] and [k] are the n x n mass and stiffness matrices for the 1st to n'th storeys.

The system therefore behaves as if a force k_0u is applied to the first storey. In the following examples, values of acceleration $_0\ddot{u}$ only are available and so the required values of $_0u$ are found by a double numerical integration.

Example 5.2

The only information given by Housner [18] for the building under consideration is that it has 15 storeys and that the periods of the first 3 modes are 1.2s, 0.4s and 0.25s. Mass and stiffness matrices for a system of the type shown in Fig. 5.5 are identical in form to those previously used in this section, although the stiffnesses in the present example are shear stiffnesses. By solving the eigenvalue problem the

periods and modal shapes for the building may be found, and by adjusting the m/k ratio the period of the first mode is made to be 1.2s when m = k/2671. The next two periods then have calculated values of 0.401s and 0.243s which agree well with the measured values given. It may therefore be deduced that, at least for the first 3 modes, the arrangement of Fig. 5.5 is a reasonable representation of the building.

Taking values given for the acceleration of the basement for the first 2.5s of the shock, and assuming that the building is initially at rest, the velocity and displacement of the basement may be found by numerical integrations. The values of these displacements so obtained from the measured accelerations are shown in Fig. 5.6.

By using these displacements as values of $_0u$, the displacements of other parts of the building may then be found by using the method previously described. In this case a solution may be obtained without difficulty by using modal analysis with numerical integration of Duhamel's integral. The values obtained by this method for the displacement of the roof at 0.025s intervals are also shown in Fig.5.6.

A second solution of this problem using refined finite elements with the same 0.025s interval gives roof displacements which differ from those obtained by modal analysis and numerical integration by less than 1%. Because of this small difference between the two numerical solutions it is not possible to show separate results for the two methods in Fig. 5.6.

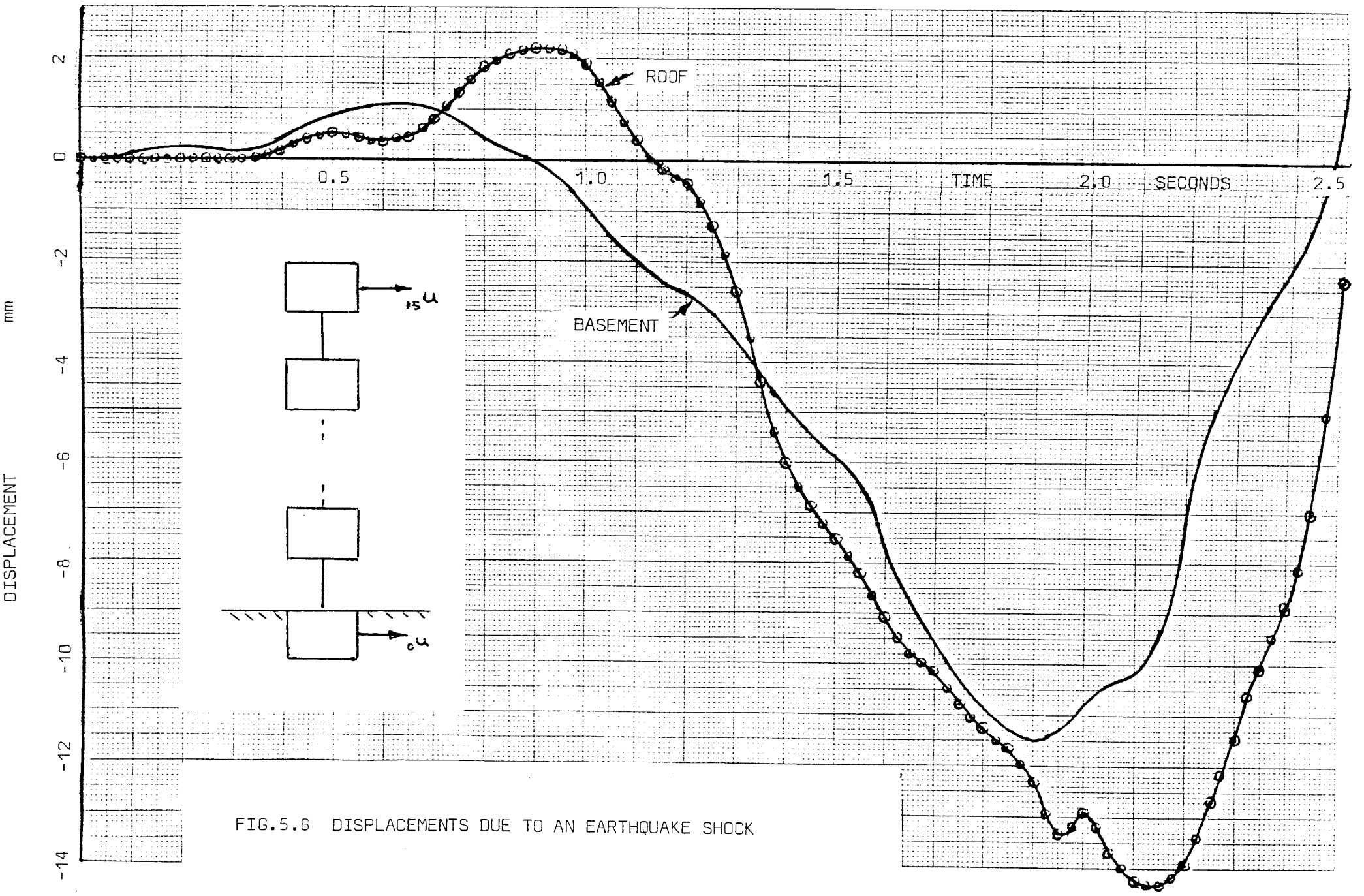


FIG.5.6 DISPLACEMENTS DUE TO AN EARTHQUAKE SHOCK

In this example no exact solution is available for comparison, but since two different methods of solution are in close agreement it seems likely that neither is in error by more than about 1%.

The computing time for these two methods of solution are:-

Modal analysis with numerical integration	36s
Refined finite element	52s

It should be noted that these times are for an I.C.L.1904S computer, and are not directly comparable with those of Fig. 5.4 which apply to a 1905 computer.

The refined finite element method is therefore a useful alternative to conventional modal analysis, its only disadvantage being the rather longer computing time. In the next example, however, the same building is used with non-zero initial conditions, and with damping added. Conventional methods then become considerably more complicated, but no increase in computing time is required to obtain a finite element solution.

Example 5.3

In addition to the acceleration of the basement of the 15-storey building used in the previous example, Housner also gives values for the acceleration of the roof. Examination of the latter in Fig. 5.7 shows that the building is not initially at rest, and values of initial conditions used are adjusted to match, as closely as possible, the given roof acceleration for the first 0.2s of the shock. The amount of

damping present is not known but Housner suggests a damping coefficient of about 5% of the critical value for buildings in general, and a damping coefficient $c = \sqrt{mk}$, giving a damping ratio of almost exactly 0.05 for the first mode is used in this example. Since, in this case, accelerations are required, the refined finite element method, which uses acceleration as a generalized displacement, seems particularly appropriate. The basement accelerations given in Fig. 5.7 have been used previously in Example 5.2 to obtain corresponding displacements, and these same displacements are used in this example. With the addition of damping, the equivalent force on the first storey is now $k_0 u + c_0 \dot{u}$, and values of ${}_0\dot{u}$ have already been calculated as a first step to finding values of ${}_0u$. Values of $k_0 u + c_0 \dot{u}$ are therefore calculated at 0.025s intervals and the roof acceleration calculated by the refined finite element method is shown in Fig. 5.7. Also shown are the measured values of roof acceleration.

Agreement between calculated and measured values of roof acceleration is not by any means perfect, but in view of the uncertain initial conditions, the amount of damping present and even the precise distribution of mass and stiffness, the differences between the two sets of results are not considered to be excessive.

It is encouraging to note that the maximum acceleration at 2.35s is predicted almost exactly in magnitude and position, and with more information concerning the system, the refined finite element method should be capable of predicting accurately the response of any building to an earthquake shock.

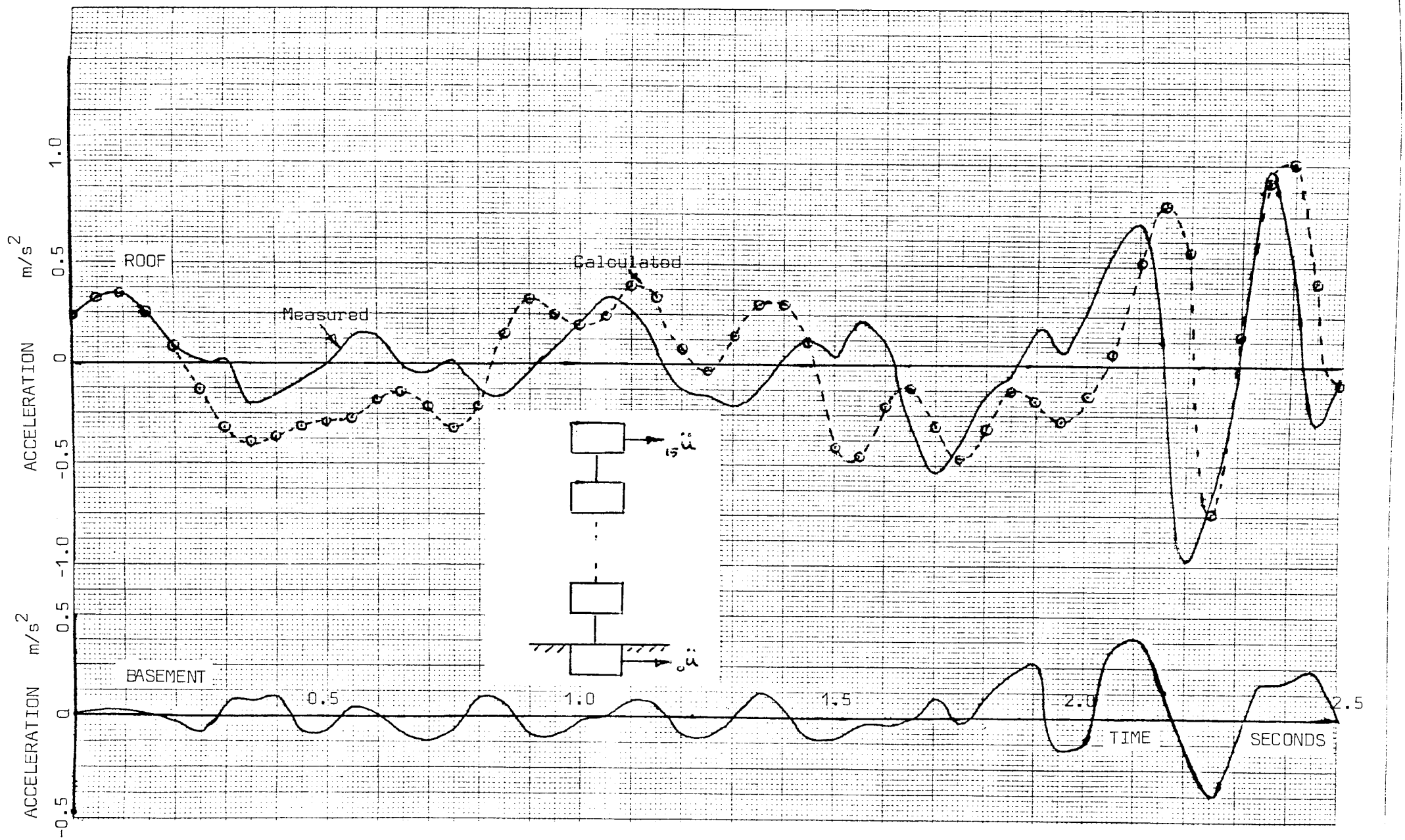


FIG.5.7 ACCELERATION DUE TO AN EARTHQUAKE SHOCK

5.5 Conclusion

The methods giving exact solutions may be very tedious except for simple systems and a limited number of forcing functions. This is particularly true of the direct solution method where the inverse Laplace transformation may present considerable difficulties. If modal analysis is used, an increase in the number of degrees of freedom does not really make the solution any more difficult, provided that the large number of calculations can be performed by a computer. Both of these methods become much more complicated if damping is present.

Of the numerical methods, the use of modal analysis with numerical integration of Duhamel's integral is, in most cases, the fastest method used, but damping makes the solution more difficult. The finite difference method is relatively simple to use, but does require special starting techniques. It is, however, faster than even numerical integration for systems with only a few degrees of freedom.

Both of the finite element methods, like the finite difference method, will give solutions for any forcing function and allow for the inclusion of damping without difficulty. The finite element methods are judged to be superior to the finite difference method because of the ease with which any starting conditions may be prescribed, and because of their improved accuracy. Except for systems with only a few degrees of freedom, both finite element methods are faster than the finite difference method for the same accuracy of solution, and the difference in computing time is particularly noticable when the refined finite element is used to give very accurate solutions. This latter is also

the obvious method to choose in problems where accelerations are required, since in this case accelerations are automatically calculated at the end of each interval.

CHAPTER 6

FREE VIBRATIONS OF DISTRIBUTED SYSTEMS

6.1 Introduction

The state of a discrete system may be described completely by a finite number of coordinates. To describe the state of a distributed system exactly in this way, an infinite number of coordinates is required, so other descriptions must be used. It may, for example, be possible to describe the shape of a vibrating beam completely and exactly by an algebraic expression of the form $u = a \sin(\pi x/\ell) \sin \omega t$. In more complicated cases the expression may consist of a number of such terms. In the general case the expression becomes an infinite series, which may of course, be used to give approximate values by considering only a finite number of terms. This is the method used by Timoshenko [20] and others.

A second alternative is to recognize that any solution for a distributed system is likely to be approximate, and to describe the state of the system as accurately as possible by using a finite number of spatial generalized displacements. The distributed system is therefore discretized in some way so that mass, stiffness and damping matrices may be obtained. The discretized distributed system is then treated in exactly the same way as a discrete system, and if the displacements vary with time any of the methods described in Chapter 5 may be used to obtain a solution in the time domain.

In this chapter, and indeed in all subsequent calculations, finite element methods are used in the time domain.

At this stage only free vibrations of distributed systems are investigated, and the various methods of solution are compared. In later chapters more difficult problems involving forcing and damping are solved by using the methods developed here for free vibrations. The main point of interest is the use of finite elements in the time domain, the method of use of these being basically the same for any spatial system. For ease of solution and comparison of methods, therefore, all the solutions obtained in this chapter are for a simply supported, uniform beam vibrating in its first mode. The exact solution for this case gives a frequency $\omega_1 = \pi^2\alpha$ where $\alpha = \sqrt{EI/\rho A}/L^2$, and a modal shape of half a sine wave. Initial values of deflection and slope required to ensure that the beam vibrates in one mode only are therefore easily found.

Two possible arrangements may be used for this system; the first, shown in Fig. 6.1 is the obvious formulation using the whole of the beam.

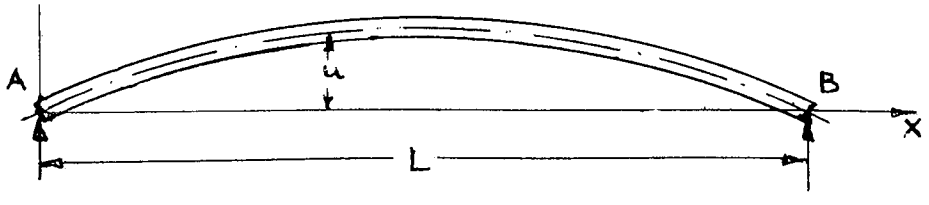


Fig. 6.1

For this arrangement the boundary conditions are $u_a = u_b = 0$ for all values of t. The lumped mass discretization of 6.2 uses matrices for the whole of the beam, but the later finite element discretizations in space produce larger matrices, and for these the half-beam of Fig. 6.2 is used. The size of the mass and stiffness matrices is thus reduced without affecting the accuracy of the results.

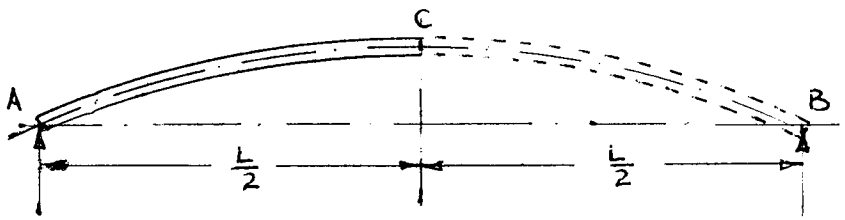


Fig. 6.2

In this case BC is a mirror image of AC; by symmetry, the slope of the beam at C is therefore always zero, and so by considering only a beam of length L/2 with boundary conditions $u_a = 0$, $u'_c = 0$, exactly the same results will be obtained as for the whole beam of length L with $u_a = u_b = 0$. Mass and stiffness matrices for this type of arrangement are given in 6.3.

For each of these arrangements the type of solution is the same. Initial conditions are chosen so that the beam has maximum displacement; the period of one cycle $2\pi/\omega_1$ is then divided into a number of intervals and displacements and velocities are then calculated at the end of each interval. The maximum range error within one cycle may then be calculated, using the definition of (3.2.1).

One difficulty encountered with distributed systems which did not previously occur with discrete systems is the tendency of the finite element solution to become unstable. Stable solutions may be obtained by using a sufficient number of temporal elements, but alternatively the modal method of solution may be used with suitable truncation to give a stable solution with fewer elements.

6.2 Lumped mass representation

The simplest method of discretizing a distributed system is that of replacing the distributed mass and stiffness by a number of discrete masses and stiffnesses. If, for example, a simply-supported beam of mass m is divided into four equal parts, the lumped mass discretization consists of the masses shown in Fig. 6.3.

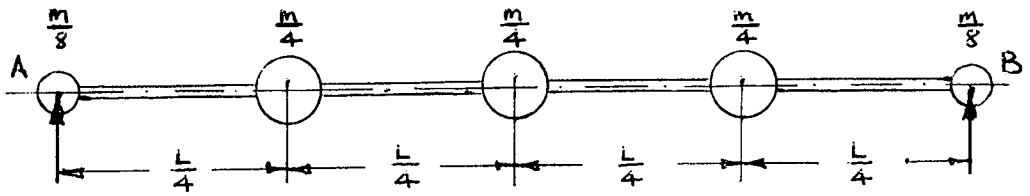


Fig. 6.3

Since the deflections at A and B are always zero the masses at these points do not appear in the mass matrix, which is therefore

$$[m] = \frac{m}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (6.2.1)$$

The stiffness matrix is more difficult to obtain and is best formed indirectly from the flexibility matrix. The coefficients of this flexibility matrix are often called influence coefficients and are readily available for a variety of end conditions. A coefficient a_{ij} represents the deflection at the i'th point due to a unit force at the j'th point. The matrix is symmetrical and its coefficients are easily evaluated by using a computer. This flexibility matrix is then inverted to obtain the stiffness matrix.

For the beam of Fig. 6.3 the flexibility matrix is

$$\frac{L^3}{768EI}\begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix} \tag{6.2.2}$$

and hence the stiffness matrix

$$[k] = \frac{192EI}{7L^3}\begin{bmatrix} 23 & -22 & 9 \\ -22 & 32 & -22 \\ 9 & -22 & 23 \end{bmatrix} \tag{6.2.3}$$

The mass and stiffness matrices (6.2.1) and (6.2.3) are then used in exactly the same way as those for the discrete systems described in Chapter 4.

As previously stated, all examples in this chapter are concerned with the first mode vibrations of a simply-supported beam. By using the appropriate initial displacements of the three masses, which are easily found from the known modal shape, and zero velocity in each case, a

solution in the time domain may be obtained by using the basic finite element. In the case where 20 such elements per cycle are used the maximum range error is found to be 0.32%. As in previous examples for discrete systems the maximum range error is in this case, and indeed in all examples in the present chapter, found to be a velocity error at the end of the cycle.

Rather surprisingly this error is less than the maximum range error of 0.51% for a single degree of freedom system using the basic temporal element with 20 intervals per cycle. The explanation for this is that temporal finite element solutions give too high a frequency (see Chapter 3), while the lumped mass spatial discretization reduces the frequency. These two discretizations therefore give errors of opposite sign resulting in a reduction of the total error.

By adjusting the numbers of masses and time intervals even smaller errors may be obtained, and with the mass discretization m/6, m/3, m/3, m/6 at L/3 spacing and 17 basic temporal elements per cycle, the values of the displacement and velocity at each of the m/3 masses are, after one cycle, $1.000000A_0$ and $-0.002641\alpha A_0$, A_0 being the initial amplitude and $\alpha = \sqrt{EI/\rho A}/L^2$, i.e. a maximum range error of only 0.03%.

This example shows that the simple lumped mass representation of a distributed system may yield surprisingly accurate results. In fact, if the beam vibrates at a single frequency, by balancing the numbers of masses and time elements, extremely accurate results may be obtained. In general, however, the motion of a beam will be more complex, and the

accuracy may be rather low unless the beam is discretized into a considerable number of masses.

6.3 Separate finite element discretizations

Although the mass and stiffness of a distributed system may be represented approximately by lumped mass and stiffness discretizations, more accurate results are obtained by using spatial finite element discretizations as described, for example, by McCallion [21]. Since slopes as well as deflections are now the unknown nodal variables, a more accurate approximation for the true shape of the beam is obtained. It is possible to avoid the introduction of powers of ℓ, the length of the element, into the mass and stiffness matrices if, instead of the actual slope u', the product u'ℓ is used as a generalized displacement. The matrices for the single element of Fig. 6.4 are given below, the order of the displacements being ${}_1u$, ${}_1u'\ell$, ${}_2u$, ${}_2u'\ell$.

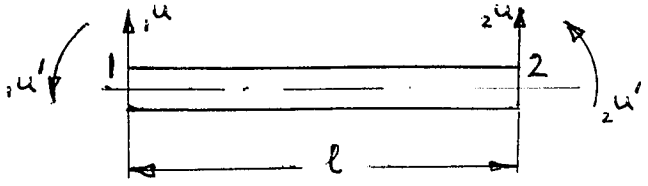


Fig. 6.4

$$[m] = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ \text{Symm} & & & 4 \end{bmatrix} \qquad (6.3.1)$$

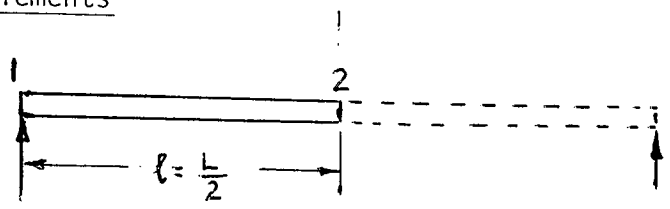
$$[k] = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ \text{Symm} & & & 4 \end{bmatrix} \qquad (6.3.2)$$

Mass and stiffness matrices may therefore be assembled for any beam discretized into any number of spatial finite elements, and any of the previously described methods may then be used to obtain a solution in the time domain. If, in particular, temporal finite elements are used, the spatial and temporal discretizations are quite separate since either of these types of discretization may be replaced by something different without affecting the feasibility of the method.

In this section both spatial and temporal finite elements are used. Since the temporal finite element uses time derivatives of the displacements, the "displacements" when both discretizations are used are u, $u'\ell$, $\dot{u}\tau$, $\dot{u}'\ell\tau$, i.e. the deflection, slope x ℓ, linear velocity x τ, and angular velocity x $\ell\tau$.

Since the same spatial discretizations for a simply-supported beam are used repeatedly in this section, the arrangements used are shown in Fig. 6.5. As noted in 6.1, matrices for one half of the beam only are necessary.

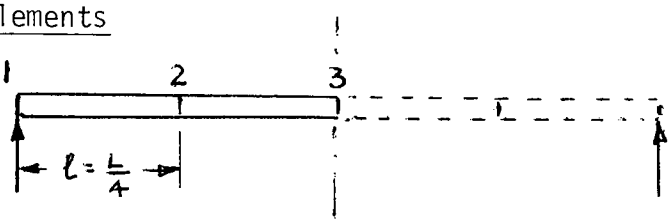
(a) 2 elements



$$_1u = {}_2u' = 0$$

$$\{u\} = \begin{Bmatrix} {}_1u'\ell \\ {}_2u \end{Bmatrix} \quad [m] = \frac{\rho AL}{840} \begin{bmatrix} 4 & 13 \\ 13 & 156 \end{bmatrix} \quad [k] = \frac{8EI}{L^3} \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

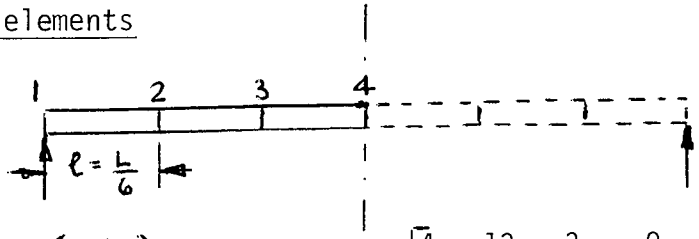
(b) 4 elements



$$_1u = {}_3u' = 0$$

$$\{u\} = \begin{Bmatrix} {}_1u'\ell \\ {}_2u \\ {}_2u'\ell \\ {}_3u \end{Bmatrix}, \quad [m] = \frac{\rho AL}{1680} \begin{bmatrix} 4 & 13 & -3 & 0 \\ & 312 & 0 & 54 \\ & & 8 & 13 \\ \text{Symm} & & & 156 \end{bmatrix}, \quad [k] = \frac{64EI}{L^3} \begin{bmatrix} 4 & -6 & 2 & 0 \\ & 24 & 0 & -12 \\ & & 8 & -6 \\ \text{Symm} & & & 12 \end{bmatrix}$$

(c) 6 elements



$$_1u = {}_4u' = 0$$

$$\{u\} = \begin{Bmatrix} {}_1u'\ell \\ {}_2u \\ {}_2u'\ell \\ {}_3u \\ {}_3u'\ell \\ {}_4u \end{Bmatrix} \quad [m] = \frac{\rho AL}{2520} \begin{bmatrix} 4 & 13 & -3 & 0 & 0 & 0 \\ & 312 & 0 & 54 & -13 & 0 \\ & & 8 & 13 & -3 & 0 \\ & & & 312 & 0 & 54 \\ & & & & 8 & 13 \\ \text{Symm} & & & & & 156 \end{bmatrix}$$

Fig. 6.5 (see also p.99)

$$[k] = \frac{216EI}{L^3}\begin{bmatrix} 4 & -6 & 2 & 0 & 0 & 0 \\ & 24 & 0 & -12 & 6 & 0 \\ & & 8 & -6 & 2 & 0 \\ & & & 24 & 0 & -12 \\ & & & & 8 & -6 \\ \text{Symm} & & & & & 12 \end{bmatrix}$$

Fig. 6.5 (see also p.98)

It should be noted that by imposing the boundary condition that the central slope is zero all the even modes are suppressed, so that the only frequencies which appear with these discretizations are ω_1, ω_3 etc.

By using these spatial discretizations, the equations of motion for a free vibration of the discretized system are $[m]\{\ddot{u}\} + [k]\{u\} = \{0\}$ When the eigenvalue problem is solved for the mass and stiffness matrices of Fig. 6.5, the natural frequencies and modal shapes are not quite the same as those found from an exact solution. These differences are due to the approximate nature of the spatial discretizations.

Even when the actual initial conditions of a distributed system are those required for one particular mode of vibration, as found from an exact solution, subsequent displacements will contain contributions from other modes when a spatial discretization is used. For example, a simply-supported beam vibrating in its first mode is known to have an exact shape of half a sine wave. If, however, the initial deflections

and slopes obtained from this shape are used in a system having the spatial discretization shown in Fig. 6.5(a), a small, but significant, third mode contribution also appears. These spurious additions inevitably lead to errors in the time domain which may be examined by using the approximate spatial discretization of Fig. 6.5 and obtaining an exact solution in time for the resulting equations of motion.

Using the transform $\{u\} = [v_N]\{p\}$ where $[v_N]$ is the normalized modal matrix, the equations of motion may be uncoupled to give

$$\{\ddot{p}\} + [\omega^2]\{p\} = \{0\}$$

The solution of each of these equations is of the form $p = A \cos \omega t + B \sin \omega t$, where the constants A and B are found from initial conditions $\{u_0\}$ and $\{\dot{u}_0\}$ using $\{p_0\} = [v_N]^{-1} u_0$ and $\{\dot{p}_0\} = [v_N]^{-1}\{\dot{u}_0\}$.

Finally, displacements in the original coordinate system are found from $\{u\} = [v_N]\{p\}$.

When a simply-supported beam is initially at rest in the shape of half a sine wave it is known to vibrate in its first mode only, but by using the spatial discretizations of Fig. 6.5 and obtaining an exact solution in time by the method described above, differences appear between calculated and correct values. When range errors are calculated over one complete cycle the largest value of this error is, in all cases investigated, found to occur at the end of the cycle in the angular velocity at the support. This is hereafter described as the "ultimate error" defined as the greatest value of the maximum range error at the

end of one complete cycle. Values of these ultimate errors (which are errors in the time domain) due to the use of approximate spatial discretizations are shown in Table 6.1.

When solutions in the time domain are obtained by using temporal finite element methods instead of the exact solution described above, an additional source of error is introduced. Since separate spatial and temporal discretizations are now used, it is not surprising to find that range errors in the time domain have higher maximum values than when an exact time solution is obtained. Ultimate errors at the end of one cycle may be obtained as before, and these are shown in Table 6.1 together with the maximum range errors due to temporal discretizations only, previously calculated and shown in Fig. 3.2.

Finite element discretizations in space, unlike lumped mass discretizations give frequencies which are too high, as do temporal finite element discretizations. It may indeed be seen that in most cases the ultimate error in the last column of Table 6.1 is very nearly equal to the sum of the ultimate error due to spatial discretization alone and the maximum range error due to temporal discretization alone.

Table 6.1 also shows that accuracy is considerably increased by using more spatial elements, but only slightly increased by an increase in the number of temporal elements. Unfortunately it is found that by using a constant number of, for example, 20 temporal elements per cycle, an increase in the number of spatial elements may give an unstable solution. This problem was not encountered previously with

Number of spatial elements for whole beam	Ultimate error due to spatial discretization %	Type of temporal element	Number of intervals per cycle	Maximum range error due to temporal discretization %	Ultimare error due to spatial and temporal discretizations %
2	2.68	Basic	20	0.50	3.26
		Refined	20	0.0002	2.55
		Basic	80	0.03	2.74
		Refined	80	Negligible	2.68
4	0.16	Basic	80	0.03	0.20
		Refined	80	Negligible	0.16
6	0.03	Basic	180	0.006	0.04
		Refined	180	Negligible	0.03

Table 6.1

either single- or multi-degree of freedom discrete systems. The next section shows that instability of solutions is not confined to distributed systems, although it is here that it is most likely to occur.

6.4 Stability of temporal finite element solutions

In the previous section it was noted that when spatial and temporal finite element discretizations are used together, the solution may not be stable. This instability is shown as a continual increase in both displacements and velocities to enormous values instead of the alternating increase and decrease required for a vibrating system. Since this instability did not appear in any of the previous solutions for discrete systems it might appear that the cause of the instability is the separate use of two finite element discretizations. It is shown below that this is not the case, and that unstable solutions can also occur for discrete systems; the reason for their not having done so becomes clear when the condition for stability is examined.

As a first step to finding the condition for a stable solution, values were taken from the previous section showing the minimum number of basic elements per cycle required to give a stable solution for various spatial discretizations. These figures were obtained by trial and error, the number of intervals being reduced until the solution became unstable. The results are shown in Table 6.2.

Number of spatial elements for whole beam	Minimum number of intervals per cycle
2	11
4	65
6	162

Table 6.2

The figures given in Table 6.2 apply only when the basic temporal element is used, but similar results were obtained for the refined element, the minimum number of intervals per cycle required for a stable solution being about 20% less than for the basic element. Examination of these results shows no obvious connection between the number of spatial elements and the number of intervals per cycle. It is therefore necessary to examine more closely the conditions under which a solution becomes unstable.

Since the use of finite elements in the time domain inevitably results in an approximate solution, even if the correct values of displacement and velocity are used at the beginning of an interval, the final values must be incorrect. When the usual step-by-step method of solution is employed these final conditions become the initial (incorrect) conditions for the next interval. There are further approximations in this next interval, and, under certain conditions, the initial errors are magnified and so the solution becomes unstable. It should be noted that, even when the solution is stable, errors may increase during part of the cycle, but errors of opposite sign then appear to produce a partial cancellation. In the case of an unstable solution the initial error seems to have a critical magnitude which later errors of opposite sign cannot correct.

The application of the basic finite element to a single degree of freedom system has been examined in detail in Chapter 3 where expressions for final displacements are given in terms of initial conditions in (3.2.7).

By using the method described in 6.3 for obtaining an exact solution in the time domain for a free vibration, the coupled equations of a multi-degree of freedom system may be reduced to a number of uncoupled equations. There will in fact be two equations for each mode of vibration which, for a free vibration, are of exactly the same form as those of (3.2.7) for a single degree of freedom system. The only difference between these equations and the present equation is that co-ordinates u and v ($=\dot{u}\tau$) are now replaced by principal coordinates p and q($=\dot{p}\tau$). If the number of time intervals in a cycle is reduced until an unstable solution is just obtained, it is found that instability first occurs as the beam passes through its mid-position, and this instability first shows as an impossibly high value of velocity, i.e. $\dot{u} > \omega A_0$ where A_0 is the amplitude. The second equation of (3.2.7) has the modified form for the r'th mode of

$$q_1 = \left(\frac{c^2-a^2}{ad+bc}\right) p_o - \left(\frac{ab+cd}{ad+bc}\right) q_o \qquad (6.4.1)$$

where a, b, c and d are as given in (3.2.5), m being replaced by $M_{rr} = 1$ and k by $K_{rr} = \omega_r^2$.

Now in mid-position, $p_o = 0$ and q_o has its maximum possible value. Hence, if there is to be no instability $q_1 \leq q_o$

$$\text{or} \quad -\frac{ab+cd}{ad+bc} \leq 1 \qquad (6.4.2)$$

But a, b, c and d are functions of $\omega\tau$, and if the r'th mode is not to exhibit instability the angular frequency to be used is that of the r'th mode ω_r. Then by substituting the appropriate functions of $\omega_r\tau$

into (6.4.2) the very simple condition obtained is

$$(\omega_r \tau)^2 \leq 42 \tag{6.4.3}$$

Since ω_r may be any one of the natural frequencies of the system it is clear that (6.4.3) is least likely to be satisfied when ω_r is the highest natural frequency $\hat{\omega}$. Thus, for a stable solution

$$\tau \leq \sqrt{42}/\hat{\omega} \tag{6.4.4}$$

If the system vibrates harmonically with an angular frequency ω_s, $\tau = 2\pi/n\omega_s$ where n is the number of intervals per cycle of the vibration.

Then, from (6.4.4), $n \geq 2\pi\hat{\omega}/\sqrt{42}\omega_s$

or $n \geq 0.9695\hat{\omega}/\omega_s$. (6.4.5)

The natural frequencies of the finite element discretizations of Fig. 6.5 are easily calculated by using a standard computer procedure, and if, for each representation, the highest frequency ($\hat{\omega}$) is substituted in (6.4.5), together with $\omega_s = \pi^2\alpha$, where $\alpha = \sqrt{EI/\rho A}/L^2$, for the first mode, minimum values of n may be calculated in each case. These values are 10.8, 64.8 and 161.6 respectively for the 2, 4 and 6 element discretizations, and these agree exactly (to the nearest integer) with the values previously found by trial and error given in Table 6.2.

It should be noted that this stability problem is not peculiar to temporal finite elements. Instability may also occur in a finite difference solution and, according to Leech et al.[22], $\tau \leq 2/\omega_n$ for a

stable solution, but τ should be no greater than one-sixth of this critical value if high frequency detail is required.

It may now be seen why no instability problems have been previously encountered. For a single degree of freedom system having only one natural frequency, $\hat{\omega} = \omega_s$, and from equation (6.4.5) slightly less than one interval per cycle will give a stable solution. Since many more intervals than this were used in Chapter 2 to obtain reasonable accuracy, all solutions were stable.

Although the multi-degree of freedom systems of Chapter 4 have more than one frequency, they are of the same order of magnitude, even for the 10 degree of freedom system the ratio ω_{10}/ω_1 being only 13.2. For the simply supported beam of the present chapter however, the frequencies are much more widely spaced; for the 6 element discretization, for example the ratio $\omega_{11}/\omega_1 = 166.7$. Thus, for the earlier discrete systems, the number of intervals required to give reasonable accuracy also ensured a stable solution, but because of the wider range of frequencies in a distributed system many more intervals may be required to give a stable solution than are needed for a given accuracy.

Thus for distributed systems, it appears that a choice must be made between two undesirable alternatives. On the one hand, a small number of spatial elements, having only a few degrees of freedom, and requiring a reasonable number of time intervals may be used. This will give a rather inaccurate solution, so alternatively, a more elaborate space discretization, with many degrees of freedom may be used. This gives

a more accurate solution but requires a large number of time intervals for stability. The large number of intervals will increase the accuracy only slightly, but will considerably increase computing time.

A way out of this dilema may be found however, by using what may be called the truncated modal method. To obtain a high degree of accuracy it is necessary to use an appropriate number of spatial elements giving, in general, a considerable number of natural frequencies. Then by using the method described in 6.3, it is possible to uncouple the equations so that there are two equations for each mode of vibration. Since any instability in the solution is caused by the highest frequencies, the solution may be truncated by discarding the equations for these highest frequencies. It is unfortunately now necessary to solve the eigenvalue problem to obtain the eigenvalues and normalized eigenvectors of the system, but this presents no difficulty if a standard computer procedure is used. The modal mass and stiffness matrices are then

$$[M] = \begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} \omega_1^2 & 0 & & & \\ 0 & \omega_2^2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \omega_n^2 \end{bmatrix}$$

If the system has initial displacements $\{u_0\}$, then the initial values of the principal coordinates are given by:-

$$\{p_0\} = [u_N]^{-1}\{u_0\} \tag{6.4.7}$$

where $[u_N]$ is the matrix of normalized eigenvectors.

The determination of $[u_N]^{-1}$ is possible without matrix inversion working from the definition of a normalized eigenvector

$$[u_N]^t[m][u_N] = [I] \tag{6.4.8}$$

from which it may be shown that

$$[u_N]^{-1} = ([m][u_N])^t \tag{6.4.9}$$

For an n-degree of freedom system, the finite element method applied to the time domain will then give n pairs of uncoupled equations. It is an easy matter to discard those equations involving the higher, unwanted frequencies. Because part of the solution is lost there may be some small reduction in accuracy, but in most cases the higher frequencies account for only a small part of the total displacement.

Further values of the principal coordinates which are retained are then found in the usual way and finally the values of displacements and velocities in the original coordinate system are found from

$$\{u\} = [u_N]\{p\} \tag{6.4.10}$$

Unfortunately it is necessary to solve the eigenvalue problem as a preliminary step in using this truncated modal method, which rather defeats the object of using temporal finite element methods. It should therefore be made clear that a finite element solution <u>can</u> be obtained by the methods previously used, and without using modal methods. This may however require the use of a considerable number of elements to achieve stability and the truncated modal method <u>may</u> therefore be used to obtain a solution using fewer elements.

For the 2-element discretization of Fig. 6.5 the solution of the eigenvalue problem gives $\omega_1 = 9.908559\alpha$ and $\omega_3 = 110.139655\alpha$ where $\alpha = \sqrt{EI/\rho A}/L^2$. The matrix of normalized eigenvectors and its inverse are

$$[u_N] = \frac{1}{\sqrt{\rho AL}} \begin{bmatrix} 3.165856 & 16.672653 \\ 2.015921 & -1.822273 \end{bmatrix}, \quad [u_N]^{-1} = \sqrt{\rho AL} \begin{bmatrix} 0.046274 & 0.423381 \\ 0.051192 & -0.080393 \end{bmatrix} \tag{6.4.11}$$

and similar matrices may be obtained for the 4- and 6-element spatial discretizations.

Ultimate errors resulting from the use of the truncated modal method, together with those previously obtained are shown in Table 6.3, in which the second column shows the modes remaining after truncation.

Number of spatial elements	Modes used in solution	Number of intervals/cycle	Ultimate error %
2	1, 3	20	3.26
4	1, 3	20	0.48
2	1, 3	80	2.74
4	1, 3	80	0.20
4	1, 3, 5, 7	80	0.20

Table 6.3

By comparing the first and second results it is clear that instead of using the 2-element discretization without modal truncation, the 4-element discretization with truncation after the first two frequencies should be used to improve the accuracy of the solution. It should be noted that the second solution would be unstable without the use of truncation. A similar increase in accuracy is shown in the next pair of results, where 80 intervals per cycle are used instead of the previous 20. Finally the last two results show the same error (to the nearest 0.01%), thus demonstrating that the contribution of the two highest frequencies is negligible in this case.

It should be remembered that the errors in Table 6.3 apply only to a beam vibrating freely in its first mode. For the higher modes of vibration and for forced vibrations where all modes may be excited the truncated modal method may give rather different results. This same method is, however, used for forced vibration problems in Chapter 7, and is again found to be very satisfactory.

Since, when the truncated modal method is used, only the lowest frequencies and modal shapes are required, it is not necessary to find all the natural frequencies. It may, in some cases, be much easier to evaluate $[m]^{-1}$ instead of $[k]^{-1}$, but this, unfortunately, normally gives the highest frequencies first. A method suggested by Downs [23] does, however, give the lowest frequencies without evaluating $[k]^{-1}$.

6.5 Improvements in spatial discretization

It has been demonstrated in the previous section that, when separate spatial and temporal finite element discretizations are used, the major source of inaccuracy is likely to be due to the spatial discretization. It has also been shown that accuracy may be improved by increasing the number of spatial elements, but only at the expense of increasing the possibility of obtaining an unstable solution. Other methods of obtaining a more accurate approximation for the true shape of the distributed system are therefore worth considering.

Since modal shapes have been introduced into the solution, in cases where the exact values of frequency and the eigenvectors are known the solution can be made to start with these values instead of the approximate values derived from the approximate mass and stiffness matrices. The frequencies and modal shapes for a simply-supported beam are well known, so that for the 2-element discretization of Fig. 6.5 the exact values of frequency may be used. These are $\omega_1 = \pi^2\alpha$, $\omega_3 = 9\pi^2\alpha$ where $\alpha = \sqrt{EI/\rho A}/L^2$. Similarly, for the first and third modes of a simply-supported beam the exact modal shapes are given by

$$[u_N] = \frac{1}{\sqrt{\rho AL}} \begin{bmatrix} \pi & 3\pi \\ 2 & -2 \end{bmatrix} \text{ and hence } [u_N]^{-1} = \sqrt{\rho AL} \begin{bmatrix} 1/4\pi & 3/8 \\ 1/4\pi & -1/8 \end{bmatrix}$$

If these values are used instead of the approximate values derived from the previous finite element matrices, a more accurate solution may be obtained for a beam vibrating in its first mode. The initial conditions for this mode are easily found from the model shape,i.e. the end slope is π/L of the central deflection for a 2-element spatial discretization.

In this case, exactly the same results are obtained if a solution is obtained without truncation, or if the second mode is not included, since the second mode is orthogonal to the first. When a finite element solution is obtained in the time domain, using 80 intervals per cycle, the ultimate error in one cycle is, in this case 0.03%,which is exactly the error due to the use of temporal finite elements given by Fig. 3.2, and is much less than the ultimate error of 2.74% previously obtained by using spatial finite elements.

In cases where exact values of natural frequencies and modal shapes are known, their use enables accuracy to be considerably improved. Unfortunately, in complicated structures these values must usually be found from finite element discretizations or similar methods, so that the use of exact values for frequencies and modal matrices is not usually possible.

A second method which gives improved accuracy is that using frequency-dependent mass and stiffness matrices.Przmieniecki [12] derives expressions for these matrices, not as constants, but as quantities which vary with frequency so that $[m] = [m_0] + \omega^2[m_2] + \ldots\ldots$ and $[k] = [k_0] + \omega^4[k_4] + \ldots\ldots$, where $[m_0]$ and $[k_0]$ are the usual finite element mass and stiffness matrices. Taking values for $[m_2]$ and $[k_4]$ given by Przmieniecki and ignoring terms containing higher powers of ω, the greatest changes in the mass and stiffness matrices for the 2-element discretization of Fig. 6.5 are about 2.8% and 0.1% respectively using the exact value of ω_1. This value of ω_1 is not quite the same as that which would be obtained by using the frequency-dependent matrices

to obtain the lowest value of ω, but even when only $[m_0]$ and $[k_0]$ are used the error in ω_1 is only 0.39%. When the $[m_2]$ and $[k_4]$ matrices are added the error in ω_1 is much less than this value.

Using these modified mass and stiffness matrices for the 2-element discretization, with 80 intervals per cycle in the finite element time solution, the ultimate error for one cycle of a free vibration in the first mode is only 0.06%. This is higher than the 0.03% previously obtained by using exact values in the modal matrix, but very much better than the 2.74% error for the unmodified matrices $[m_0]$ and $[k_0]$.

In general, however, a beam vibrates at many frequencies, and, for a particular values of ω only that part of the solution contributed by the appropriate mode is improved in accuracy by the use of time-dependent matrices, and contributions for other modes are little affected. This may be shown by using the values of [m] and [k] as determined previously taking $\omega = \omega_1$ and allowing the beam to vibrate at its third frequency ω_3. It is found that the errors in this case are almost the same as those obtained when the unmodified matrices are used.

If one particular mode of a vibration predominates, a considerable improvement in the accuracy of a solution could be achieved by using the appropriate value of frequency to evaluate the mass and stiffness matrices. If no one mode predominates it might be possible to use the modal method previously described with different corrections applied to the mass and stiffness matrices for each frequency, but this would be a very involved method.

A rather similar method of improving the spatial discretization is suggested by Cohen and McCallion [24], who use a shape function.

$u = a(1+\lambda x^4/24)+b(x+\lambda x^5/120)+c(x^2+\lambda x^6/360)+d(x^3+\lambda x^7/840)$

but again the value of λ required to give best results depends on the frequency, for example $\lambda = 0.155$ for the first mode and 2.5 for the second mode for the 5-element discretization used in reference [24].

It appears that this modification also can only improve the accuracy of the solution for one particular mode.

A third possible approach is to use refined spatial elements which are the spatial equivalents of the refined temporal elements derived in 2.5. Mass and stiffness matrices for these refined spatial elements are given by Pestel [7], the generalized displacements now being u, u' and u" resulting in 6 x 6 matrices. By introducing u'" as an additional displacement, 8 x 8 matrices are derived in the same reference. The derivation of these matrices is achieved in a rather different way to that employed in 2.2, by using the properties of Hermitian polynomials, but the two methods give identical results.

Using the 6 x 6 matrices given below, a further solution may be obtained for the free vibration of a simply-supported beam.

$$[m] = \frac{\rho A \ell}{55440}\begin{bmatrix} 21720 & 3732 & 281 & 6000 & -1812 & 181 \\ & 832 & 69 & 1812 & -532 & 52 \\ & & 6 & 181 & -52 & 5 \\ & & & 21720 & -3732 & 281 \\ & & & & 832 & -69 \\ \text{Symm} & & & & & 6 \end{bmatrix} \qquad (6.5.1)$$

$$[k] = \frac{EI}{70\ell^3}\begin{bmatrix} 1200 & 600 & 30 & -1200 & 600 & -30 \\ & 384 & 22 & -600 & 216 & -8 \\ & & 6 & -30 & 8 & 1 \\ & & & 1200 & -600 & 30 \\ & & & & 384 & -22 \\ & \text{Symm} & & & & 6 \end{bmatrix} \qquad (6.5.2)$$

These matrices are used in exactly the same way as the 4 x 4 matrices (6.3.1) and (6.3.2), and the only complication in the solution of vibration problems is that, in order to start a vibration in a particular mode, since u" is now a nodal displacement its initial value is required at each node. These values are, of course, easily found for a simply-supported beam vibrating in its first mode.

Using a two-element discretization for the whole beam, but making use of the symmetry of the arrangement as before, a solution may be obtained for the free vibrations of the simply-supported beam. Without using modal truncation, with 80 intervals per cycle, the ultimate error is 0.44% which is considerably less than the error of 2.74% previously obtained by using the two element discretization of Fig. 6.5 with simpler spatial elements. By using the 4-element discretization of Fig. 6.5 it is, however, possible to reduce the error to 0.20%.

It therefore appears that, for a given degree of accuracy a choice must be made between using a few refined spatial elements and a greater number of basic elements. Because of the added complexity of the refined elements, and their tendency to give unstable solutions due

to their increased number of degrees of freedom, only basic spatial elements are used in subsequent calculations. Since the object of these calculations is to show the validity of the methods used, if increased accuracy is required the basic elements may, of course, be replaced by refined elements by using the appropriate mass and stiffness matrices.

6.6 Combined space-time discretization

Before proceding to the study of forced vibrations of continuous systems, one further form of finite element discretization is used to investigate free vibrations. This method is unique in that it gives a complete solution in space and time, and not merely a solution in space for a particular value of time which the step-by-step method produces.

Section 6.3 shows that a continuous system may be discretized into separate spatial and temporal finite elements, and that once the mass and stiffness matrices have been generated, the method of solution is exactly the same as for a discrete system. In this present section, combined space-time finite elements are used, and since the element is continuous in space and time it may be used for continuous systems only.

Oden [25] shows that dimensions of space and time need not be separated and that a longitudinally vibrating bar may be discretized into two-dimensional triangular space-time elements by using the simple shape function $u = a + bx + ct$. This gives continuity of displacement

(but not of velocity) between adjacent elements along the length of the bar. A beam, however, requires continuity of slope as well as of displacement between adjacent elements and the shape function must include at least one term containing x^2 since u" must not, in general, be zero. A much more elaborate shape function than the above is therefore required.

There are obvious advantages in using a rectangular element, since the modal displacements etc. are then always measured at the same points on the beam at the beginning and end of the time interval. An element of the type shown in Fig. 6.6 is therefore suitable.

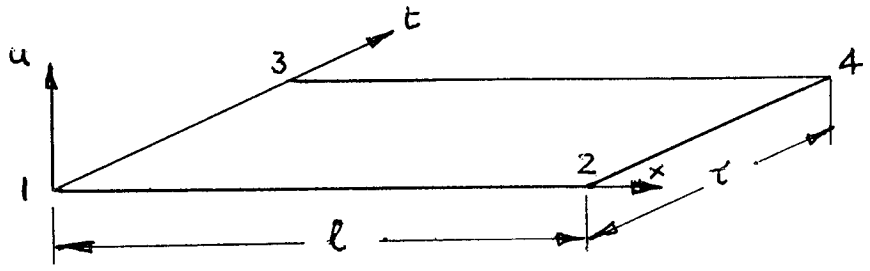


Fig. 6.6

In this element, initial values of "nodal" coordinates are given at 1 and 2 and final values at 3 and 4. The coordinates will be, in general, linear and angular displacements and their time derivatives, although in some cases not all of these may be included.

The polynomial function used to describe the shape is

$$u = \Sigma a_i x^j t^k \qquad i = 1,2,3.....\ n$$
$$j,k = 0,1,2..... \tag{6.6.1}$$

Derivatives in space and time may be expressed in terms of the same a_i so that the generalized coordinates at the nodes u_i are given by

$$\{u\} = [X]\{a\} \tag{6.6.2}$$

where [X] is an n x n matrix having coefficients which are products of the type $rx_i t_i$, r being a coefficient resulting from any differentiation, or being unity if no differentiation is involved, and x_i and t_i are the values of x and t at the nodes, i.e. $x_i = 0$ or ℓ and $t_i = 0$ or τ.

The coefficients a_i may thus be determined from

$$\{a\} = [X]^{-1}\{u\} \tag{6.6.3}$$

By substitution into (6.6.1) followed by the appropriate differentiations, the velocity and curvature at any point in the element may be expressed in terms of the u_i. Then by using Hamilton's principle, for a free vibration

$$\frac{\partial}{\partial u_i}\left(\int_0^{\tau}\int_0^{\ell} \frac{1}{2}\rho A\dot{u}^2\, dxdt - \int_0^{\tau}\int_0^{\ell} \frac{1}{2} EI(u'')^2 dxdt\right) = 0 \tag{6.6.4}$$

"Mass" and "stiffness" matrices [m*] and [k*] may thus be derived in the usual way. Two n x n matrices will thus be generated giving n equations

$$([m^*] - [k^*])\{u_i\} = \{0\} \tag{6.6.5}$$

The algebraic processes required in this method are thus a matrix inversion to find the a_i, differentiation to find $\dot{u}$ and u", multiplication of n functions of a_i in pairs followed by the double integration

and summation. These processes are, however, fairly easily arranged so that computer calculations are possible.

It is then necessary, for checking purposes, to choose a suitable space-time shape function which is capable of giving a satisfactory solution in the case of a system having a known exact solution. Once again the system used is a simply-supported beam vibrating freely in its first mode, and once again use is made of the symmetry of the arrangement to reduce the size of the matrices used. Details of the various functions tested are given below.

(i) To maintain continuity of deflection and slope between adjacent space elements a minimum of 4 terms is necessary to describe the shape of an element in space, and to allow a linear velocity at least two time-dependent terms are required. Combining these two requirements gives a shape function

$$u = (b_0+b_1x+b_2x^2+b_3x^3)(c_0+c_1t)$$

or $$u = a_1+a_2x+a_3t+a_4x^2+a_5xt+a_6x^3+a_7x^2t+a_8x^3t \qquad (6.6.6)$$

The 8 coefficients are functions of 8 generalized coordinates which are the deflection and slope at the two ends of the element at the beginning and end of the time interval, i.e. values of u and u' at points 1, 2, 3 and 4 in Fig. 6.6. The "mass" and "stiffness" matrices [m*] and [k*] may then be derived as described previously in this section, giving the following results.

$$[m^*] = \frac{\rho A \ell}{2520\tau} \begin{bmatrix} 936 & 132 & 324 & -78 & -936 & -132 & -324 & 78 \\ & 24 & 78 & -18 & -132 & -24 & -78 & 18 \\ & & 936 & -132 & -324 & -78 & -936 & 132 \\ & & & 24 & 78 & 18 & 132 & -24 \\ & & & & 936 & 132 & 324 & -78 \\ & & & & & 24 & 78 & -18 \\ & & & & & & 936 & -132 \\ \text{Symm} & & & & & & & 24 \end{bmatrix}$$

$$[k^*] = \frac{EI\tau}{252\ell^3} \begin{bmatrix} 1008 & 504 & -1008 & 504 & 504 & 252 & -504 & 252 \\ & 336 & -504 & 168 & 252 & 168 & -252 & 84 \\ & & 1008 & -504 & -504 & -252 & 504 & -252 \\ & & & 336 & 252 & 84 & -252 & 168 \\ & & & & 1008 & 504 & -1008 & 504 \\ & & & & & 336 & -504 & 168 \\ & & & & & & 1008 & -504 \\ \text{Symm} & & & & & & & 336 \end{bmatrix}$$

With a shape function of the type of (6.6.6), no variation of velocity in the time domain is possible in a single element. Continuity of velocity between adjacent elements is therefore impossible so that the step by step method previously used cannot be employed in this case. Instead, the whole space-time continuum must be considered as an entity and values of final and intermediate coordinates must all be found together. This may be demonstrated by considering the space-time discretization for the half-beam shown in Fig. 6.7.

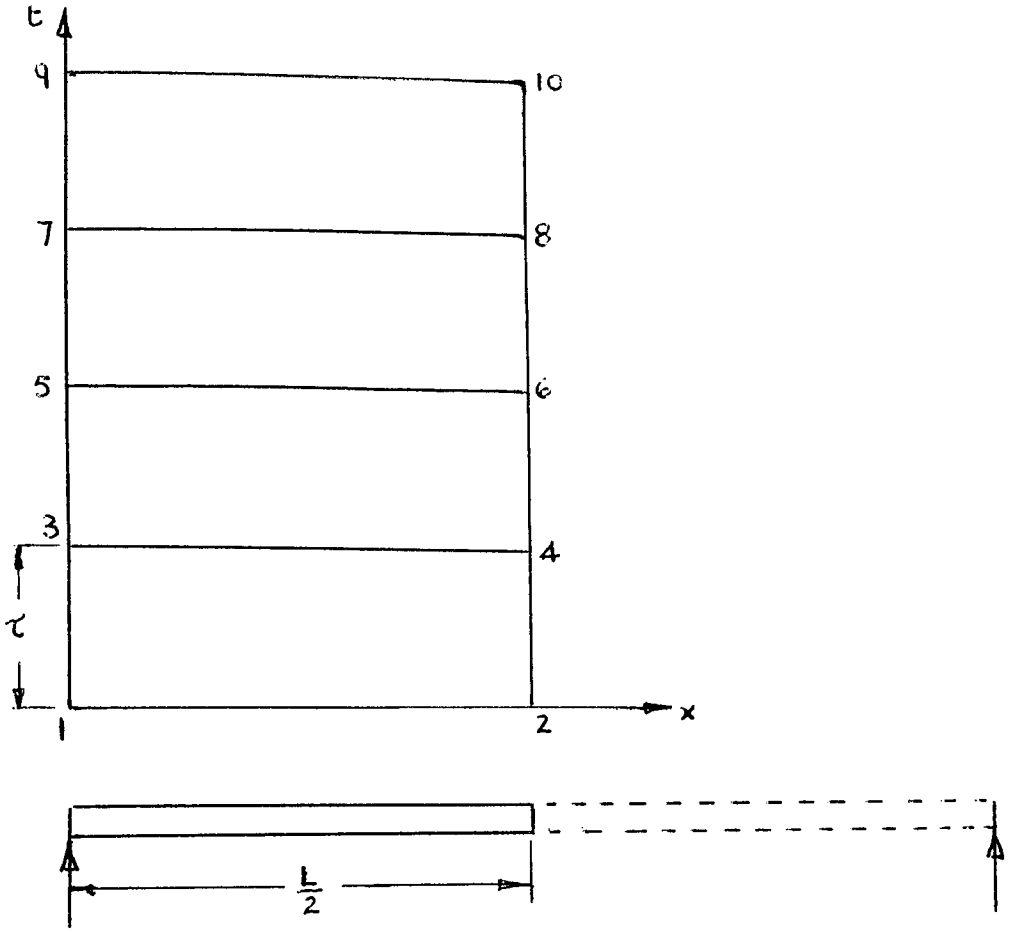


Fig. 6.7

Since $u_1 = u_3 = u_5 \ldots\ldots = 0$

and $u_2' = u_4' = u_6' \ldots\ldots = 0$ (due to symmetry of the first mode)

and u_1' and u_2 are known initial conditions, only values of u_3', u_4, u_5', u_6 ..... are to be determined. Thus if there are r elements only 2r values are required and therefore only 2r equations are needed. In this case it is found that initial and final displacements (u_2 and u_{10} in Fig. 6.7) must be treated as constants and meaningful equations are obtained only from the partial derivatives with respect to u_1', u_3', u_4, u_5', $u_6 \ldots\ldots u_9'$. Hence the 4 elements of Fig. 6.7 give the required 8 equations.

As in previous methods used in this chapter the beam is initially taken to be at the extremity of its motion; initial values of u_1^1 and u_2 are thus known. Values of u_3^1, u_5^1, u_7^1, u_9^1 and u_4, u_6, u_8, u_{10} may then be calculated by using equations (6.6.5). Values of maximum range errors in deflection and slope for the discretization of Fig. 6.7 and similar arrangements are given in Table 6.4.

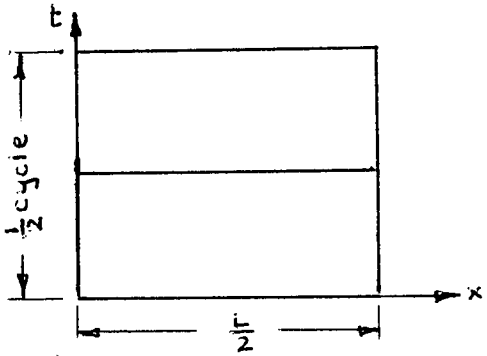
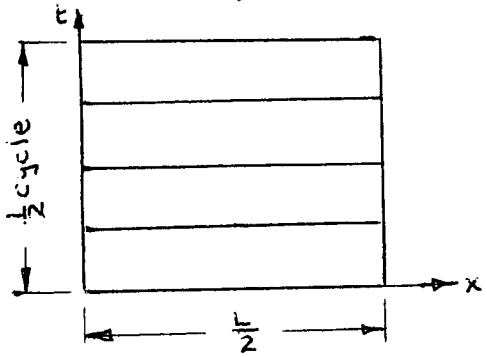
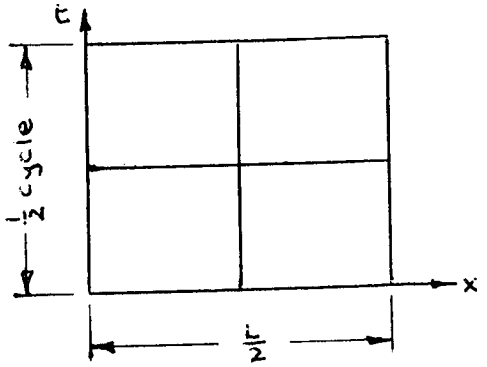
	Representation	Maximum range error	
		Deflection	Slope
(a)		8.6%	9.1%
(b)		3.0%	2.9%
(c)		8.4%	8.8%

Table 6.4

A comparison of arrangements (a) and (b) shows that accuracy is improved by increasing the number of elements per cycle, while if (a) and (c) are compared it will be seen that there is hardly any reduction in the maximum error when elements of length L/4 are used instead of L/2.

While the results shown in Table 6.4 seem quite satisfactory for the small numbers of elements used, further investigation shows that very unsatisfactory results are obtained if the final position of the beam is not one for which all velocities are zero. Using 4 elements as in (b), but for only one quarter of a cycle, the final slope shows a range error of 65%, while if the 4 elements occupy three-quarters of a cycle the final deflection range error is 25%. Errors of these magnitudes seem likely to occur whenever initial and/or final velocities are not zero. The explanation of this appears to be that because velocity does not appear as a nodal parameter, non-zero velocities must not occur as boundary conditions.

Since, in general, initial and final velocities will not be zero, this slope function must be regarded as unsatisfactory in spite of the encouraging results obtained when initial and final velocities are zero.

(ii) Since, without the explicit use of velocity, the previous shape function is unsatisfactory, a more elaborate space-time element which uses velocity as a generalized displacement is required. If deflection, slope and linear velocity are to be included as generalized displacements a shape function of the following type may be used.

$$u = (b_0+b_1x+b_2x^2+b_3x^3)(c_0+c_1t+c_2t^2)$$

or $$u = a_1+a_2x+a_3t+a_4x^2+a_5xt+a_6t^2+a_7x^3+a_8x^2t+a_9xt^2+a_{10}x^3t+a_{11}x^2t^2+a_{12}x^3t^2 \quad (6.6.7)$$

This is not a symmetrical function with respect to x and t, but with only slight modifications of 6.6.7 many symmetrical functions may be obtained. Two of these are:

$$u = a_1+a_2x+a_3t+a_4x^2+a_5xt+a_6t^2+a_7x^3+a_8x^2t+a_9xt^2+a_{10}t^3+a_{11}x^3t+a_{12}xt^3 \quad (6.6.8)$$

$$u = a_1+a_2x+a_3t+a_4x^2+a_5xt+a_6t^2+a_7x^3+a_8x^2t+a_9xt^2+a_{10}t^3+a_{11}x^3t^2+a_{12}x^2t^3 \quad (6.6.9)$$

It is found, however, that for function (6.6.7) the determinant of the [X] matrix of (6.6.2) is zero and so the inverse $[X]^{-1}$ required in (6.6.3) does not exist. The unsymmetrical nature of the function may cause $|X|$ to be zero, since the symmetrical arrangements of (6.6.8) and (6.6.9) have non-zero determinants and therefore {a} may be found from (6.6.3). "Mass" and "stiffness" matrices may also be obtained from these two functions, but in both cases all combinations of the 12 equations for a single element give meaningless results for the free vibrations of a simply-supported beam.

(iii) It has been shown previously in 6.3 that separate 4-term space and time shape functions give satisfactory results. It should therefore be possible to combine these separate discretizations to give a space-time function of the form

$$u = (b_1+b_2x+b_3x^2+b_4x^3)(c_1+c_2t+c_3t^2+c_4t^3)$$

which may be given as a 16-term function

$$u = a_1+a_2x+a_3t+a_4x^2+a_5xt+a_6t^2+a_7x^3+a_8x^2t+a_9xt^2+a_{10}t^3+a_{11}x^3t+a_{12}x^2t^2$$
$$+a_{13}xt^3+a_{14}x^3t^2+a_{15}x^2t^3+a_{16}x^3t^3 \qquad (6.6.10)$$

In this case also the [X] matrix has an inverse, and "mass" and "stiffness" matrices may be obtained. These are given in Appendix A. In this case the generalized displacements are the deflection, slope, linear velocity and angular velocity at each corner of the rectangular space-time element shown in Fig. 6.8.

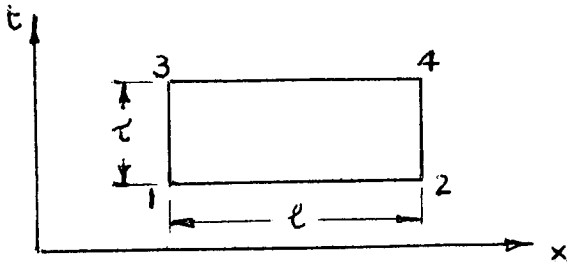


Fig. 6.8

Since linear and angular velocities are now nodal parameters, continuity of these quantities is assured at each node, and a step-by-step method of solution in time may be used. By taking $\ell = L/2$ and τ equal to 1/20 of the period of the free vibrations (first mode) of a simply-supported beam, it is necessary to consider only one element at a time since, in Fig. 6.8, the initial conditions at nodes 1 and 2 are known, and therefore the final generalized displacements at 3 and 4 may be found by using equations (6.6.5). The resulting values calculated are found to be identical to those obtained in 6.3 by using separate space and time discretizations.

Since a satisfactory space-time element is now available it should be possible to obtain solutions by coupling elements together in the time direction. The disadvantage of this procedure is that much more data has to be stored than when the previous step-by-step method is used. Thus, for a space-time field of the type shown below in Fig. 6.9 for half a simply-supported beam, if there are n time intervals there are 4n equations, each having 4(n + 1) coefficients to be stored. Since these coefficients are obtained from the "mass" and "stiffness" matrices, each of these will have 16n(n+1) coefficients. The total number of values to be stored is therefore 32n(n + 1). For the arrangement of Fig. 6.9, where n = 4, the number of coefficients to be supplied is thus 640, compared with only 64 if the step-by-step method is used with the space-time element, and only 8 for the step-by-step method with separate space and time discretizations.

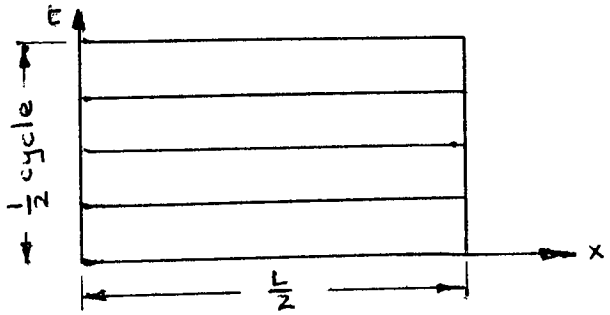


Fig. 6.9

Unfortunately the space-time discretization of Fig. 6.9 gives an unstable solution, presumably due to the use of too large a time interval. An increase in the number of elements in the time direction would require even more data input. This additional work does not seem justified since perfectly satisfactory solutions can be obtained much more easily by using a step-by-step method of solution with either space-time or separate space and time elements.

6.7 Conclusion

Spatial discretization of continuous systems may be carried out, quite independently of the method of solution in the time domain. The lumped-mass representation of a continuous system is not, in general, as satisfactory as a finite element discretization since it is likely to give a less accurate solution. In addition, a flexibility matrix is first obtained for the lumped-mass system, while the finite element system gives the stiffness matrix required for a temporal finite element solution. The use of spatial finite elements therefore avoids a matrix inversion.

The use of temporal finite elements may result in an unstable solution, the instability being caused by the higher natural frequencies of the system. Since, to increase accuracy more spatial elements may be used, thus increasing the number of degrees of freedom, instability may result. This may be avoided by using the truncated modal method (although this requires that all the frequencies and modal shapes are known) and discarding the contribution of the higher frequencies. This method gives satisfactory results, and accuracy is improved, for a given number if frequencies included in the solution, by increasing the number of spatial elements. This is because, in the finite element spatial discretization the highest frequencies become increasingly inaccurate. The contributions of the highest frequencies, in addition to being generally small, are therefore likely to be inaccurate and a probable cause of instability in the solution. Except in unusual circumstances the absence of these frequencies is likely to be of considerable advantage.

It is perhaps worth repeating that this truncated modal method is not an essential method of solution when temporal finite elements are used, but is merely a refinement of the original method to obtain a stable solution without using a large number of temporal elements.

Various methods of improving the spatial discretization are available, and of these the refined spatial element using u" as a generalized displacement is probably the most obvious. It also gives good results if suitable precautions are taken to avoid instability in the solution.

The use of space-time elements appears to have the advantage that all values of the nodal displacements in both space and time are obtained simultaneously. This same result can, however, be obtained more simply by using separate space and time discretizations and coupling the elements in time as described in 6.3 and 2.4. The space-time element requires considerable care in the choice of shape function to obtain a solution, and in practice requires large matrices to be assembled and stored, and even so may give an unstable solution. Since perfectly satisfactory results may be obtained much more easily by using separate space and time elements, the extra complications of the space-time element method are hardly worth while.

CHAPTER 7

FORCED VIBRATIONS OF DISTRIBUTED SYSTEMS

7.1 Introduction

The motion of a distributed system which is made to vibrate by external forces is, in general, much more complicated than the motion when the system vibrates freely. It is possible, by choosing certain initial conditions, to allow a system to vibrate freely in one mode only, but, in a forced vibration, all modes may be excited. To obtain an accurate solution for a forced vibration, as many modes as possible should be included in the calculations. To obtain a stable solution with a reasonable number of elements, the truncated modal method is again used and so the above requirement may conflict with the need to discard the higher frequencies, and so a compromise may be necessary. In 7.2 the effects of various degrees of truncation are examined, and it is shown that truncation errors may be quite moderate.

When spatial finite elements are used, only nodal forces and couples may be used in the solution, and so any distributed force must be discretized to give the required generalized forces. Examples of vibrations due to distributed forces are given in 7.4.

Since all solutions must be stable, it is necessary to determine the size of the time interval required, and although this may be found

exactly for a free vibration from (6.4.4) this condition does not necessarily apply for a forced vibration. As a result of preliminary tests it appears, however, that (6.4.4) and (6.4.5) do give, quite accurately, the size and number of intervals respectively for a stable solution for forced vibrations also.

As in the previous chapter, it is intended here to show that temporal finite elements may be used to solve vibration problems, and to obtain an estimate of the accuracy likely to be obtained. It is again, therefore, necessary to choose simple systems and forces so that an exact solution is available for comparison. Once again it must be stated that finite element solutions are possible for much more difficult cases than those examined in this chapter, but it may then be very difficult to obtain an exact solution.

Separate temporal and spatial finite element discretizations, as described in 6.3, using the basic temporal element and its spatial equivalent, are used throughout the present chapter. If increased accuracy is required the more elaborate elements could of course be used by employing exactly the same methods of solution.

7.2 Discrete forces

Solutions for forced vibrations of distributed systems may be obtained by using exactly the same methods as those used for discrete systems. The mass and stiffness matrices must of course be available, and these are obtained by using spatial finite elements. Any modal force or couple then appears in the force vectors $\{F_0\}$ and $\{F_1\}$ of (4.2.4) and the solution then proceeds in the usual way.

Since it may be necessary to use the truncated modal method to obtain a stable solution, the effect of any truncation on the accuracy of the solution is first examined. A suitable example for this investigation is that of a simply-supported beam with a central harmonic force F cos Ωt. The solution for this case is given by Timoshenko [20] as an infinite series for the deflection u at a distance x from one end.

$$u = \frac{2FL^3}{\pi^4 EI} \sum_{i=1,3}^{\infty} \frac{1}{i^4-(\Omega/\omega_1)^2} \sin\frac{i\pi}{2} \sin\frac{i\pi x}{L} \cos \Omega t \qquad (7.2.1)$$

and by differentiation the slope is given by

$$u' = \frac{2FL^2}{\pi^3 EI} \sum_{i=1,3}^{\infty} \frac{i}{i^4-(\Omega/\omega_1)^2} \sin\frac{i\pi}{2} \cos\frac{i\pi x}{L} \cos \Omega t \qquad (7.2.2)$$

where ω_1 is the lowest natural frequency of the beam.

It will be seen that the series for u' converges more slowly than that for u, so that, for a given number of terms, the error in the slope is likely to be greater than that for the deflection. In Chapter 6 it was noted that, for the free vibrations of a beam, the maximum error in the slope usually occurs at the ends of the beam. It may therefore be tentatively assumed that this also applies for a forced vibration. By evaluating the series (7.2.2) at x = 0, and including sufficient terms to give an answer correct to 8 figures, values of the end slope may be calculated for various values of Ω/ω_1. By including only 1,2,3 and 4 terms the percentage errors in the end slope for these degrees of truncation may then be calculated. These errors for frequencies up to 10 times the first natural frequency are shown in Fig. 7.1. It must be made

clear that these errors are due to truncation only, and are additional to any errors which may be introduced by discretizations.

It is immediately obvious that when only the first mode (one term of the series) is used, there are enormous errors except at frequencies up to about $1.5\omega_1$. As the number of modes included is increased there is a rapid increase in accuracy at all frequencies, and when the first 3 modes are used, the truncation error does not exceed 2.4% in the frequency range under consideration and when the first 4 modes are included the maximum error is just over 1%. The sign of the error depends on the number of modes used, as consecutive terms in the series (7.2.2) have opposite signs and in addition there is a change of sign of the appropriate term when $\Omega = \omega_i$. It is therefore not possible to state that, in general, the truncation of the series will give too large or too small a value for u' as either of these possibilities may occur.

It will also be seen that there is a rapid reduction in the truncation error as the forcing frequency approaches a natural frequency, provided that the mode corresponding to this natural frequency is included. This reduction is the result of the rapid increase of $1/[1 - (\Omega/\omega_i)^2]$ as $\Omega \to \omega_i$ so that the term containing this reciprocal is much larger than any of the terms discarded.

For forcing frequencies up to $2\omega_1$ an examination of Fig. 7.1 shows that when 1,2,3 and 4 modes are included the maximum truncation errors are 9, 1.6, 0.55 and 0.25% respectively. Thus if errors greater than

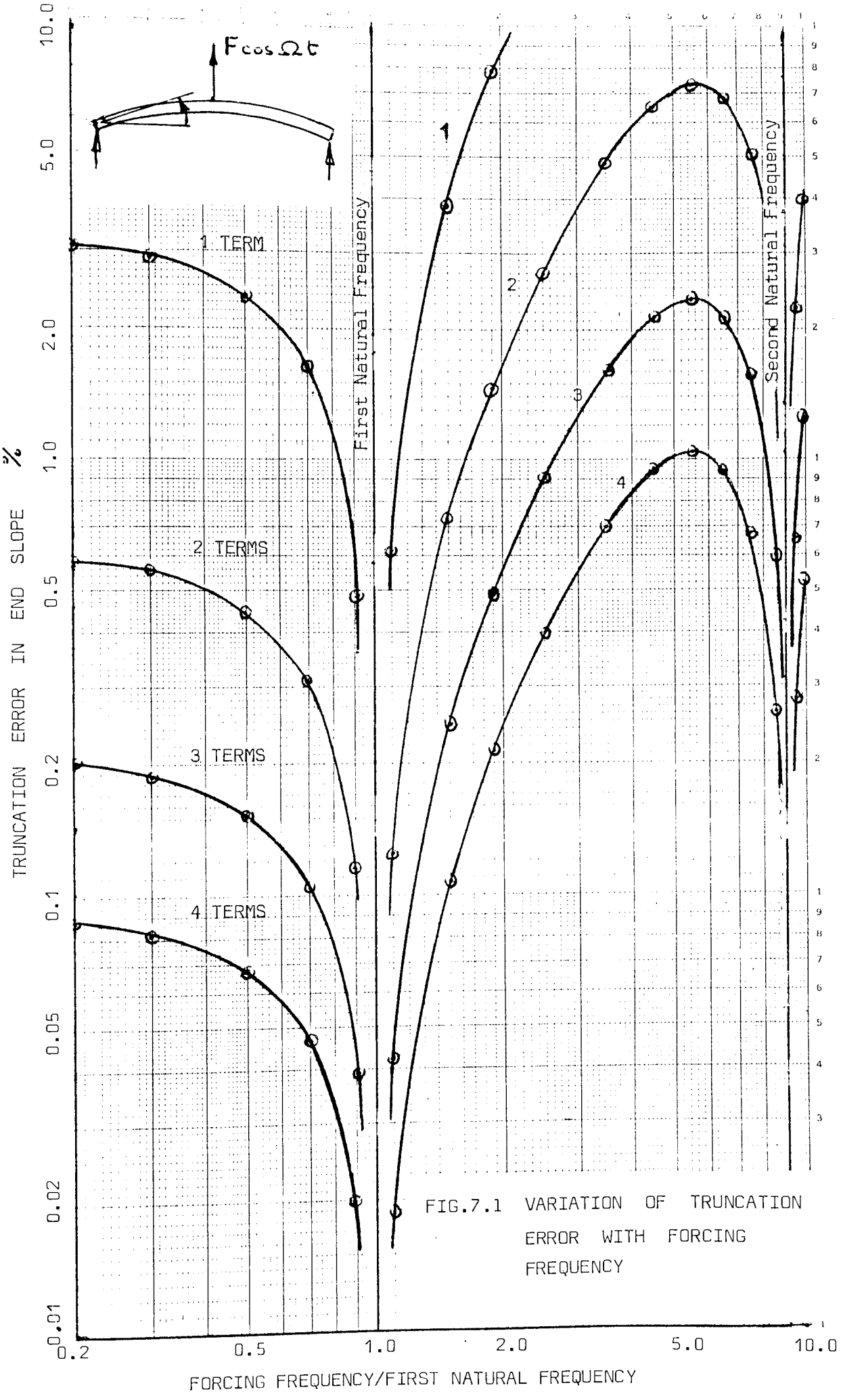
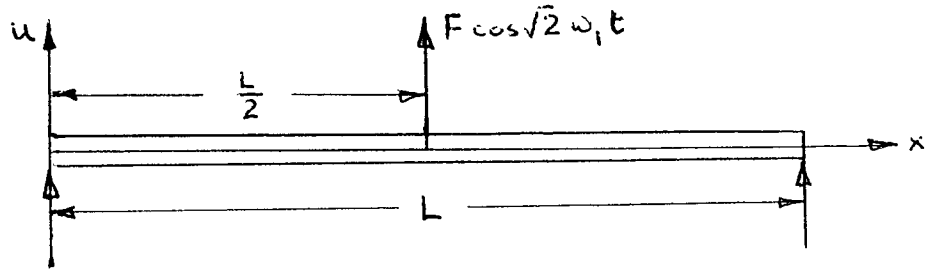


FIG.7.1 VARIATION OF TRUNCATION ERROR WITH FORCING FREQUENCY

1% are regarded as unacceptable, at least 3 modes must be included in the solution, while for forcing frequencies up to $10\omega_1$, 4 modes are required to ensure that the truncation error does not exceed 1%.

Example 7.1



For a simply-supported beam with a harmonic central force, the steady-state solution may be found by using spatial and temporal finite elements. Since the system is symmetrical, any of the spatial discretizations of Fig. 6.5 may be used, provided that only one half of the force is applied to the half-beam. Thus, for example, for the 4-element discretization of Fig. 6.5(b) the force vector is $[0 \quad 0 \quad 0 \quad 0.5F\cos\sqrt{2}\omega_1 t]^t$ and the required values of the force and its time derivative at any time t may be calculated in the usual way.

Since a steady-state solution is required, the beam must start a cycle with the correct values of generalized displacements. If the beam is initially in its highest position all velocities are zero, and the correct values of deflection and slope may be calculated from (7.2.1) and (7.2.2). These values may be found from Table 7.1.

$\frac{x}{L}$	Deflection Factor $\frac{2FL^3}{\pi^4 EI}\cos\sqrt{2}\omega_1 t$	Slope Factor $\frac{2FL^2}{\pi^3 EI}\cos\sqrt{2}\omega_1 t$
0	0	-1.03196790
0.1667	-0.51176117	-0.87078555
0.25	-0.71684679	-0.68686047
0.3333	-0.86770019	-0.46074774
0.25	-0.98500403	0

Table 7.1

Using the 4- and 6-element space discretizations of Fig. 6.5 and the basic temporal finite element, with 40 elements per cycle in each case, it is found that the maximum range error within one cycle, as in (3.2.1), is the error in the end slope at the end of the cycle. Values of this ultimate error for various degrees of truncation, obtained by using the truncated modal method of solution described in 6.4, are given in Table 7.2

Number of spatial elements	Modes used in solution	Number of intervals/ cycle	Ultimate error %
4	1, 3, 5	40	0.38
4	1,3,5,7	40	Unstable sol.
6	1,3,5,7	40	0.25

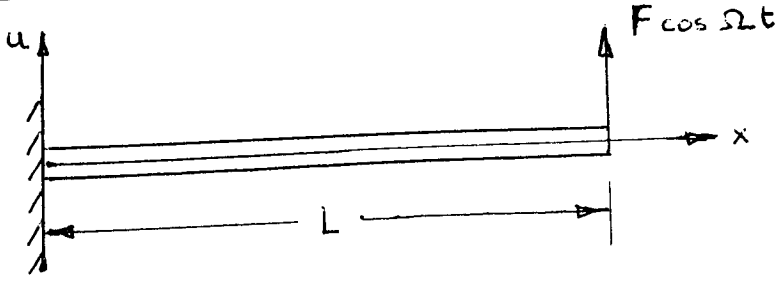
Table 7.2

It will be seen from Fig. 7.1 that for the first result in Table 7.2 the truncation error is 0.19%, and for the second and last results 0.09%. The difference between the ultimate errors shown and the truncation errors is accounted for by the approximate nature of the spatial and temporal discretizations. It is also interesting to note that the second solution is unstable while the last, using the same number of modes, is stable. This is because for the 4-element spatial discretization the value of ω_7 is 37% high, while for the 6-element representation the error in ω_7 is only 3%. Since instability is caused by the highest frequencies, this very inaccurate value for the 4-element model is sufficiently high to be the cause of the unstable solution.

Solutions may be obtained for other forcing frequencies in this example, and very similar results to those of Table 7.2 are obtained with a forcing frequency of $5\omega_1$.

Solutions may be obtained for other types of beam, although at this stage care is needed in choosing examples for which exact solutions may be obtained in order to check the accuracy of the finite element solution. An exact solution is obtainable for the cantilever of the next example.

Example 7.2



In this case the beam may be treated as a case where the shear force at the right hand end of the beam is known. Thus the usual form of solution for a free vibration may be used, i.e.

$u = C_1 \sin mx + C_2 \cos mx + C_3 \sinh mx + C_4 \cosh mx$ where

where $m^4 = \Omega^2 \rho A/EI$

Using the boundary conditions

$x = 0 \qquad u = 0$ and $u' = 0$

$x = L \qquad u''' = -\frac{F}{EI} \cos \Omega t$ and $u'''' = 0$

the values of the coefficients may be obtained and hence

$$u = \frac{-F \cos \Omega t}{2EIm^3(1+ \cos m\ell \cosh m\ell)} \left[\begin{array}{l} (\cos m\ell + \cosh m\ell)(-\sin mx + \sinh mx) \\ +(\sin m\ell + \sinh m\ell)(\cos mx - \cosh mx) \end{array} \right]$$

Choosing, for convenience a value $\Omega = \sqrt{EI/\rho AL^4} (=0.5333\omega_1)$, $mL = 1.$ and for this frequency

$$u = \frac{FL^3 \cos \Omega t}{3.667460\ EI} [2.083383\ (\sin mx - \sinh mx) + 2.016672\ (\cosh mx - \cos mx)]$$

and

$$u' = \frac{FL^2 \cos \Omega t}{3.667460EI} [2.083383\ (\cos mx - \cosh mx) + 2.016672\ (\sinh mx + \sin mx)]$$

Hence initial values of deflection and slope may be calculated to ensure that a steady state vibration is obtained. The cantilever may be discretized in space, and results are given below for 2- and 3-element discretizations. Temporal finite elements are also used with the truncated modal method. In one cycle, the maximum error again occurs in the slope at the end of the beam. The errors for two spatial discretizations are given in Table 7.3.

Number of spatial elements	Modes used in solution	Number of intervals cycle	Ultimate error %
2	1, 2, 3	80	1.1
3	1,2,3,4	160	0.67

Table 7.3

It should be noted that in order to obtain a stable solution, when the fourth mode is included the number of intervals per cycle is increased from 80 to 160. Reference to Fig. 3.2 shows that this reduces the temporal discretization range error from 0.03% to 0.008%, so that the effect on the total error is very small. It will be seen that the ultimate errors in Examples 7.1 and 7.2 are of the same order.

It should be made clear that although, to obtain exact solutions readily, the previous examples in this chapter have been concerned with steady state solutions, the finite element method automatically gives a complete solution, which includes the transient part of the solution if this is present. The following example should make this clear.

Example 7.3

A simply-supported beam, which is initially at rest has a discrete force of F sin Ωt applied at its centre. Taking $\Omega = \sqrt{2}\omega_1$, values of deflection and slope may be calculated from the expression for the complete solution given by Timoshenko [20].

$$u = \frac{2FL^3}{\pi^4 EI} \sum_{i=1}^{\infty} \frac{1}{i^4} \sin \frac{i\pi}{2} \sin \frac{i\pi x}{L} (\sin \Omega t - \frac{\Omega}{\omega_i} \sin \omega_i t)/[1-(\Omega/\omega_i)^2]$$

Exact values of the displacements are not, in this case, required to start the finite element solution, since all values are initially zero, but later exact values must be calculated to determine the accuracy of the finite element solution. When the 4-element space discretization of Fig. 6.5(b) is used with 40 temporal elements per cycle of the force the inclusion of modes 1, 3 and 5 gives an ultimate error of 0.64% for one cycle of the force. This is higher than the ultimate error of 0.38% previously obtained in Example 7.1 for the steady state vibrations of the same beam, and the increase is almost certainly due to the more complicated shape of the beam for the transient vibration. Once again accuracy could be increased, if necessary, by using more spatial elements and including more modes in the solution, and it is indeed found that by using 6 spatial elements with nodes 1, 3 and 5 the ultimate error is reduced to 0.31%.

7.3 The mode acceleration method

As an alternative to merely discarding those parts of the solution derived from the higher frequencies (which may be necessary to obtain a stable solution), the remaining part of the solution may be suitably modified. This method was first proposed by Williams [26], and is described by Anderson [27], and consists of finding the response of a system separately to external and internal forces so that the total response is

$$\{u\} = \{u\}_{ex} + \{u\}_{in} \tag{7.3.1}$$

For the external forces $[k]\{u\}_{ex} - \{F\}_{ex} = \{0\}$

or $$\{u\}_{ex} = [k]^{-1}\{F\}_{ex} \tag{7.3.2}$$

For internal forces $-[m]\{\ddot{u}\}_{in} - [k]\{u\}_{in} = \{0\}$ and hence if the forces are normalized by using the transformation $\{P\} = [u_N]^t\{Q\}$ a set of uncoupled equations of the form $\ddot{p}_i + \omega_i^2 p_i = P_i$ are produced since both the [M] and [K] matrices are diagonal. The equations containing the higher frequencies may again be discarded and the remaining equations will then give the p_i's as functions of t. Reverting to the original coordinate system by using the transformation $\{u\}_{in} = [u_N]\{p\}$, it is then possible to obtain the complete response of the system using any number of the possible modes of vibration.

Example 7.4

The application of the mode acceleration method may be illustrated by considering the case of a beam subjected to a central step force F. Making use of the symmetry of the system only one half of the beam need be used provided that only one half of the force is used. Fig. 7.2 shows this arrangement with the two degrees of freedom.

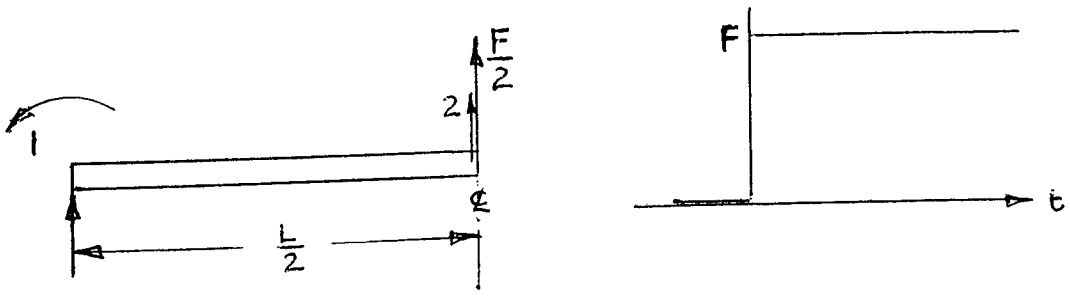


Fig. 7.2

(a) First an exact solution of the discretized system shown in Fig. 7.2 is given. The modal matrix for the one element is

$$[u_N] = \frac{1}{\sqrt{\rho AL}} \begin{bmatrix} 3.165856 & 2.015921 \\ 16.672653 & -1.822273 \end{bmatrix}$$

and hence using the transformation $\{P\} = [u_N]^t\{Q\}$

where ${}_1Q = 0$ and ${}_2Q = \frac{F}{2}$ gives

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{F}{\sqrt{\rho AL}} \begin{Bmatrix} 1.007961 \\ -0.911136 \end{Bmatrix} \tag{7.3.3}$$

For a step force the solution of the equation of motion is $p_i = \frac{P_i}{K_{ii}} (1 - \omega_i t)$ and in this case

$$p_1 = 0.0101087 \frac{FL^3\sqrt{\rho AL}}{EI} (1 - \cos \omega_1 t)$$

$$p_2 = 0.00007511 \frac{FL^3\sqrt{\rho AL}}{EI} (1 - \cos \omega_2 t) \tag{7.3.4}$$

The response in the original coordinate system is then found by using $\{u\} = [u_N]\{p\}$ so that

$${}_1u = 0.032003 \frac{FL^3}{EI} (1 - \cos \omega_1 t) - 0.0012523 \frac{FL^3}{EI} (1 - \cos \omega_2 t)$$

$${}_2u = 0.020378 \frac{FL^3}{EI} (1 - \cos \omega_1 t) + 0.0001368 \frac{FL^3}{EI} (1 - \cos \omega_2 t) \tag{7.3.5}$$

(b) By discarding the second term in each of equations (7.3.5) a truncated solution is obtained giving, for example, for the central deflection ${}_2u = 0.020378 \frac{FL^3}{EI} (1 - \cos \omega_1 t)$

(c) Using the mode acceleration method the response to the external forces is first found. The $[k]^{-1}$ matrix is, of course, the flexibility matrix and it is easily shown that $\begin{Bmatrix} {}_1u \\ {}_2u \end{Bmatrix}_{ex} = \frac{FL^3}{EI} \begin{Bmatrix} \frac{1}{32} \\ \frac{1}{48} \end{Bmatrix}$ (7.3.6)

If the second mode is discarded, the solution for internal forces is

$$\begin{Bmatrix} {}_1u \\ {}_2u \end{Bmatrix}_{in} = \frac{1}{\sqrt{\rho AL}} \begin{bmatrix} 3.165856 & 2.015921 \\ 16.672653 & -1.822273 \end{bmatrix} \begin{Bmatrix} -1.007961\ F/\omega_1{}^2 \cos \omega_1 t \\ 0 \end{Bmatrix} \tag{7.3.7}$$

Thus the total response is given by the sum of (7.3.6) and (7.3.7) giving, for the central deflection

$${}_2u = (0.020833 \quad -0.020378 \cos \omega_1 t) \frac{FL^3}{EI}$$

Now at $t = \pi/\omega_1$ the central deflection is exactly $FL^3/24EI$, and the true errors in the central deflection at this value of t may thus be found. These are found to be:-

(a) Two-mode solution 1.5% error

(b) Truncated solution 2.2% error

(c) Mode acceleration solution 1.1% error.

Similar results are obtained for the end slope.

The use of the mode acceleration method does therefore give a more accurate result than the truncated method, and even a more accurate result than the original two-mode solution. This latter result is presumably due to the inaccurate value of ω_2 obtained when only two finite elements are used, and the difference would probably be reduced if more elements were used. If the results for (b) and (c) are compared it will be seen that by using mode acceleration the error is reduced by a factor of 2 compared with the truncated modal result. While this is a useful gain in accuracy, an error of over 1% is still rather high and a more

elaborate space discretization using more elements is needed for higher accuracy. From the results of previous examples, it should be possible to reduce errors to considerably less than 1% by using 4 elements and truncating the solution at 3 modes. This seems an easier way to increase accuracy than by using the mode acceleration method which would not be easy to fit into the existing method of solution.

7.4 Distributed forces

When a method of solution for discrete forces is known, by the addition of a preliminary calculation a solution for distributed forces is readily found. Hamilton's principle is again used, and it may be recalled that this requires the work done by the forces to be known. For a discretized distributed system with discrete forces this is given by

$$W = [F]^t\{u\} \tag{7.4.1}$$

where $\{F\}$ and $\{u\}$ are the force and displacement vectors respectively.

For a distributed force $p = p(x)$ applied to an element of length ℓ the work done is

$W = \int_0^\ell p\, u\, dx$, and using Hermitian interpolation both p and u may be discretized in space as

$$p = [\psi_1\ \psi_2\ \psi_3\ \psi_4]\ [{}_1p\ {}_1p'_\ell\ {}_2p\ {}_2p'_\ell]^t$$

and $$u = [\psi_1\ \psi_2\ \psi_3\ \psi_4]\ [{}_1u\ {}_1u'_\ell\ {}_2u\ {}_2u'_\ell]^t$$

so that in terms of the loading and displacement vectors $\{p\}$ and $\{u\}$

$$W = \int_0^\ell [\psi]\{p\}[\psi]\{u\}dx \tag{7.4.2}$$

The components of the loading vector may be observed in Fig. 7.3

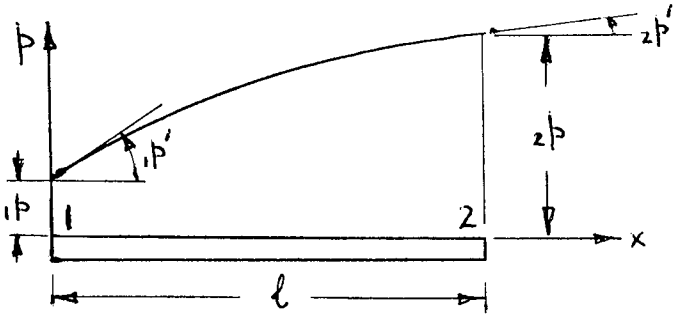


Fig. 7.3

When the partial differentiation required by the use of Hamilton's principle is carried out, it is observed that the contributions of the distributed forces to the resulting equations are very similar in form to those from discrete forces, and indeed become identical to discrete force expressions if equivalent nodal forces and couples are calculated from (7.4.3)

$$\begin{Bmatrix} {}_1F \\ {}_1M/\ell \\ {}_2F \\ {}_2M/\ell \end{Bmatrix} = \frac{\ell}{420} \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ \text{Symm.} & & & 4 \end{bmatrix} \begin{Bmatrix} {}_1p \\ {}_1p'\ell \\ {}_2p \\ {}_2p'\ell \end{Bmatrix} \qquad (7.4.3)$$

where ${}_1F$, ${}_2F$, ${}_1M$, ${}_2M$ are the equivalent nodal forces and couples.

Time derivatives of these discrete equivalents are also required and these are found by differentiating equations (7.4.3) and so introducing ${}_1\dot{p}$ etc. These rates of change may be found by exactly the same method as that used to find velocities using equations (2.2.7)

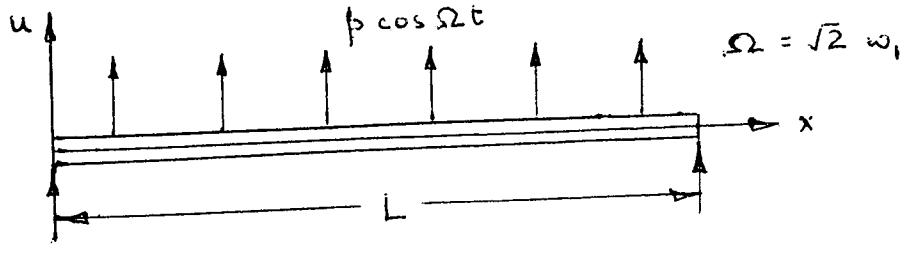
$$\text{i.e. } {}_1\dot{p} = [\dot{\psi}_1 \ \dot{\psi}_2 \ \dot{\psi}_3 \ \dot{\psi}_4][{}_1p_0 \ {}_1\dot{p}_{0T} \ {}_1p_1 \ {}_1\dot{p}_{1T}]^t \qquad (7.4.4)$$

and the same interpolation may be used to find the other time derivatives.

For each element of a beam the initial values of ${}_1p$ ${}_2p$, their space derivatives and the time derivatives of all four quantities are required. In addition, the corresponding eight values at the end of the time interval are needed, giving a total of 16 values for each element. If p is a known algebraic function of space and time these quantities may usually be evaluated without difficulty. Alternatively, if only numerical values of p are available for various values of x and t, the numerical interpolation methods given in Appendix A may be used.

It should be noted that since equations (7.4.3) may be used to find all the above equivalent forces and their time derivatives, the space and time discretizations are again automatically quite separate.

Example 7.5



In this example for a simply-supported beam, a distributed force, which is uniform along the whole length of the beam, varies harmonically in time. An exact solution is given by Timoshenko [20].

$$u = \frac{4pL^4}{\pi^5 EI} \sum_{i=1,3}^{\infty} \frac{1}{i^5 - i(\Omega/\omega_1)^2} \sin \frac{i\pi x}{L} \cos \Omega t \qquad (7.4.5)$$

$$u' = \frac{4pL^3}{\pi^4 EI} \sum_{i=1,3}^{\infty} \frac{1}{i^4 - (\Omega/\omega_1)^2} \cos \frac{i\pi x}{L} \cos \Omega t \qquad (7.4.6)$$

Values of deflection and slope at various points on the beam may thus be calculated for any value of t. The summations of the series (7.4.5) and (7.4.6) are given in Table 7.4

$\frac{x}{L}$	Deflection Factor $\frac{4pL^4}{\pi^5 EI} \cos\sqrt{2}\omega_1 t$	Slope Factor $\frac{4pL^3}{\pi^4 EI} \cos\sqrt{2}\omega_1 t$
0	0	-0.98500388
0.25	-0.70437820	-0.71684678
0.5	-1.00394537	0

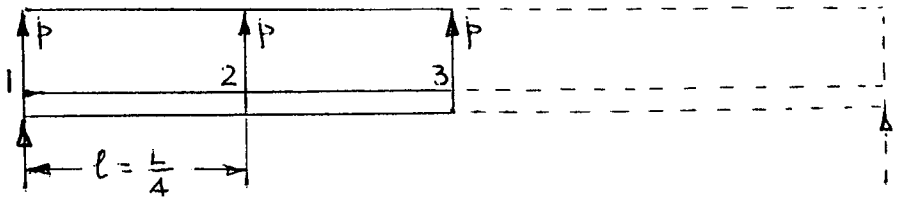
Table 7.4

An alternative form of the solution for this particular problem which may be obtained by the summation of the series (7.4.5) is given by Seto [28], stated here in a slightly different form as

$$u = \frac{pL^4}{\pi^4 EI} \left(\frac{\omega_1}{\Omega}\right)^2 \left[\frac{\cos\{\pi/L(L/2-x)\sqrt{\Omega/\omega_1}\}}{2\cos\{\pi/2\sqrt{\Omega/\omega_1}\}} + \frac{\cosh\{\pi/L(L/2-x)\sqrt{\Omega/\omega_1}\}}{2\cosh\{\pi/2\sqrt{\Omega/\omega_1}\}} - 1 \right] \cos \Omega t \qquad (7.4.7)$$

It may be noted that the sum of the first 30 terms of (7.4.5) agrees with the deflection calculated from (7.4.7) to at least 8 figures for the values of x/L given in Table 7.4.

A finite element solution may also be obtained by using the values of deflection and slope at t = 0 from Table 7.4 as initial conditions to obtain the steady-state solution. In this case, the 4-element spatial discretization of Fig. 6.5(b) is used, and a preliminary calculation is required to obtain the force vector from equations (7.4.3).



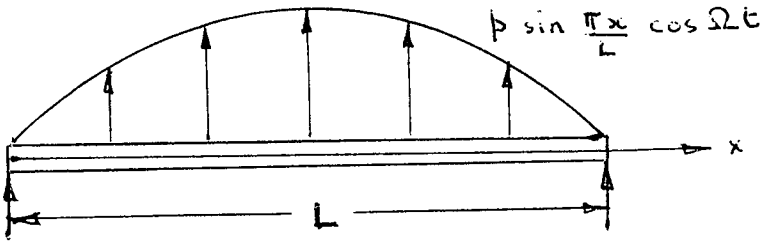
Since the displacements ${}_1u$ and ${}_3u'$ are zero, the corresponding generalized forces are not required; the remainder are found from the assembled matrix for the two elements for which ${}_1p = {}_2p = {}_3p = p$; ${}_1p' = {}_2p' = {}_3p' = 0$

$$\begin{Bmatrix} {}_1M/\ell \\ {}_2F \\ {}_2M/\ell \\ {}_3F \end{Bmatrix} = \frac{\ell}{420} \begin{bmatrix} 22 & 4 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \end{bmatrix} \begin{Bmatrix} p \\ 0 \\ p \\ 0 \\ p \\ 0 \end{Bmatrix}$$

and hence ${}_1M/\ell = pL/48$, ${}_2F = pL/4$, ${}_2M/\ell = 0$, ${}_3F = pL/8$. These values are then used in exactly the same way as discrete forces, and by using the basic temporal finite element with 40 intervals per cycle, and discarding the 7th mode of vibration, the ultimate error for one cycle is found to be 0.30%. This is very similar to the error of 0.38% previously obtained for a discrete force using the same spatial and temporal discretizations.

Example 7.6

Finite element solutions are also readily obtained for non-uniform distributed forces, although exact solutions required for comparison may be much more difficult. If, however, the rate of loading varies sinusoidally in space and harmonically in time it may be shown that the beam vibrates in its first mode only.



The solution for this example is

$$u = \frac{pL^4}{\pi^4 EI} \sin\frac{\pi x}{L} \cos \Omega t. \left(\frac{1}{1-(\frac{\Omega}{\omega_1})^2}\right)$$

If the same spatial discretization as that of Example 7.5 is used, the rates of loading and their spatial derivatives are now:-

$${}_1p = 0, \quad {}_2p = p/\sqrt{2}, \quad {}_3p = p$$

$${}_1p' = \pi p/L, \quad {}_2p' = \pi p/\sqrt{2}L, \quad {}_3p' = 0$$

and by substituting these values into (7.4.3), the nodal forces are

$${}_1M/\ell = 0.00634994pL, \quad {}_2F = 0.169540pL$$

$${}_2M/\ell = 0.00898017pL, \quad {}_3F = 0.119883pL.$$

By using these values in conjunction with the basic temporal element with 40 intervals per cycle, when $\Omega = \sqrt{2}\,\omega_1$ it is found that,

even when only the first 3 modes are included, the ultimate error for one cycle is 0.05%. This very small error is due to the fact that only one mode of vibration is excited.

7.5 Conclusion

Finite element solutions are readily obtained for distributed systems with either discrete or distributed forces. The truncated modal method of solution may be used to obtain a stable solution without any excessive number of time intervals, although several modes should be included to obtain a reasonably accurate solution. Although the stability condition (6.4.4) was obtained for a free vibration, it also gives a very good estimate of the minimum time interval for a variety of forcing functions.

For the same spatial discretization, and the same amount of truncation, errors for free and forced vibrations appear to be similar for distributed systems, as they were earlier shown to be for discrete systems.

The methods used in this chapter would seem to be particularly useful in obtaining transient solutions. A finite element solution for a transient vibration is no more difficult than a steady state solution, while with other methods of solution it may be much more difficult.

CHAPTER 8

DAMPED VIBRATIONS OF DISTRIBUTED SYSTEMS

8.1 Introduction

The response of an undamped distributed system has been examined in the previous chapter, and it has been shown that for a harmonic force the steady-state response is also harmonic, the motion being either in phase, or half a cycle out of phase, with the force. There are no particular difficulties in using the finite element method of solution, and probably the most difficult task is to find the shape of the vibrating beam, using an exact method, to check the accuracy of the finite element method.

With the introduction of damping, additional complications occur; in particular the lag between the force and the displacement is now an unknown quantity, and may not have the same value at all points in the system. This makes an exact solution even more difficult, although it will be shown that damping is quite easily incorporated in the finite element method of solution.

Perhaps an even more fundamental difficulty lies in defining the exact type of damping being investigated and the construction of a suitable model for the system. Two such models are examined below.

8.2 External viscous damping

If a beam is assumed to vibrate in a viscous medium, then the damping forces will be proportional to velocity and in the opposite direction. A suitable approximate model for this case will then be as shown in Fig. 8.1.

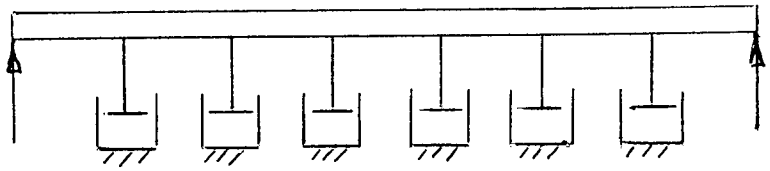


Fig. 8.1

This is only an approximate representation as, if the beam is completely immersed in the fluid, the damping will be distributed and not concentrated into discrete dampers as shown. The partial differential equation for the forced vibrations of a beam subjected to this type of damping is

$$EI \frac{\partial^4 u}{\partial x^4} + c \frac{\partial u}{\partial t} + \rho A \frac{\partial^2 u}{\partial t^2} = p(x,t) \qquad (8.2.1)$$

where c is the damping coefficient per unit length.

A solution of this equation may be obtained, if p is periodic in time, by using Fourier series in space and time.

Considering the case of a simply-supported beam acted on by a distributed force $p \cos \Omega t$ per unit length, where p is uniform along the length of the beam, the Fourier series consists of sine terms only

due to symmetry, and hence it may be assumed that the steady-state solution is given by

$$u = \sum_{i=1,3}^{\infty} \sum_{j=1,2}^{\infty} (a_{ij} \cos j\Omega t + b_{ij} \sin j\Omega t) \sin \frac{i\pi x}{L} \qquad (8.2.2)$$

The uniformly distributed force is in effect half a square wave in space and hence

$$p = \sum_{i=1,3}^{\infty} \frac{4p}{i\pi} \sin \frac{i\pi x}{L} \qquad (8.2.3)$$

By finding the appropriate partial derivatives of (8.2.2) and using these and (8.2.3) in (8.2.1) it is found, by equating coefficients of cos $j\Omega t$ and sin $j\Omega t$ respectively, that in this case only the a_{i1} and b_{i1} coefficients are non-zero, and are therefore designated a_i and b_i. The values of these may be found from the pairs of equations

$$\begin{aligned} a_i[EI(\tfrac{i\pi}{L})^4 - \rho A\Omega^2] + b_i\, c\Omega &= \frac{4p}{i\pi} \\ b_i[EI(\tfrac{i\pi}{L})^4 - \rho A\Omega^2] - a_i\, c\Omega &= 0 \end{aligned} \qquad (8.2.4)$$

$$\text{and hence} \quad a_i = \frac{4pL^4}{i^5\pi^5 EI} \left(\frac{1/\beta_i}{\frac{1}{\beta_i^2} + (\frac{c}{\rho A})^2 \frac{\Omega^2}{\omega_i}^4} \right) \qquad (8.2.5)$$

$$\text{and } b_i = \frac{c}{\rho A} \frac{\Omega}{\omega_i^2} \beta_i a_i$$

where $\beta_i = 1/[1 - (\Omega/\omega_i)^2]$

The deflection at any point on the beam may be found from

$u = A(x) \cos \Omega t + B(x) \sin \Omega t$

$$\text{where} \quad A(x) = \sum_{1,3}^{\infty} a_i \sin(i\pi x/L)$$

$$\text{and } B(x) = \sum_{1,3}^{\infty} b_i \sin(i\pi x/L) \qquad (8.2.6)$$

and similar series may be obtained to find the slope.

The addition of distributed damping to the finite element solution is not difficult, as for uniformly distributed damping it is fairly obvious that the damping matrix is proportional to the mass matrix. The two matrices are in fact identical except for the factor of $\rho A\ell$ in the mass matrix and $c\ell$ for the damping matrix. If the truncated modal method is again used, the damping matrix must be normalized from

$$[C_N] = [u_N]^t[c][u_N]$$

Since [c] is proportional to [m], in this case $[C_N]$ is a diagonal matrix.

$[C_N]$ may not be a diagonal matrix, but since, in practical situations, damping is usually light, Meirovitch [14] suggests that a diagonal matrix may be assumed without much loss of accuracy. It is not, however, necessary to make any such simplification when using the finite element method, and it makes no difference to the method of solution if $[C_N]$ is not a diagonal matrix.

Example 8.1

Consider a simply supported beam acted on by a distributed force $p \cos \sqrt{2}\, \omega_1 t$ per unit length and subjected to distributed damping, of the type shown in Fig. 8.1, of $\rho A \omega_1/\sqrt{2}$ per unit length. This gives a damping ratio of $1/2\sqrt{2}$ and a phase lag of 135° for the first mode. By summing the first 50 terms of the series of (8.2.6), starting conditions for the steady-state vibration are obtained. It is found that the lag varies slightly along the length of the beam from 134.1° at the end to 135.2° at the centre.

By using 4 space elements, 40 intervals per cycle and discarding the last mode, a finite element solution may be obtained, and after one cycle the ultimate error is 0.22%, which is rather less than the error of 0.30% for an undamped system subject to the same distributed force which was obtained in Example 7.5.

8.3 Internal viscous damping

If the damping is caused, not by motion relative to a surrounding viscous medium, but by internal friction in the beam material, a different model may be constructed. It is now assumed that the damping forces are proportional to the rate of change of curvature of the beam giving a differential equation.

$$EI \frac{\partial^4 u}{\partial x^4} + c_1 \frac{\partial}{\partial t}\left(EI \frac{\partial^4 u}{\partial x^4}\right) + \rho A \frac{\partial^2 u}{\partial t^2} = p(x,t) \qquad (8.3.1)$$

Thus for the i'th mode of a simply-supported beam

$$EI \frac{i^4\pi^4}{L^4} u + c_1 EI \frac{i^4\pi^4}{L^4} \frac{du}{dt} + \rho A \frac{d^2u}{dt^2} = p_i \qquad (8.3.2)$$

It will be seen that the coefficient $\frac{du}{dt}$ is equal to the coefficient of u multiplied by the damping coefficient c_1. The damping matrix is therefore proportional to the stiffness matrix and $[C_N]$ is thus again a diagonal matrix. It will also be seen that if $p(x,t) = p \cos \Omega t$, equation (8.3.1) will have a solution very similar to that of (8.2.1) the only difference being that c is replaced by $c_1 EI \frac{i^4\pi^4}{L^4}$ which is equal to $c_1\omega_i^2\rho A$.

Then, for example, the coefficient a_i in (8.2.5) is replaced by

$$\frac{4pL^4}{i^5\pi^5 EI} \left(\frac{1/\beta_i}{\frac{1}{\beta_i^2} + c_1^2\Omega^2} \right) \qquad (8.3.3)$$

The responses for external and internal viscous damping can therefore be made identical for one mode only, by adjusting the value of c or c_1, and will differ for all other modes. If, however, the agreement is made for the lowest frequency, differences between the results for the two types of damping are likely to be very small as the contributions of the higher frequencies become rapidly smaller.

This type of damping is easily incorporated into the finite element solution since the damping matrix is now given by $[c] = c_1[k]$.

8.4 Hysteretic damping

In both of the previous sections it has been assumed that damping forces are proportional to velocity and that for a given amplitude the energy loss per cycle is therefore proportional to the frequency. Meirovitch [14] and other authors agree that this does not agree with observed experimental results and that the damping force is more likely to be independent of frequency, and thus the energy loss per cycle depends only on the amplitude. Bishop and Johnson [29] state that the assumption is closer to that of actual material behaviour than that of viscous damping and show that results may be obtained for hysteretic damping calculations by using a complex modulus $E(1 + j\mu)$.

For a single degree of freedom system with a hysteretic damper the equation of motion then becomes

$$m \frac{d^2u}{dt^2} + k(1 + j\mu)u = F(t) \qquad (8.4.1)$$

since the stiffness k is proportional to E.

The steady-state response is thus found from

$$u = \frac{1}{mD^2+k(1+j\mu)} \; F(t)$$

and if $F(t) = F \cos \Omega t$ the solution is

$$u = \left(\frac{1/\beta}{1/\beta^2 + \mu^2} \cos \Omega t + \frac{\mu}{1/\beta^2 + \mu^2} \sin \Omega t \right) \frac{F}{k} \qquad (8.4.2)$$

where $\beta = 1/[1 - (\Omega/\omega)^2]$

It is then fairly obvious that if a distributed force p cos Ωt acts on a simply-supported beam, for the i'th mode

$$u = \left(\frac{1/\beta_i}{1/\beta_i^2 + \mu^2} \cos \Omega t + \frac{\mu}{1/\beta_i^2 + \mu^2} \sin \Omega t \right) \frac{p_i L^4}{\pi^4 EI} \qquad (8.4.3)$$

and by comparing the coefficient of cos Ωt given in (8.3.3) with the first term of (8.4.3) it will be seen that $c_1^2\Omega^2$ has been replaced by μ^2. or c_1 by $\frac{\mu}{\Omega}$.

It is found that this provides a most convenient method of handling hysteretic damping. The procedure consists of solving the equivalent viscous damping problem and then replacing the viscous damping coefficient c or c_1 by μ/Ω if the system is forced to vibrate with an angular frequency Ω. If many frequencies are present it is necessary to use modal analysis and replace c by μ/ω_i for each mode.

This again presents no problems in the finite element method as modal analysis may already be used to obtain a stable solution. For a hysteretically damped beam the equation of motion is

$$EI \frac{\partial^4 u}{\partial x^4} + \frac{\mu}{\omega} \frac{\partial}{\partial t} \left(EI \frac{\partial^4 u}{\partial x^4} \right) + \rho A \frac{\partial^2 u}{\partial t^2} = p(x,t) \qquad (8.4.4)$$

and for the i'th mode

$$EI \frac{i^4\pi^4}{L^4} u + \frac{\mu}{\omega_i} EI \frac{i^4\pi^4}{L^4} \frac{du}{dt} + \rho A \frac{d^2u}{dt^2} = p_i \qquad (8.4.5)$$

and by comparison with (8.3.2) it will be seen that if a solution for internal viscous damping is available, it is only necessary to replace

c_1 by μ/ω_i. The modal damping matrix in the finite element solution is therefore obtained from $[c_N] = \mu$

$$\begin{bmatrix} K_{11}/\omega_1 & 0 & - & - \\ 0 & K_{22}/\omega_1 & & \\ 0 & 0 & \ddots & K_{nn}/\omega_n \end{bmatrix}$$

Example 8.2

The same values as those of Example 8.1 are used, i.e. a simply supported beam with a distributed force p cos $\sqrt{2}\,\omega_1 t$, but with hysteretic damping instead of external viscous damping. In order to obtain the same first mode response as in Example 8.1 a value of μ equal to 1 must be used in the present example. This is a much higher value than is likely to be encountered with any material (for example, from data given by Lazan and Goodman [30], a figure of the order of 0.001 would be appropriate for structural steel), but by using this high value, any errors in the method of solution should be made very obvious.

An exact solution is obtained by replacing c by $\mu\omega_i\rho A/\Omega$ in (8.2.5) giving

$$a_i = \frac{4pL^4}{i^5\pi^5 EI}\left(\frac{1/\beta_i}{\frac{1}{\beta_i^2} + \mu^2}\right)$$

$$\text{and } b_i = \mu\beta_i a_i \qquad (8.4.6)$$

Values of A(x) and B(x) are then obtained by summation of the series (8.2.6).

The finite element solution is readily obtained by using the damping matrix $[c] = \frac{\mu}{\Omega}[k]$. This solution is then obtained in the usual way using 4 space elements, 40 time elements per cycle, and discarding the fourth mode. After one cycle the ultimate error is 0.20% which is slightly better than the result for viscous damping in Example 8.1 although the difference is not significant.

A comparison of the exact solutions of Examples 8.1 and 8.2 shows very little difference for the two types of damping. For example, the central deflection is found from the following

External viscous damping $u = -0.503945 \cos \Omega t + 0.499947 \sin \Omega t$

Hysteretic damping $u = -0.501920 \cos \Omega t + 0.498028 \sin \Omega t$

The very small differences between the coefficients in the two cases are perhaps rather surprising initially, since the two types of damping are not strictly comparable. It should be remembered that the external damping forces are related to deflection, while the hysteretic damping forces are proportional to curvature. For a particular mode of vibration the curvature is however directly proportional to deflection; by far the largest damping forces are those associated with the first mode, and the two damping parameters are arranged to give identical results for the first mode. The small differences are therefore due to the effects of the higher frequencies, and by examination of (8.2.5) and (8.4.6) it will be seen that the viscous damping forces decay more rapidly than the hysteretic forces as the frequency rises.

It should be noted that although the damping parameters have been matched in these two examples, the two types of damping will give nearly the same results only at this one forcing frequency. If this frequency is increased the system with viscous damping will be more heavily damped than that with hysteretic damping, and a reduction of forcing frequency will produce the opposite effect.

8.5 The transient response of a hysteretically damped system

It has already been pointed out that the finite element method gives a transient solution as readily as a steady-state solution for an undamped system, and this also holds for a damped system. The difficulty here is not to obtain a finite element solution , but to obtain an exact solution for comparison. However by choosing a suitable forcing function an exact solution is possible.

Example 8.3

If a simply-supported beam, which is initially at rest in its equilibrium position, has a uniformly distributed force p suddenly applied, an exact solution is possible. Considering first the case of external viscous damping, by comparison with the known solution for a single degree of freedom system the solution is

$$u = \sum_{i=1,3}^{\infty} \frac{4pL^4}{i^5\pi^5 EI}\left[1 - e^{\frac{-c}{2\rho A}t}\left\{\cos\sqrt{\omega_i^2 - \left(\frac{c}{2\rho A}\right)^2}.t + \frac{\frac{c}{2\rho A}}{\sqrt{\omega_i^2 - \left(\frac{c}{2\rho A}\right)^2}} \sin\sqrt{\omega_i^2 - \left(\frac{c}{2\rho A}\right)^2}.t\right\}\right] \quad (8.5.1)$$

For internal viscous damping c is replaced by $c_1\omega_i^2\rho A$ and hence

$$u = \sum_{i=1,3}^{\infty} \frac{4pL^4}{i^5\pi^5 EI}\left[1-e^{-\frac{c_1\omega_i^2}{2}}\left\{\cos\left(\omega_i\sqrt{1-\left(\frac{c_1\omega_i}{2}\right)^2}\,t\right)\right.\right.$$

$$\left.\left. + \frac{\frac{c_1\omega_i}{2}}{\sqrt{1-\left(\frac{c_1\omega_i}{2}\right)^2}}\sin\left(\omega_i\sqrt{1-\left(\frac{c_1\omega_i}{2}\right)^2}\,t\right)\right\}\right] \qquad (8.5.2)$$

Finally for hysteretic damping c_1 is replaced by μ/ω_i giving

$$u = \sum_{i=1,3}^{\infty} \frac{4pL^4}{i^5\pi^5 EI}\left[1-e^{-\frac{\mu\omega_i}{2}}\left\{\cos\left(\omega_i\sqrt{1-\left(\frac{\mu}{2}\right)^2}\,t\right)\right.\right.$$

$$\left.\left. + \frac{\mu}{2\sqrt{1-\left(\frac{\mu}{2}\right)^2}}\sin\left(\omega_i\sqrt{1-\left(\frac{\mu}{2}\right)^2}.t\right)\right\}\right] \qquad (8.5.3)$$

It is then a simple matter to find the space and time derivatives of (8.5.3) to obtain the slope and velocity. When the summation of the four series is carried out, using the first 50 terms of each series, at intervals of 1/40 of the period of the lowest undamped frequency for a value of μ of 1 as in Example 8.2, values of deflection, slope, linear and angular velocities may be found.

A finite element solution is obtained by the method described in Example 8.2 using 4 space elements, 40 intervals per cycle of the lowest frequency and discarding the last mode. It is thus possible to compare the exact and finite element solution for the same values of time.

With this type of loading, the value most likely to be required is the maximum deflection, and this is found to occur at a time of

approximately 0.575 x $2\pi/\omega_1$. The maximum true error in the displacements as given by the finite element solution for this time is 0.07%. Since the velocities at this time are all very small, percentage true errors would mean very little, so the two solutions are again compared for $t = 2\pi/\omega_1$. The true displacement error is almost the same as before, but the maximum true velocity error is 0.23% which is very similar to errors previously found for steady-state solutions.

8.6 Conclusion

The effect of damping in a distributed system may be allowed for in exactly the same way as in a discrete system. In a distributed system two types of viscous damping and hysteretic damping have been used to generate the appropriate damping matrices and so to obtain solutions. By using temporal finite elements, solutions may be obtained either by direct solution of the equations produced by this method, or by using modal analysis, in which case truncation may be used to give a stable solution. Although, in the examples used, the modal damping matrices were diagonal, this condition is not necessary to obtain a finite element solution. If the modal damping matrix is not diagonal solution by other methods may be difficult.

In this and the two preceding chapters a simplified representation of a beam has been used, in that no allowance has been made for rotary inertia or shear effects. Allowance for rotary inertia alone is readily made by a suitable addition to the mass matrix, but Timoshenko shows that shearing effects have more effect on the natural frequencies than rotary inertia [20]. Shear effects may be allowed for by

modifying the mass matrix as shown by Severn [31], but since the cross-sections are warped by the shear stresses, rotary inertia corrections now become rather complicated. It is however possible to modify both the mass and stiffness matrices to allow for combined rotary inertia and shear effects. There seems to be some doubt concerning the value of the shear correction factor to be used in Severn's stiffness matrix, which is basically concerned with Timoshenko's factor k. Both Cowper [32] and Love [33] think that Timoshenko's value of 0.833 (for ν=1/3) is too low and suggest a value of 0.851. The use of spatial finite elements to obtain the natural frequencies gives very good agreement with Cowper's results when a factor of 1/5 is used in Severn's stiffness matrix. An exact solution for the static deflection of a uniformly loaded simply-supported beam may also be obtained by using the principles of elasticity given by, for example, Timoshenko and Goodier [34], and once again the use of the factor of 1/5 in the stiffness matrix gives close agreement with the elasticity solution.

A detailed description of the modification to the mass matrix is not appropriate here, since this is a spatial discretization problem and not concerned with temporal finite elements. In any case rotary inertia and shear effects normally have little effect on the response of a beam, except for short deep beams. The spatial finite element discretization can then be suitably adjusted to give very accurate results even for these unusual cases.

CHAPTER 9

NON-LINEAR SYSTEMS

9.1 Introduction

This thesis is concerned principally with the behaviour of linear systems, that is systems in which masses, stiffnesses and damping coefficients remain constant. Since, however, in many practical situations these parameters are not constant, any work of this nature would be incomplete without some reference to this type of behaviour. No attempt is made to include a complete analysis of non-linear systems, but it is shown that the finite element method can be applied to systems of this type. For convenience, the simplest possible non-linear vibration, i.e. the free vibration of an undamped single degree of freedom system, is considered, and it is shown that the finite element method does give a solution in this case. By comparison with previous results for linear systems, there seems to be no reason why finite element methods should not work equally well for multi-degree of freedom systems and for those which include damping.

Since damping is not included in the present investigation, only two parameters are involved, i.e. mass and stiffness, and each of these may be a function of time and/or displacement. If an unsymmetrical shaft rotates, its bending stiffness is a function of time, and a system with reciprocating masses will have a varying moment of inertia.

Pashricha and Carnegie [37] point out that this variation increases the speed range over which resonance effects are experienced.

These are time-dependent variations, and another common case is that in which stiffness varies with displacement, e.g. if the stress-strain relationship is non-linear.

All these cases are examined, and since the time-dependent variations are, at least in principle, the easiest to handle, these are investigated first.

9.2 Time-dependent stiffness and mass

An obvious method of approach for systems of this type, is to use the same method as for a linear system with a constant average value for the varying parameter within each interval. This is not very satisfactory as there are then discontinuities in the value of the parameter at the end of each interval.

Much better results can be expected if Hermitian interpolation is used to obtain the value of the varying parameter within each interval. Considering, first of all varying stiffness, the value of the stiffness k within an interval is given by

$$k = \psi_1 k_0 + \psi_2 \dot{k}_0 \tau + \psi_3 k_1 + \psi_4 \dot{k}_1 \tau \qquad (9.2.1.)$$

where $\psi_1 = 1 - 3(t/\tau)^2 + 2(t/\tau)^3$ etc.

and k_0 k_1 $\dot{k}_0$ $\dot{k}_1$ are the stiffnesses and their time derivatives at the beginning and end of the interval.

Thus using Hamilton's principle for a free undamped vibration as $\delta \int_0^\tau (T-U)dt = 0$ where U is the strain energy, the contribution of the strain energy to the integral is

$$\int_0^\tau U dt = \int_0^\tau \frac{1}{2} ku^2$$

$$= \frac{1}{2} \int_0^\tau (\psi_1 k_0 + \psi_2 \dot{k}_0 \tau + \psi_3 k_1 + \psi_4 \dot{k}_1 \tau)(\psi_1 u_0 + \psi_2 \dot{u}_0 \tau + \psi_3 u_1 + \psi_4 \dot{u}_1 \tau)^2 dt \quad (9.2.2)$$

Proceeding in exactly the same manner as for a linear system and obtaining the partial derivatives with respect to u_0 and u_1, the contributions of the stiffness to the two equations are

$$([A_2]\{k\})^t\{u\} \quad \text{and} \quad ([A_4]\{k\})^t\{u\} \qquad (9.2.3)$$

where $[A_2] = \frac{\tau}{5040}\begin{bmatrix} 194 & 32 & 70 & -18 \\ 32 & 6 & 16 & -4 \\ 70 & 16 & 86 & -18 \\ -18 & -4 & -18 & 4 \end{bmatrix}$ and $[A_4] = \frac{\tau}{5040}\begin{bmatrix} -86 & -18 & -70 & 16 \\ -18 & -4 & -18 & 4 \\ -70 & -18 & -194 & 32 \\ 16 & 4 & 32 & -6 \end{bmatrix}$

These functions replace the $[h^I]$ terms of (2.2.13) in equations (2.2.14), so that for a system having a varying stiffness

$$(\frac{m}{30\tau}[3 \quad 4 \quad -3 \quad -1] - ([A_2]\{k\})^t)\{u\} = \{0\}$$

and

$$(\frac{m}{30\tau}[3 \quad -1 \quad -3 \quad 4] - ([A_4]\{k\})^t)\{u\} = \{0\}$$

There are thus two equations from which the displacement and velocity u_1 and $\dot{u}_1$ at the end of the interval may be found in exactly the same way as for a linear system. It is of course necessary to know the stiffness

and its rate of change at the beginning and end of each interval to evaluate the functions in 9.2.3.

Example 9.1

The general solution for a second-order differential equation with variable coefficients is extremely difficult. A study of the literature did however show that a few solutions have been obtained for differential equations of a suitable type. The solution used here is given by Rektorys [35], re-written in terms of the present author's symbols.

If a single degree of freedom system has a constant mass m and a time-dependent stiffness given by $k = \frac{b^2}{(a^2+t^2)^2}$ where a and b are constants, and of constant mass m = 1, the equation of motion for a free undamped vibration is

$$\ddot{u} = - \frac{b^2}{(a^2+t^2)^2} \; u \qquad (9.2.4)$$

having the solution

$$u = \sqrt{a^2+t^2}(C_1 \cos p + C_2 \sin p) \qquad (9.2.5)$$

where $p = \frac{\sqrt{a^2+b^2}}{a} \tan^{-1} \frac{t}{a}$

Taking initial conditions $u_0 = A$ and $\dot{u}_0 = 0$ it is found that $C_1 = \frac{A}{a}$ and $C_2 = 0$.

If, in addition, values are assigned to a and b i.e. a = 1 and b = 10 it is easily seen that

$$u = A\sqrt{1 + t^2} \cos (\sqrt{101} \tan^{-1} t) \qquad (9.2.6)$$

and hence

$$\dot{u} = A[t \cos(\sqrt{101} \tan^{-1} t) - \sqrt{101} \sin(\sqrt{101} \tan^{-1} t)]/\sqrt{1+t^2} \qquad (9.2.7)$$

For comparison with results from the finite element solution, values of u and $\dot{u}$ are calculated from equations (9.2.6) and (9.2.7), and values of u are shown in Fig. 9.1. It will be seen that, as a result of the reduction of stiffness, both the amplitude and period increase with time.

For a finite element solution, values of k and $\dot{k}$ are required; these are readily found by substituting the chosen values of a and b into the expression for k, so that $k = 100/(1+t^2)^2$ and by differentiation $\dot{k} = -400t/(1+t^2)^3$. By using the appropriate value of t at the beginning and end of each interval the stiffness vector{k} in (9.2.3) is obtained. The coefficients of the components of the displacement vector {u} in (9.2.3) then replace the corresponding stiffness terms in the [G] and [H] matrices of (2.2.14), giving two equations from which values of u_1 and $v_1 (=\dot{u}_1\tau)$ may be calculated.

When a time interval of 0.02 units (approximately 1/37 of the period for the first cycle) is used, the finite element solution agrees extremely well with the exact solution. For example, the values of the maximum negative and positive displacements after half and one complete cycle respectively are:

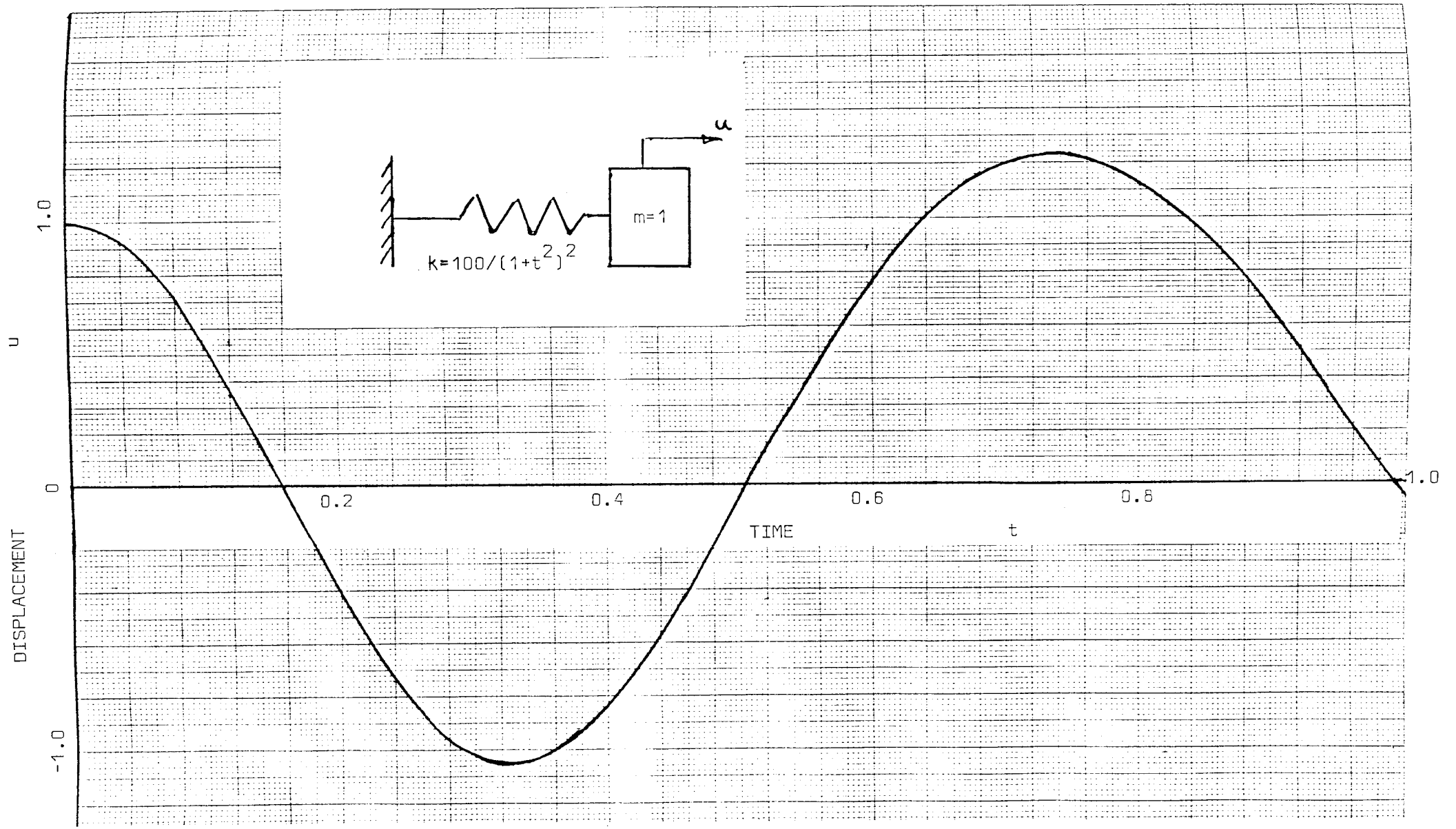


FIG.9.1 FREE NON-LINEAR VIBRATIONS FOR k=k(t)

	Finite element	Exact solution
$\frac{1}{2}$ cycle	-1.049444	-1.049507
1 cycle	1.234582	1.235185

There is thus a difference of the order of only 0.1% after one cycle, and as usual this could be reduced by using a smaller time interval.

Separate variation of mass alone may be treated in the same way as variation of stiffness. Hamilton's principle is again used, but with a varying mass the kinetic energy term in equation (2.2.9) must be modified by discretizating m to give

$$\int_0^\tau T dt = \int_0^\tau \frac{1}{2} m\dot{u}^2 dt$$

$$= \frac{1}{2}\int_0^\tau (\psi_1 m_0+\psi_2 \dot{m}_0\tau+\psi_3 m_1+\psi_4 \dot{m}_1\tau)(\dot{\psi}_1 u_0+\dot{\psi}_2 \dot{u}_0\tau+\dot{\psi}_3 u_1+\dot{\psi}_4 \dot{u}_1\tau)^2 dt \qquad (9.2.8)$$

while the strain energy term remains as in (2.2.9) since k is now constant.

As previously, two equations are required to find the values of u_1 and $\dot{u}_1$ and these are obtained from the partial differential coefficients of the integral of (2.2.9) with respect to $\dot{u}_0$ and $\dot{u}_1$. The contributions of (9.2.8) to these equations are respectively

$$([B_2]\{m\})^t\{u\} \qquad \text{and} \qquad ([B_4]\{m\})^t\{u\} \qquad (9.2.9)$$

$$\text{where } [B_2] = \frac{1}{840\tau}\begin{bmatrix} -12 & 6 & 96 & -18 \\ 86 & 7 & 26 & -5 \\ 12 & -6 & -96 & 18 \\ -14 & -1 & -14 & 1 \end{bmatrix}$$

$$[B_4] = \frac{1}{840\tau}\begin{bmatrix} 96 & 18 & -12 & -6 \\ -14 & -1 & -14 & 1 \\ -96 & -18 & 12 & 6 \\ 26 & 5 & 86 & -7 \end{bmatrix}$$

and $\{m\} = [m_0 \ \dot{m}_0\tau \ m_1 \ \dot{m}_1\tau]^t$

The coefficients of k are obtained from the second and fourth rows of the $[h^I]$ matrix of (2.2.13).

The replacement of the mass coefficients in the [G] matrix of (2.2.14) by the modification of (9.2.9) unfortunately leads to incorrect results for u_1 and $\dot{u}_1$. This is due to the nature of the problem which is one of varying mass. If, for example the mass is increasing, the additional mass must be added with the same velocity as that of the vibrating mass, i.e. if the mass is increasing at a rate $\dot{m}$ when the velocity is $\dot{u}$, momentum must be supplied at a rate $\dot{m}\dot{u}$ which is equivalent to the application of a force $F = \dot{m}\dot{u}$. The contribution of this force term to the integral of (2.2.9) is found by discretizing m, u and u giving

$$\int_0^\tau (\dot{\psi}_1 m_0 + \dot{\psi}_2 \dot{m}_0\tau + \dot{\psi}_3 m_1 + \dot{\psi}_4 \dot{m}_1\tau)(\dot{\psi}_1 \underline{u}_0 + \dot{\psi}_2 \underline{\dot{u}}_0\tau + \dot{\psi}_3 \underline{u}_1 + \dot{\psi}_4 \underline{\dot{u}}_1\tau)$$

$$\times (\psi_1 u_0 + \psi_2 \dot{u}_0\tau + \psi_3 u_1 + \psi_4 \dot{u}_0\tau)dt \qquad (9.2.10)$$

The first two brackets in (9.2.10) replace F in (2.2.9), and as explained in 2.2 the underlined quantities in the second bracket of (9.2.10) must be treated as constants during partial differentiation. By differentiating (9.2.10) with respect to $\dot{u}_0$ and $\dot{u}_1$ two further sets of coefficients of {u} are obtained, i.e.

$$([C_2]\{m\})^t \quad \text{and} \quad ([C_4]\{m\})^t \tag{9.2.11}$$

$$\text{where } [C_2] = \frac{1}{840\tau} \begin{bmatrix} 108 & 6 & -108 & 18 \\ 6 & 7 & -6 & -1 \\ -108 & -6 & 108 & -18 \\ 18 & -1 & -18 & 5 \end{bmatrix}$$

$$\text{and } [C_4] = \frac{1}{840\tau} \begin{bmatrix} -108 & -18 & 108 & -6 \\ -18 & -5 & 18 & 1 \\ 108 & 18 & -108 & 6 \\ -6 & 1 & 6 & -7 \end{bmatrix}$$

The final correct form of the equations for varying mass and constant stiffness is

$$\begin{aligned} (([B_2]+[C_2])\{m\})^t - \frac{k\tau}{420}[\,22 \quad 4 \quad 13 \quad -3])\{u\} &= \{0\} \\ (([B_4]+[C_4])\{m\})^t - \frac{k\tau}{420}[-13 \quad -3 \quad -22 \quad 4])\{u\} &= \{0\} \end{aligned} \tag{9.2.12}$$

where the coefficients of k are taken from (2.2.14).

Example 9.2

The differential equation of Example 9.1 can also result from variation of mass instead of stiffness so that if $m = (1+t^2)^2$ and $k = 100$, for a free vibration $(1+t^2)^2 \ddot{u} = -100u$. This is the same equation as that of Example 9.1 and will therefore have the same exact solution.

A finite element solution may also be obtained using equations (9.2.12) with 50 time intervals, each of 0.02 units. The finite element solution not only agrees well with the exact solution, but differs from the previous variable stiffness finite element solution only in the sixth figure.

It has thus been shown that the finite element method will give very accurate results for variation of stiffness or mass alone, and it only remains to combine these two solutions to obtain a general solution for the case where both stiffness and mass vary simultaneously. This is quite simply achieved by substituting the stiffness terms from (9.2.3) into equations (9.2.12), and substituting for the [B] and [C] matrices to give

$$\left[\left(\frac{1}{840\tau}\begin{bmatrix} 96 & 12 & -12 & 0 \\ 92 & 14 & 20 & -6 \\ -96 & -12 & 12 & 0 \\ 4 & -2 & -32 & 6 \end{bmatrix}\{m\}\right)^t - \left(\frac{\tau}{5040}\begin{bmatrix} 194 & 32 & 70 & -18 \\ 32 & 6 & 16 & -4 \\ 70 & 16 & 86 & -18 \\ -18 & -4 & -18 & 4 \end{bmatrix}\{k\}\right)^t\right]\{u\}=\{0\}$$

$$\left[\left(\frac{1}{840\tau}\begin{bmatrix} -12 & 0 & 96 & -12 \\ -32 & -6 & 4 & 2 \\ 12 & 0 & -96 & 12 \\ 20 & 6 & 92 & -14 \end{bmatrix}\{m\}\right)^t - \left(\frac{\tau}{5040}\begin{bmatrix} -86 & -18 & -70 & 16 \\ -18 & 4 & -18 & 4 \\ -70 & -18 & -194 & 32 \\ 16 & 4 & 32 & -6 \end{bmatrix}\{k\}\right)^t\right]\{u\}=\{0\} \qquad (9.2.13)$$

Example 9.3

If $m = 1+t^2$ and $k = \frac{100}{1+t^2}$, for a free vibration $(1+t^2)\ddot{u} = -\frac{100}{1+t^2}u$,

which is the same differential equation as that of Examples 9.1 and 9.2 so that the exact solution for this case of variable mass and stiffness is already known.

A finite element solution is readily obtained by using equations (9.2.13), again with 50 time intervals of 0.02 units. Once again the finite element solution agrees extremely well with the exact solution, and differs only in the sixth figure from the finite element results of Examples 9.1 and 9.2.

Although, because of difficulties in obtaining an exact solution, only one non-linear problem has been solved, it has been shown that the finite element method gives very accurate results in this case. Since both the function and the numerical values are arbitrary it is very unlikely that there is anything special about this particular example, and the method should work with any system in which stiffness and/or mass are functions of time.

Only free vibrations have been considered, but forced vibrations can be solved by adding the force terms in (2.2.14) to (9.2.13) without alteration. Linear damping can also be added in the same way, and non-linear damping may be allowed for by discretizating c and proceeding in the same manner as when considering variations of stiffness and mass.

9.3 Displacement - dependent stiffness

In many practical cases stiffness may vary with displacement, either because of non-linearity in material behaviour or as a result of the geometry of the system. The finite element method of solution is again able to allow for this variation in stiffness. If the mass is constant, only the strain energy of the system differs from that in a linear system, and must now be found from

$$U = \int_0^u kx\,dx \qquad \text{where } k = F/x \tag{9.3.1}$$

The stiffness k may now be found in the usual way by Hermitian interpolation, i.e.

$$k = \psi_1 k_0 + \psi_2 k_0' \delta + \psi_3 k_1 + \psi_4 k_1' \delta \tag{9.3.2.}$$

where k_0 and k_1 are the values of k corresponding to displacements u_0 and u_1, k_0' and k_1' are their rates of change with respect to displacement, $\delta = u_1 - u_0$ and $\psi_1 = 1-3(x/\delta)^2+2(x/\delta)^3$ etc.

Hence by substituting (9.3.2) into (9.3.1), this integration may be performed independently of any integration with respect to time, giving an equivalent stiffness for the interval $u_0 \leqslant u \leqslant u_1$.

$$K = \frac{1}{2} k_0 + \frac{1}{12} k_0' \delta + \frac{1}{2} k_1 - \frac{1}{12} k_1' \delta \tag{9.3.3}$$

Now, if k is a known function of u, values of k_0 and k_0' may be found, but if the usual method of solution described in 2.3 is used an immediate difficulty appears in that, since the displacement u_1 at the end of a prescribed time interval τ is unknown, k_1 and k_1' are also

unknown quantities. A rather more elaborate form of solution is therefore needed.

It has been shown in equation (3.2.7) that the final displacement u_1 may be predicted from the initial displacement and initial velocity. This equation is of the form $u_1 = Au_0 + Bv_0$ where $v_0 = \dot{u}_0\tau$ and A and B are functions of τ. Thus it should be possible to prescribe a final value of u and find the time τ required to reach this position. This can indeed be done and the required manipulation of equation (3.2.7) gives

$$\frac{p^4\dot{u}_0\tau^5}{28} + \frac{p^4(7u_0-2u_1)}{28}\tau^4 - \frac{13p^2\dot{u}_0\tau^3}{7} - \frac{p^2(96u_0+9u_1)}{14}\tau^2+15\dot{u}_0\tau+15(u_0-u_1) = 0 \qquad (9.3.4)$$

where $p^2 = K/m$, the appropriate value for K being found from (9.3.3)

Equation (9.3.4) may be solved by approximate methods, and since a reasonable estimate of the value of τ can be made, Newton's method should give rapid convergence. This method is indeed found to give 8 figure accuracy in about 5 iterations.

Once the value of τ is known the final velocity may be found from equation (3.2.7) by replacing k/m by p^2. It should be noted that since, in general, the value of τ will be different for each interval, it is not possible to use the composite variable $v_1(=\dot{u}_1\tau)$ but the actual velocity $\dot{u}_1$ must be used.

This method should also give satisfactory results for linear systems, although it would not be so used because of its complexity. However, an initial check on the accuracy of the method may be obtained by repeating the calculations of Example 2.1 for the free undamped vibrations of a linear system. In this case $k_0 = k_1 = k$ and $k_0' = k_1' = 0$ so that equation (9.3.3.) gives $K = k$. Thus $p^2 = k/m = \omega^2$. Taking initial conditions $u_0 = A$ and $\dot{u}_0 = 0$, and dividing one cycle into 10 intervals so that $u_1 = A \cos 2\pi/10$ etc., the times for each of these intervals are equal for an exact solution.

By using the usual step-by-step finite element method, equation (9.3.4) may be used to find these 10 time intervals. Because of the approximate nature of the solution it might be expected that the times of the individual intervals would be slightly different, but rather surprisingly they are all found to be identical (to 6 figures) but slightly low. The resulting time for one cycle is calculated as $6.2636\sqrt{m/k}$ which is 0.3% low.

This is perhaps a rather artificial problem as the period is known to be $2\pi\sqrt{m/k}$, so that the result is assumed in calculating values of displacement. A much more logical method is to divide the total travel into a number of displacement increments, which may or may not be equal, and to proceed without making any assumptions concerning time. When this procedure is followed, using 10 equal displacement increments per cycle (values of u being A, 0.6A, 0.2A etc.), the value obtained for the period, $6.2104\sqrt{m/k}$, is 1.2% low. This increase in error compared with the equal time interval solution results from

the fact that as the time interval increases the error also increases, and the increased errors in the large intervals are not compensated for by corresponding reduced errors in the small intervals.

This error can of course be reduced by increasing the number of intervals, and with 80 intervals per cycle the true error in the period is only -0.07%.

Since the method has been found to be satisfactory for linear systems, the next step is to obtain a finite element solution for a non-linear system for which an exact solution is available.

Example 9.4

Taking a spring with the non-linear characteristic $F = au^3$ where a is a constant

$$k = au^2$$

and $k' = 2au$

Hence, by substitution in equation (9.3.3) and letting $p^2 = K/m$

$$p^2 = \frac{a}{m}\left[(u_0^2 + u_1^2)/2 - (u_0 - u_1)^2/6\right]$$

By choosing convenient values of u, (which may be equally or unequally spaced), the value of p^2 may thus be found for each interval and the corresponding time is then calculated from equation (9.3.4)

Timoshenko [20] gives an exact solution for the free vibrations of a mass m attached to a spring having the non-linear characteristic

given above, and shows that the period is given by

$$t = \frac{4\sqrt{2}}{\beta A} \int_0^1 \frac{dr}{\sqrt{1-r^4}} \qquad (9.3.5)$$

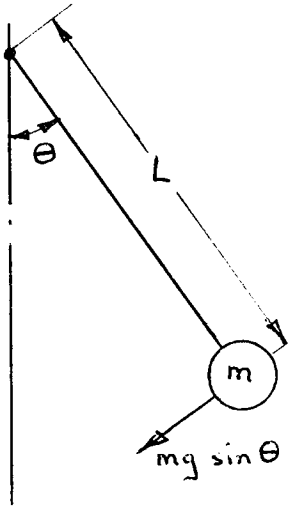
where A is the amplitude, $\beta^2 = a/m$ and $r = u/A$.

The integral in (9.3.5) may be evaluated by reference to tables of gamma functions given by, for example, Rektorys [35], giving $t = 7.4163/\beta A$.

By using 80 equal increments of displacement in a finite element solution, the value of the period obtained is $7.4388/\beta A$. This represents an error of +0.30%, which is higher than the error in the previous linear system because there is a further approximation in discretizing the stiffness of the non-linear system.

It has, however, been shown that the finite element method does give a satisfactory solution for a non-linear system.

Example 9.5



A well-known case of a non-linear system for which an exact solution is available is that of a simple pendulum having a large amplitude. A finite element solution may be obtained by considering the force mg sinθ and the displacement Lθ. The equivalent stiffness is given by

$$k = \frac{mg}{L} \frac{\sin\theta}{\theta}$$

and

$$k' = \frac{mg}{L}\left(\frac{\theta \cos\theta - \sin\theta}{\theta^2}\right)$$

If these expressions are used in equation 9.3.3 and (9.3.4), by using 80 equal displacement intervals per cycle, periods may be obtained for various amplitudes. These are given in Table 9.1.

An exact solution for this problem is given by Timoshenko [20] in terms of an elliptic integral, the period being

$$t = 4\sqrt{\frac{L}{g}}\, F(k,\frac{\pi}{2}) \qquad (9.3.6)$$

where $F(k,\frac{\pi}{2})$ is an elliptic integral of the first kind, and $k = \sin(A/2)$, A being the amplitude. The value of this integral may be found by reference to, for example Abramowitz and Stegun [36] who give

$$F(k, \tfrac{\pi}{2}) = \frac{\pi}{2}\left[1 + (\tfrac{1}{2})^2 k^2 + (\tfrac{1.3}{2.4})^2 k^4 + \ldots\ldots\right]$$

The exact values of the periods thus calculated from (9.3.6) are also given in Table 9.1.

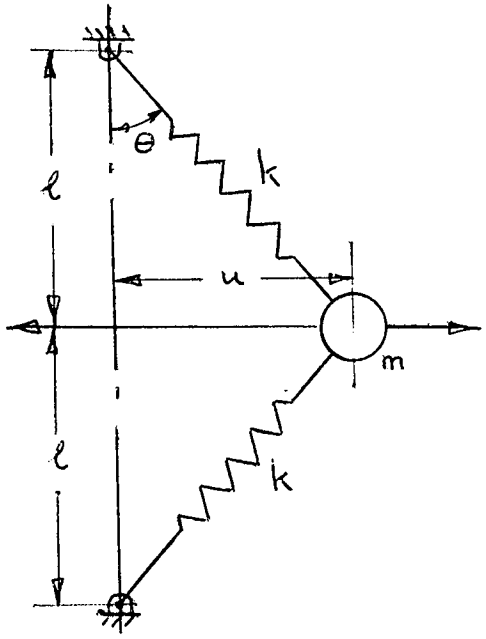
Amplitude	Period/$\sqrt{\frac{L}{g}}$	
degrees	Finite element solution	Exact solution
5	6.2798	6.2862
10	6.2887	6.2952
20	6.3245	6.3314
30	6.3699	6.3926
60	6.7313	6.7430
90	7.3916	7.4163
120	8.5724	8.6260

Table 9.1

It will be seen that the error in the period found by the finite element method is in all cases less than 1%, and is much less than this value for small amplitudes.

Finite element solutions may be obtained for the vibrations of much more complicated systems than those of Examples 9.4 and 9.5, provided that it is possible to obtain the values of k and k' required in equation (9.3.3). The next example concerns the free vibrations of a system for which a finite element solution is obtained without difficulty. Other methods of solution may require the introduction of approximations which can be justified for small displacements only.

Example 9.6



In this example the non-linearity is due to the geometry of the system, and it is readily shown that the horizontal force required to give a displacement u is $2k(1 - \cos\theta)u$ in the case where the spring forces are zero when u = 0. It therefore follows that the values of stiffness and its derivative required in equation (9.3.3) to give a finite element solution are $2k(1 - \cos\theta)$ and $k/\ell \sin 2\theta \cos\theta$ respectively where $\theta = \tan^{-1}(u/\ell)$. Thus equation (9.3.3) may be used to find the value for K for any pair of displacements u_0 and u_1, and equation (9.3.4) then gives the approximate value of τ.

In this case 100 elements per cycle are used to obtain finite element solutions for free vibrations of varying amplitudes. It is found to be most convenient to measure the angular amplitude α(maximum value of θ), and Fig. 9.2 shows how the period of the vibration varies with α.

A solution for this problem is obtained by Timoshenko [20] who uses an approximate expression for the horizontal force which gives a

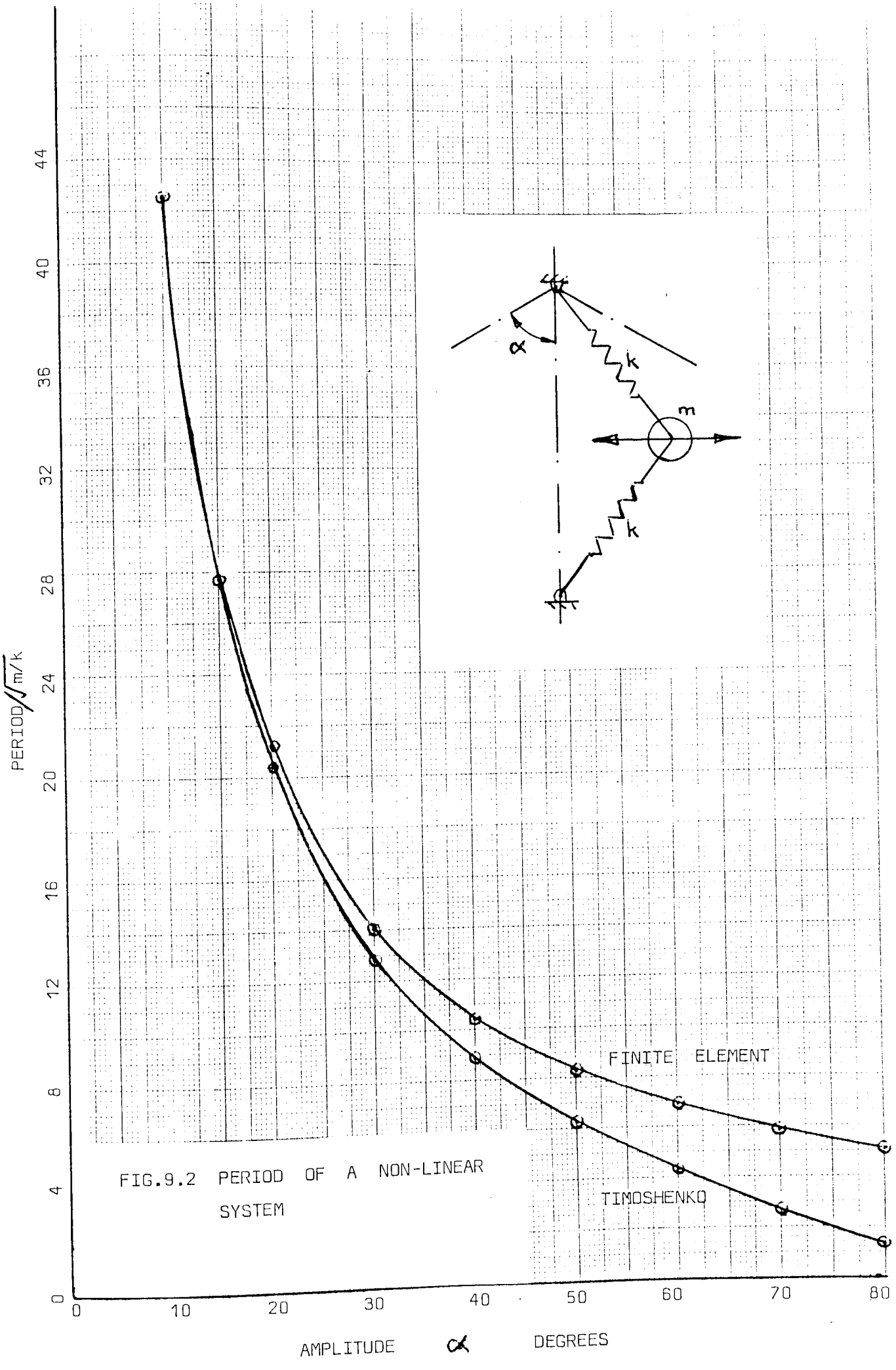


FIG.9.2 PERIOD OF A NON-LINEAR SYSTEM

cubic relationship between force and displacement. This example is then solved in exactly the same way as Example 9.4 where this same relationship appears. This approximation is reasonably accurate for small values of θ only. Fig. 9.2 shows the finite element results and also the results calculated from Timoshenko's approximate solution, and it is obvious that, except for small angles, the two sets of results differ considerably. Although it is likely that the finite element solution is the more accurate, a more definite confirmation would be more satisfying. Because of the complicated relationship between displacement u and acceleration

$$\text{i.e. } m\frac{d^2u}{dt^2} = -2ku\left(1 - \frac{1}{\sqrt{1+u^2}}\right) \tag{9.3.7}$$

it appears that an exact solution for the period will be very difficult to obtain. It is, however, comparatively easy, by equating the sum of the strain energy of the springs and the kinetic energy of the mass to zero, to show that the velocity of the mass is

$$\frac{du}{dt} = \pm \sqrt{2[(\sec\alpha - 1)^2 - (\sec\theta - 1)^2]} \tag{9.3.8}$$

and, in particular, in mid position when $\theta = 0$

$$\left(\frac{du}{dt}\right)_{max} = \pm\sqrt{2}(\sec\alpha - 1) \tag{9.3.9}$$

Inspection of the finite elements results shows that, in the first cycle, the two values of maximum velocity (of opposite sign) differ from each other by less than 0.0001%, even for values of α as large as 80°, and by less than 0.01% from the predictions of equation (9.3.9). Since,

in the finite element solution, the velocity is calculated directly from the time interval, it seems reasonable to deduce that the average error in the calculated times is not greater than 0.01%, and that the values obtained for the periods are extremely accurate.

This example gives some indication of the wide range of non-linear problems which can be solved by the finite element method.

9.4 Conclusion

By suitably modifying the usual finite element method of solution, non-linear vibration problems for freely vibrating single degree of freedom systems may be solved. Since the damping coefficient may be discretized in exactly the same manner as stiffness and mass, non-linear damping effects may also be allowed for. The characteristics of any forcing function are independent of the properties of the system, and therefore the solution may be extended to deal with forced damped non-linear systems having one degree of freedom.

Multi-degree of freedom systems present no additional problems when time-dependent variations are considered, but solutions may be more difficult to obtain when displacement-dependent parameters are involved In this case, the method of solution used in 9.3 may not be the best, as only one final value of displacement may be specified, leaving the other displacements and the interval time as unknown variables to be found.

It may therefore be preferable to use the methods of Chapter 4 using a specified time interval, and finding the displacements by an iterative method. The application of finite element methods to multi-degree of freedom non-linear systems may well repay further study.

CHAPTER 10

CONCLUSION

Of the various methods available for obtaining the response of a mechanical system to an external stimulus one of the methods giving an exact solution would obviously be preferred. Unfortunately, except for relatively easy applications, these methods become very tedious. One of the numerical methods of solution must therefore be employed for more complicated systems and for "difficult" forcing functions.

Probably the most widely used of these numerical methods is that based on finite difference approximations and these may be used exclusively as, for example, by Geers and Sobel [38] employing numerical integration techniques as suggested by Henrici [39]. Alternatively, a mixture of spatial finite elements and finite difference time discretization as used by Wu and Witmer [40] may be suitable.

Although finite difference methods have been found to give satisfactory solutions for a wide range of problems they do have the disadvantage that boundary or initial conditions normally require special procedures.

Finite element methods, however, appear to have no such disadvantages; boundary or initial conditions may be explicitly stated and

this method of discretization is suitable for use in both space and time. In the time domain, even the basic finite element is more accurate than the finite difference approximation using the same number of steps, and the refined finite element can be expected to give even more accurate results.

For distributed systems, separate discretizations in space and time give very satisfactory results, and, indeed, provided that mass, stiffness and damping matrices are available, a solution may always be obtained by using temporal finite elements.

The finite element solution may be unstable, but the necessary condition for stability has been established, and so it may be possible to prevent instability by increasing the number of temporal elements, or intervals, in a given time. If this requires an excessive number of intervals the truncated modal method may be used to remove the contributions of the higher frequencies, which cause the instability, from the solution. This will, of course, reduce the accuracy of the solution, but if the method is used with care, the reduction in accuracy should be small. It is also necessary, when using modal analysis, to solve the eigenvalue problem to find natural frequencies and modal shapes, a process which is not normally required. This presents no particular difficulties as standard computer procedures are available, but it does increase computing time. On balance, however, there is probably a saving in computation time when the truncated modal method is used compared with the use of the greater number of intervals required to avoid instability.

It is found that by using finite elements in the time domain transient solutions are obtained without difficulty and without any restriction on the forcing function. Various types of damping may also be incorporated in the solution.

The majority of the examples have been solved by using a step-by-step process, but, alternatively, values of all variables could be calculated simultaneously by coupling the finite elements in time. This procedure does give some improvement in accuracy, but much larger matrices have to be stored. If increased accuracy is required it is much more simply obtained by increasing the number of elements, or, if the basic element has been used initially, by replacing it with the refined element.

For distributed systems only it is possible to replace the separate space and time finite elements by combined space-time elements. Although this appears to be an attractive method of obtaining simultaneously the complete history of a system in space and time, in practice the choice of a suitable element is very difficult and the compilation of matrices is extremely laborious. Since there appears to be no particular advantage to be obtained by using this method, the extra complications appear to rule it out as a practical method of solution for vibration problems.

The principal aim of this work has been to investigate methods of solution in the time domain, and so no attempt has been made to deal with difficult spatial systems. The temporal discretizations can however be

applied to any system which can be discretized in space. Vibration problems of non-uniform beams may, for example, be solved by using the stiffness matrix for a tapered element given by Howard [41]. Allowance for axial forces may be made by using the matrices derived by Paz and Dung [42], while Tada and Lee [43] deal with large displacements by using spatial finite elements. The beam need not even be straight if the curved finite element of Davis et al. [44] is used.

The method of solution in time is in no way changed if 2- and 3-dimensional systems are to be studied, and provided that mass and stiffness matrices are available temporal finite elements may be used.

In these spatial discretizations, the matrices may be large and the determination of eigenvalues and eigenvectors as described by Jennings and Orr [45] may be of interest. If the truncated modal method of solution is to be used, only the lowest frequencies are required and these are the values obtained by using the methods of Wright and Miles[46].

Perhaps one of the most interesting aspects of temporal finite element methods is the relative ease with which they can be modified to give solutions for non-linear systems. Both displacement- and time-dependent variations of parameters have been studied, and although solutions have been obtained for single degree of freedom systems only, there seems to be no reason why the same principles should not be applied to multi-degree of freedom systems. The methods to be used are not, perhaps, immediately obvious in all cases, and a study of this problem would probably be of considerable interest.

APPENDIX A

A.1 Approximate derivatives for forcing functions

If a number of values of an applied force, F_0, F_1... at equal time intervals τ, are known, a simple approximation for the first derivative is $\dot{F}_i \simeq (F_{i+1} - F_{i-1})/2\tau$. The errors introduced by this approximation are likely to be greater than those due to the use of temporal finite elements.

A more accurate approximation is obtained from 5 values of F at $t = 0, \tau, 2\tau, 3\tau, 4\tau$ and by assuming that

$$F = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \qquad (A.1.1)$$

By substituting values $F = F_0$ when $t = 0$, $F = F_1$ when $t = \tau$ etc. equations of the form

$$\{F\} = [T]\{a\} \qquad (A.1.2)$$

are obtained, and hence

$$\{a\} = [T]^{-1}\{F\} \qquad (A.1.3)$$

In this case [T] is a 5 x 5 matrix, is a function of τ only, and is easily inverted. When the values of {a} are substituted into (A.1.1), the required derivative is found by differentiation with respect to t.

$$\text{i.e. } \dot{F} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 \qquad (A.1.4)$$

Since values of $\dot{F}$ may be required at any of the 5 nodes ($t = 0, \tau, 2\tau, 3\tau, 4\tau$), by substituting these values of t into (A.1.4), 5 different approximations are obtained.

$$\begin{Bmatrix} \dot{F}_0 \\ \dot{F}_1 \\ \dot{F}_2 \\ \dot{F}_3 \\ \dot{F}_4 \end{Bmatrix} = \frac{1}{12\tau} \begin{bmatrix} -25 & 48 & -36 & 16 & -3 \\ -3 & -10 & 18 & -6 & 1 \\ 1 & -8 & 0 & 8 & -1 \\ -1 & 6 & -18 & 10 & 3 \\ 3 & -16 & 36 & -48 & 25 \end{bmatrix} \begin{Bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \qquad \text{(A.1.5)}$$

These approximations are identical to the finite difference approximations given, for example, by Salvadori and Baron [13], and are found to give errors of the same order as the basic finite element errors when 20 intervals per cycle are used.

When the refined finite element is used, both the first and second derivatives of F are required, and although approximations for $\ddot{F}$ can be obtained by differentiating (A.1.4), the errors due to the approximations are much larger than those introduced by the refined finite element. More accurate approximations are obtained by assuming that

$$F = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \qquad \text{(A.1.6)}$$

Seven values of F are now required, but otherwise the procedure is exactly the same as for the simpler polynomial of (A.1.1).

Approximations for $\dot{F}$ and $\ddot{F}$ are now required at 7 nodes, i.e.

$$\begin{Bmatrix} \dot{F}_0 \\ \dot{F}_1 \\ \dot{F}_2 \\ \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix} = \frac{1}{180\tau} \begin{bmatrix} -441 & 1080 & -1350 & 1200 & -675 & 216 & -30 \\ -30 & -231 & 450 & -300 & 150 & -45 & 6 \\ 6 & -72 & -105 & 240 & -90 & 24 & -3 \\ -3 & 27 & -135 & 0 & 135 & -27 & 3 \\ 3 & -24 & 90 & -240 & 105 & 72 & -6 \\ -6 & 45 & -150 & 300 & -450 & 231 & 30 \\ 30 & -216 & 675 & -1200 & 1350 & -1080 & 441 \end{bmatrix} \begin{Bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{A.1.7}$$

$$\begin{Bmatrix} \ddot{F}_0 \\ \ddot{F}_1 \\ \ddot{F}_2 \\ \ddot{F}_3 \\ \ddot{F}_4 \\ \ddot{F}_5 \\ \ddot{F}_6 \end{Bmatrix} = \frac{1}{180\tau^2} \begin{bmatrix} 812 & -3132 & 5265 & -5080 & 2970 & -972 & 137 \\ 137 & -147 & -255 & 470 & -285 & 93 & -13 \\ -13 & 228 & -420 & 200 & 15 & -12 & 2 \\ 2 & -27 & 270 & -490 & 270 & -27 & 2 \\ 2 & -12 & 15 & 200 & -420 & 228 & -13 \\ -13 & 93 & -285 & 470 & -255 & -147 & 137 \\ 137 & 1972 & 2970 & -5080 & 5265 & -3132 & 812 \end{bmatrix} \begin{Bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{A.1.8}$$

Errors in the approximations for $\ddot{F}$ are greater than those for $\dot{F}$, but are found to be of the same order as the refined element discretization errors when 20 intervals per cycle are used.

A.2 Matrices for the refined temporal element

$$[h^{\circ}] = \frac{1}{1260\tau}\begin{bmatrix} 1800 & 270 & 15 & -1800 & 270 & -15 \\ & 288 & 21 & -270 & -18 & 6 \\ & & 2 & -15 & -6 & 1 \\ & & & 1800 & -270 & 15 \\ & \text{Symm.} & & & 288 & -21 \\ & & & & & 2 \end{bmatrix}$$

$$[h^{I}] = \frac{\tau}{55440}\begin{bmatrix} 21720 & 3732 & 281 & 6000 & -1812 & 181 \\ & 832 & 69 & 1812 & -532 & 52 \\ & & 6 & 181 & -52 & 5 \\ & & & 21720 & -3732 & 281 \\ & \text{Symm.} & & & 832 & -69 \\ & & & & & 6 \end{bmatrix}$$

$$[h^{II}] = \frac{1}{5040}\begin{bmatrix} -2520 & 660 & 60 & 2520 & -660 & 60 \\ -660 & 0 & 5 & 660 & -156 & 13 \\ -60 & -5 & 0 & 60 & -13 & 1 \\ -2520 & -660 & -60 & 2520 & 660 & -60 \\ 660 & 156 & 13 & -660 & 0 & 5 \\ -60 & -13 & -1 & 60 & -5 & 0 \end{bmatrix}$$

A.3. Matrices for space-time element

The following matrices are obtained from the shape function of (6.6.10), i.e.

$$u = a_1 + a_2x + a_3t + a_4x^2 + a_5xt + a_6t^2 + a_7x^3 + a_8x^2t + a_9xt^2$$

$$+ a_{10}t^3 + a_{11}x^3t + a_{12}x^2t^2 + a_{13}xt^3 + a_{14}x^3t^2 + a_{15}x^2t^3 + a_{16}x^3t^3$$

$$[m^*] = \frac{\rho A \ell}{25200\tau}$$

$$\begin{bmatrix}
11232 & 1584 & 936 & 132 & 3888 & -936 & 324 & -78 & -11232 & -1584 & 936 & 132 & -3888 & 936 & 324 & -78 \\
 & 288 & 132 & 24 & 936 & -216 & 78 & -18 & -1584 & -288 & 132 & 24 & -936 & 216 & 78 & -18 \\
 & & 1248 & 176 & 324 & -78 & 432 & -104 & -936 & -132 & -312 & -44 & -324 & 78 & -108 & 26 \\
 & & & 32 & 78 & -18 & 104 & -24 & -132 & -24 & -44 & -8 & -78 & 18 & -26 & 6 \\
 & & & & 11232 & -1584 & 936 & -132 & -3888 & -936 & 324 & 78 & -11232 & 1584 & 936 & -132 \\
 & & & & & 288 & -132 & 24 & 936 & 216 & -78 & -18 & 1584 & -288 & -132 & 24 \\
 & & & & & & 1248 & -176 & -324 & -78 & -108 & -26 & -936 & 132 & -312 & 44 \\
 & & & & & & & 32 & 78 & 18 & 26 & 6 & 132 & -24 & 44 & -8 \\
 & & & & & & & & 11232 & 1584 & -936 & -132 & 3888 & -936 & -324 & 78 \\
 & & & & & & & & & 288 & -132 & -24 & 936 & -216 & -78 & 18 \\
 & & & & & & & & & & 1248 & 176 & -324 & 78 & 432 & -104 \\
 & & & & & & & & & & & 32 & -78 & 18 & 104 & -24 \\
 & & & & & & & & & & & & 11232 & 1584 & -936 & 132 \\
 & & SYMM. & & & & & & & & & & & 288 & 132 & -24 \\
 & & & & & & & & & & & & & & 1248 & -176 \\
 & & & & & & & & & & & & & & & 32
\end{bmatrix}$$

$$[k^*] = \frac{EI_\tau}{2520\ell^3}$$

$$\begin{bmatrix}
11232 & 5616 & 1584 & 792 & -11232 & 5616 & -1584 & 792 & 3888 & 1944 & -936 & -468 & -3888 & 1944 & 936 & -468 \\
 & 3744 & 792 & 528 & -5616 & 1872 & -792 & 264 & 1944 & 1296 & -468 & -312 & -1944 & 648 & 468 & -156 \\
 & & 288 & 144 & -1584 & 792 & -288 & 144 & 936 & 468 & -216 & -108 & -936 & 468 & 216 & -108 \\
 & & & 96 & -792 & 264 & -144 & 48 & 468 & 312 & -108 & -72 & -468 & 156 & 108 & -36 \\
 & & & & 11232 & -5616 & 1584 & -792 & -3888 & -1944 & 936 & 468 & 3888 & -1944 & -936 & 468 \\
 & & & & & 3744 & -792 & 528 & 1944 & 648 & -468 & -156 & -1944 & 1296 & 468 & -312 \\
 & & & & & & 288 & -144 & -936 & -468 & 216 & 108 & 936 & -468 & -216 & 108 \\
 & & & & & & & 96 & 468 & 156 & -108 & -36 & -468 & 312 & 108 & -72 \\
 & & & & & & & & 11232 & 5616 & -1584 & -792 & -11232 & 5616 & 1584 & -792 \\
 & & & & & & & & & 3744 & -792 & -528 & -5616 & 1872 & 792 & -264 \\
 & & & & & & & & & & 288 & 144 & 1584 & -792 & -288 & 144 \\
 & & & & & & & & & & & 96 & 792 & -264 & -144 & 48 \\
 & & & & & & & & & & & & 11232 & -5616 & -1584 & 792 \\
 & & & & & & & & & & & & & 3744 & 792 & -528 \\
 & & & & & & & & & & & & & & 288 & -144 \\
 & & & & & & & & & & & & & & & 96
\end{bmatrix}$$

SYMM.

APPENDIX B

B.1. Computer program using the basic temporal element

Prog. 1 is a computer program written to solve problems concerning the response of discrete or discretized continuous systems to external forces. This program incorporates the approximations of (A.1.5) for the time derivatives of the forces, and values of forces only are required. Viscous damping may also be included in the solution.

The method of solution may be seen by reference to equations (4.2.4) which may be written in the alternative form

$$[H]\begin{Bmatrix}\{u_1\}\\ \{v_1\}\end{Bmatrix} = [A]\begin{Bmatrix}\{F\}\\ \{\dot{F}_\tau\}\end{Bmatrix} - [G]\begin{Bmatrix}\{u_0\}\\ \{v_0\}\end{Bmatrix} \qquad (B.1.1)$$

F being a function of F_0 and F_1

Since all quantities on the right-hand side of equations (B.1.1) are known, these equations may be further simplified to

$$[H]\{u_1\} = \{B\} \qquad (B.1.2)$$

where $\{B\}$ is a vector of constants.

These equations may then be solved by Gauss elimination, reducing the [H] matrix to an upper triangular matrix. The value of the last variable is then found, and by back substitution values of all the other variables may be calculated.

Since this calculation must be repeated for each step, any unnecessary computation should be eliminated. The necessary operations on the matrix [H] of (B.1.2) need be performed once only, and since both sides of each equation must be treated in exactly the same way the factors used may be stored as a lower triangular matrix. These factors are then used to perform the required operations on the {B} vector of (B.1.2). These calculations are necessary since the values of {B} will change at each step.

The data input required to use this program is:

NM = Number of degrees of freedom

ND = 0 if no damping, any other integer if system is damped

NF = Number of nodes at which forces are applied

NT = Number of time intervals used

DT = Size of time interval τ

M[I,J] = Mass matrix, line by line

K[I,J] = Stiffness matrix

C[I,J] = Damping matrix

MV = Number of nodes having initial displacement and/or velocity

R = Number of any such node

U[R] = Initial displacement

U[R+NM] = Initial velocity

FC[P] = Number of a node at which a force is applied

F[P,R] = List of NT + 1 forces at this node

The output is a list of displacements and velocities for each value of time, i.e.

t $\quad {}_1u \quad {}_1\dot{u} \quad {}_2u \quad {}_2\dot{u}$ etc.

PROGRAM 1 BASIC ELEMENT

```
'BEGIN' 'REAL' MM,KK,CC,G0,G1,G2,H0,H1,H2,T3,T4,DF0,DF1,SUM,T,
      DT,PI;
     'INTEGER' NM,N,NF,I,J,P,R,S,ND,NT,IN,MV,NS;
      NM:=READ; N:=2*NM; ND:=READ; NF:=READ; NT:=READ;
     'BEGIN' 'REAL' 'ARRAY' M,K,C[1:NM,1:NM],G,H[1:N,1:N],
      U,FF[1:N],L[2:N,1:N-1],F[1:NF+2,0:NT];
     'INTEGER' 'ARRAY' FC[1:NF+2];
      DT:=READ;
     'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
     'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
      G[I,J]:=H[I,J]:=C[I,J]:=0;
     'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
     'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
     M[I,J]:=READ;
     'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
     'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
      K[I,J]:=READ;
     'IF' ND=0 'THEN' 'GOTO' UND;
     'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
     'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
      C[I,J]:=READ;
UND:  T3:=30*DT; T4:=DT/420;
      G0:=3/T3; G1:=22*T4; G2:=-6/60; H0:=-G0; H1:=13*T4; H2:=-G2;
     'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
     'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
     'BEGIN' G[I,J]:=G0*M[I,J]-G2*C[I,J]-G1*K[I,J];
             H[I,J]:=H0*M[I,J]-H2*C[I,J]-H1*K[I,J]; 'END';
      G0:=4/T3; G1:=4*T4; G2:=0; H0:=-1/T3; H1:=-3*T4; H2:=-1/60;
     'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
     'FOR' J:=NM+1 'STEP' 1 'UNTIL' N 'DO'
     'BEGIN' G[I,J]:=G0*M[I,J-NM]-G2*C[I,J-NM]-G1*K[I,J-NM];
             H[I,J]:=H0*M[I,J-NM]-H2*C[I,J-NM]-H1*K[I,J-NM]; 'END';
      G0:=3/T3; G1:=-13*T4; G2:=6/60; H0:=-G0; H1:=-22*T4;
      H2:=-6/60;
     'FOR' I:=NM+1 'STEP' 1 'UNTIL' N 'DO'
     'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
     'BEGIN' G[I,J]:=G0*M[I-NM,J]-G2*C[I-NM,J]-G1*K[I-NM,J];
             H[I,J]:=H0*M[I-NM,J]-H2*C[I-NM,J]-H1*K[I-NM,J]; 'END';
      G0:=-1/T3; G1:=-3*T4; G2:=1/60; H0:=4/T3; H1:=4*T4; H2:=0;
     'FOR' I:=NM+1 'STEP' 1 'UNTIL' N 'DO'
     'FOR' J:=NM+1 'STEP' 1 'UNTIL' N 'DO'
     'BEGIN' G[I,J]:=G0*M[I-NM,J-NM]-G2*C[I-NM,J-NM]-G1*K[I-NM,J-NM];
             H[I,J]:=H0*M[I-NM,J-NM]-H2*C[I-NM,J-NM]-H1*K[I-NM,J-NM];
     'END';
     'FOR' P:=1 'STEP' 1 'UNTIL' N-1 'DO'
     'FOR' I:=P+1 'STEP' 1 'UNTIL' N 'DO'
     'BEGIN' L[I,P]:=H[I,P]/H[P,P];
     'FOR' J:=P+1 'STEP' 1 'UNTIL' N 'DO'
      H[I,J]:=H[I,J]-L[I,P]*H[P,J]; 'END';
     'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO' U[I]:=0;
      MV:=READ; 'IF' MV=0 'THEN' 'GOTO' NOM;
     'FOR' P:=1 'STEP' 1 'UNTIL' MV 'DO'
     'BEGIN' R:=READ; U[R]:=READ; U[R+NM]:=READ*DT; 'END';
NOM:  'IF' NF=0 'THEN' 'GOTO' ZEF;
     'FOR' P:=1 'STEP' 1 'UNTIL' NF 'DO'
     'BEGIN' FC[P]:=READ;
```

```
      'FOR' R:=0 'STEP' 1 'UNTIL' NT 'DO' F[P,R]:=READ; 'END';
ZEF:  'FOR' IN:=1 'STEP' 1 'UNTIL' NT+1  'DO'
      'BEGIN' T:=(IN-1)*DT; NEWLINE(1); PRINT(T,5,5);
      'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
      'BEGIN' PRINT(U[I],10,6); PRINT(U[I+NM]/DT,5,6);
       FF[I]:=0; FF[I+NM]:=0; 'END';
      'IF' IN=NT+1 'THEN' 'GOTO' LAS;
      'IF' NF=0 'THEN' 'GOTO' COM;
      'FOR' P:=1 'STEP' 1 'UNTIL' NF 'DO'
      'BEGIN' I:=FC[P]; 'IF' IN 'GE' NT-1 'THEN' 'GOTO' PEN;
      'IF' IN=1 'THEN' 'BEGIN'
       DF0:=(-25*F[P,0]+48*F[P,1]-36*F[P,2]+16*F[P,3]-3*F[P,4])/12;
       DF1:=(-3*F[P,0]-10*F[P,1]+18*F[P,2]-6*F[P,3]+F[P,4])/12;
      'GOTO' FMA; 'END';
      'IF' IN=2 'THEN'
       DF0:=(-3*F[P,0]-10*F[P,1]+18*F[P,2]-6*F[P,3]+F[P,4])/12
      'ELSE'
       DF0:=(F[P,IN-3]-8*F[P,IN-2]+8*F[P,IN]-F[P,IN+1])/12;
       DF1:=(F[P,IN-2]-8*F[P,IN-1]+8*F[P,IN+1]-F[P,IN+2])/12;
PEN:  'IF' IN=NT-1 'THEN' 'BEGIN'
       DF0:=(F[P,NT-4]-8*F[P,NT-3]+8*F[P,NT-1]-F[P,NT])/12;
       DF1:=(-F[P,NT-4]+6*F[P,NT-3]-18*F[P,NT-2]+10*F[P,NT-1]
            +3*F[P,NT])/12; 'END';
      'IF' IN=NT 'THEN' 'BEGIN'
       DF0:=(-F[P,NT-4]+6*F[P,NT-3]-18*F[P,NT-2]+10*F[P,NT-1]
            +3*F[P,NT])/12;
       DF1:=(3*F[P,NT-4]-16*F[P,NT-3]+36*F[P,NT-2]-48*F[P,NT-1]
            +25*F[P,NT])/12; 'END';
FMA:   FF[I]:=(22*F[P,IN-1]+4*DF0+13*F[P,IN]-3*DF1)*T4;
       FF[I+NM]:=(-13*F[P,IN-1]-3*DF0-22*F[P,IN]+4*DF1)*T4;
      'END';
COM:  'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN' FF[I]:=-FF[I];
      'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
       FF[I]:=FF[I]-G[I,J]*U[J]; 'END';
      'FOR' J:=1 'STEP' 1 'UNTIL' N-1 'DO'
      'FOR' I:=J+1 'STEP' 1 'UNTIL' N 'DO'
       FF[I]:=FF[I]-FF[J]*L[I,J];
       U[N]:=FF[N]/H[N,N];
      'FOR' I:=N-1 'STEP' -1 'UNTIL' 1 'DO'
      'BEGIN' SUM:=FF[I];
      'FOR' J:=N 'STEP' -1 'UNTIL' I+1 'DO'
       SUM:=SUM-H[I,J]*U[J];
       U[I]:=SUM/H[I,I];
      'END';
      'END';
      'END';
LAS:  'END';
```

B.2 Computer program using refined element

Prog. 2 is very similar to Prog. 1, but since the refined element uses acceleration as a generalized displacement, this quantity is now incorporated into the program so that the vector of unknown variables $\{u\}$ is now $[[u]\ [\dot{u}\tau]\ [\ddot{u}\tau^2]]^t$. Since it is not always convenient to specify the initial acceleration as well as displacement and velocity, all initial accelerations are now calculated in the program using equations (4.4.1). The matrix inversion required to find $[m]^{-1}$ is obtained by Jordan's method.

Values of $\dot{F}$ and $\ddot{F}$ are now required, and these are obtained by using the approximations of (A.1.7) and (A.1.8).

The data input required is exactly the same as that required in Prog. 1.

The output is the same as for Prog. 1 with values of acceleration added, i.e.

$t \quad {}_1u \quad {}_1\dot{u} \quad {}_1\ddot{u} \quad {}_2u \quad {}_2\dot{u} \quad {}_2\ddot{u}$ etc.

PROGRAM 2 REFINED ELEMENT

```
'BEGIN' 'REAL' MM,KK,CC,T3,T4,DF0,DF1,SUM,T,D2F0,D2F1,
     DT,PI,FAC;
    'INTEGER' NM,N,NF,I,J,P,R,S,ND,NT,IN,MV,NS;
     NM:=READ; N:=3*NM; ND:=READ; NF:=READ; NT:=READ;
    'BEGIN' 'REAL' 'ARRAY' M,K,C[1:NM,1:NM],G,H[1:N,1:N],
     U,FF[1:N],L[2:N,1:N-1],F[1:NF+2,0:NT],D1,D2[1:NF+2],
     G0,G1,G2,H0,H1,H2[1:3,1:3],F0[1:NM],MI[1:NM,1:NM];
    'INTEGER' 'ARRAY' FC[1:NF+2];
     DT:=READ;
    'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
    'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
    'BEGIN' G[I,J]:=0; H[I,J]:=0; 'END';
    'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO' F0[I]:=0;
    'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
    'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
     M[I,J]:=READ;
    'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
    'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
     K[I,J]:=READ;
    'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
    'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO' C[I,J]:=0;
    'IF' ND=0 'THEN' 'GOTO' CAV;
    'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
    'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
     C[I,J]:=READ;
CAV: G0[1,1]:=11880/DT;  G0[1,2]:=12672/DT;  G0[1,3]:=924/DT;
     G0[2,1]:=11880/DT;  G0[2,2]:=-792/DT;   G0[2,3]:=-264/DT;
     G0[3,1]:=-660/DT;   G0[3,2]:=264/DT;    G0[3,3]:=44/DT;
     G1[1,1]:=3732*DT;   G1[1,2]:=832*DT;    G1[1,3]:=69*DT;
     G1[2,1]:=-1812*DT;  G1[2,2]:=-532*DT;   G1[2,3]:=-52*DT;
     G1[3,1]:=181*DT;    G1[3,2]:=52*DT;     G1[3,3]:=5*DT;
     G2[1,1]:=-7260;     G2[1,2]:=0;         G2[1,3]:=55;
     G2[2,1]:=7260;      G2[2,2]:=1716;      G2[2,3]:=143;
     G2[3,1]:=-660;      G2[3,2]:=-143;      G2[3,3]:=-11;
     H0[1,1]:=-11880/DT; H0[1,2]:=-792/DT;   H0[1,3]:=264/DT;
     H0[2,1]:=-11880/DT; H0[2,2]:=12672/DT;  H0[2,3]:=-924/DT;
     H0[3,1]:=660/DT;    H0[3,2]:=-924/DT;   H0[3,3]:=88/DT;
     H1[1,1]:=1812*DT;   H1[1,2]:=-532*DT;   H1[1,3]:=52*DT;
     H1[2,1]:=-3732*DT;  H1[2,2]:=832*DT;    H1[2,3]:=-69*DT;
     H1[3,1]:=281*DT;    H1[3,2]:=-69*DT;    H1[3,3]:=6*DT;
     H2[1,1]:=7260;      H2[1,2]:=-1716;     H2[1,3]:=143;
     H2[2,1]:=-7260;     H2[2,2]:=0;         H2[2,3]:=55;
     H2[3,1]:=660;       H2[3,2]:=-55;       H2[3,3]:=0;
    'FOR' R:=1,2,3 'DO'
    'FOR' S:=1,2,3 'DO'
    'FOR' I:=(R-1)*NM+1 'STEP' 1 'UNTIL' R*NM 'DO'
    'FOR' J:=(S-1)*NM+1 'STEP' 1 'UNTIL' S*NM 'DO'
    'BEGIN'
     G[I,J]:=G0[R,S]*M[I-(R-1)*NM,J-(S-1)*NM]-G2[R,S]

     *C[I-(R-1)*NM,J-(S-1)*NM]-G1[R,S]*K[I-(R-1)*NM,J-(S-1)*NM];
     H[I,J]:=H0[R,S]*M[I-(R-1)*NM,J-(S-1)*NM]-H2[R,S]
     *C[I-(R-1)*NM,J-(S-1)*NM]-H1[R,S]*K[I-(R-1)*NM,J-(S-1)*NM];
    'END';
```

```
      'FOR' P:=1 'STEP' 1 'UNTIL' N-1 'DO'
      'FOR' I:=P+1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN' L[I,P]:=H[I,P]/H[P,P];
      'FOR' J:=P+1 'STEP' 1 'UNTIL' N 'DO'
       H[I,J]:=H[I,J]-L[I,P]*H[P,J]; 'END';
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO' U[I]:=0;
       MV:=READ; 'IF' MV=0 'THEN' 'GOTO' NOM;
      'FOR' P:=1 'STEP' 1 'UNTIL' MV 'DO'
      'BEGIN' R:=READ; U[R]:=READ; U[R+NM]:=READ*DT;
      'END';
NOM:  'IF' NF=0 'THEN' 'GOTO' ZEF;
      'FOR' P:=1 'STEP' 1 'UNTIL' NF 'DO'
      'BEGIN' FC[P]:=READ;
      'FOR' R:=0 'STEP' 1 'UNTIL' NT 'DO' F[P,R]:=READ; 'END';
ZEF:  'FOR' IN:=1 'STEP' 1 'UNTIL' NT+1 'DO'
      'BEGIN' T:=(IN-1)*DT; NEWLINE(1); PRINT(T,5,5);
      'IF' IN=1 'THEN' 'BEGIN'
      'IF' NF=0 'THEN' 'GOTO' XPR;
      'FOR' P:=1 'STEP' 1 'UNTIL' NF 'DO'
      'BEGIN' I:=FC[P]; FO[I]:=F[P,0]; 'END';
XPR:  'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
      'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
       MI[I,J]:= 'IF' I=J 'THEN' 1 'ELSE' 0;
      'FOR' P:=1 'STEP' 1 'UNTIL' NM-1 'DO'
      'FOR' I:=P+1 'STEP' 1 'UNTIL' NM 'DO'
      'BEGIN' FAC:=M[I,P]/M[P,P];
      'FOR' J:=P+1 'STEP' 1 'UNTIL' NM 'DO'
       M[I,J]:=M[I,J]-FAC*M[P,J];
      'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
       MI[I,J]:=MI[I,J]-FAC*MI[P,J];
      'END';
      'FOR' P:=NM 'STEP' -1 'UNTIL' 2 'DO'
      'FOR' I:=P-1 'STEP' -1 'UNTIL' 1 'DO'
      'BEGIN' FAC:=M[I,P]/M[P,P];
      'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
       MI[I,J]:=MI[I,J]-FAC*MI[P,J];
      'END';
      'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
      'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
       MI[I,J]:=MI[I,J]/M[I,I];
      'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
      'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
       FO[I]:=FO[I]-K[I,J]*U[J]-C[I,J]*U[J+NM]/DT;
      'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
      'FOR' J:=1 'STEP' 1 'UNTIL' NM 'DO'
       U[I+2*NM]:=U[I+2*NM]+MI[I,J]*FO[J]*DT*DT;
      'END';
      'FOR' I:=1 'STEP' 1 'UNTIL' NM 'DO'
      'BEGIN' PRINT(U[I],10,6); PRINT(U[I+NM]/DT,5,6);
       PRINT(U[I+2*NM]/(DT*DT),5,6);
       FF[I]:=0; FF[I+NM]:=0; FF[I+2*NM]:=0; 'END';
     'IF' IN=NT+1 'THEN' 'GOTO' LAS;
     'IF' NF=0 'THEN' 'GOTO' COM;
     'FOR' P:=1 'STEP' 1 'UNTIL' NF 'DO'
     'BEGIN' I:=FC[P];
     'IF' IN=1 'THEN' 'BEGIN'
```

```
      DF0:=(-441*F[P,0]+1080*F[P,1]-1350*F[P,2]+1200*F[P,3]-675*F[P,4]
            +216*F[P,5]-30*F[P,6])/180;
      D2F0:=(812*F[P,0]-3132*F[P,1]+5265*F[P,2]-5080*F[P,3]
            +2970*F[P,4]-972*F[P,5]+137*F[P,6])/180;
      DF1:=(-30*F[P,0]-231*F[P,1]+450*F[P,2]-300*F[P,3]+150*F[P,4]
            -45*F[P,5]+6*F[P,6])/180;
      D2F1:=(137*F[P,0]-147*F[P,1]-255*F[P,2]+470*F[P,3]-285*F[P,4]
            +93*F[P,5]-13*F[P,6])/180;
      D1[P]:=DF1; D2[P]:=D2F1;
    'GOTO' FMA; 'END';
      DF0:=D1[P]; D2F0:=D2[P];
    'IF' IN=2 'THEN' 'BEGIN'
      DF1:=(6*F[P,0]-72*F[P,1]-105*F[P,2]+240*F[P,3]-90*F[P,4]
            +24*F[P,5]-3*F[P,6])/180;
      D2F1:=(-13*F[P,0]+228*F[P,1]-420*F[P,2]+200*F[P,3]+15*F[P,4]
            -12*F[P,5]+2*F[P,6])/180;
      D1[P]:=DF1; D2[P]:=D2F1;
    'GOTO' FMA; 'END';
    'IF' IN=NT-2 'THEN' 'BEGIN'
      DF1:=(3*F[P,NT-6]-24*F[P,NT-5]+90*F[P,NT-4]-240*F[P,NT-3]
            +105*F[P,NT-2]+72*F[P,NT-1]-6*F[P,NT])/180;
      D2F1:=(2*F[P,NT-6]-12*F[P,NT-5]+15*F[P,NT-4]+200*F[P,NT-3]
            -420*F[P,NT-2]+228*F[P,NT-1]-13*F[P,NT])/180;
      D1[P]:=DF1; D2[P]:=D2F1;
    'GOTO' FMA; 'END';
    'IF' IN=NT-1 'THEN' 'BEGIN'
      DF1:=(-6*F[P,NT-6]+45*F[P,NT-5]-150*F[P,NT-4]+300*F[P,NT-3]
            -450*F[P,NT-2]+231*F[P,NT-1]+30*F[P,NT])/180;
      D2F1:=(-13*F[P,NT-6]+93*F[P,NT-5]-285*F[P,NT-4]+470*F[P,NT-3]
            -255*F[P,NT-2]-147*F[P,NT-1]+137*F[P,NT])/180;
      D1[P]:=DF1; D2[P]:=D2F1;
    'GOTO' FMA; 'END';
    'IF' IN=NT 'THEN' 'BEGIN'
      DF1:=(30*F[P,NT-6]-216*F[P,NT-5]+675*F[P,NT-4]-1200*F[P,NT-3]
            +1350*F[P,NT-2]-1080*F[P,NT-1]+441*F[P,NT])/180;
      D2F1:=(137*F[P,NT-6]-972*F[P,NT-5]+2970*F[P,NT-4]-5080*F[P,NT-3]
            +5265*F[P,NT-2]-3132*F[P,NT-1]+812*F[P,NT])/180;
    'GOTO' FMA; 'END';
      DF1:=(-3*F[P,IN-3]+27*F[P,IN-2]-135*F[P,IN-1]+135*F[P,IN+1]
            -27*F[P,IN+2]+3*F[P,IN+3])/180;
      D2F1:=(2*F[P,IN-3]-27*F[P,IN-2]+270*F[P,IN-1]-490*F[P,IN]
            +270*F[P,IN+1]-27*F[P,IN+2]+2*F[P,IN+3])/180;
      D1[P]:=DF1; D2[P]:=D2F1;
FMA:   FF[I]:=(3732*F[P,IN-1]+832*DF0+69*D2F0
              +1812*F[P,IN]-552*DF1+52*D2F1)*DT;
       FF[I+NM]:=(-1812*F[P,IN-1]-552*DF0-52*D2F0
                  -3732*F[P,IN]+832*DF1-69*D2F1)*DT;
       FF[I+2*NM]:=(181*F[P,IN-1]+52*DF0+5*D2F0
                    +281*F[P,IN]-69*DF1+6*D2F1)*DT;
    'END';
COM: 'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
    'BEGIN' FF[I]:=-FF[I];
    'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
     FF[I]:=FF[I]-G[I,J]*U[J]; 'END';
```

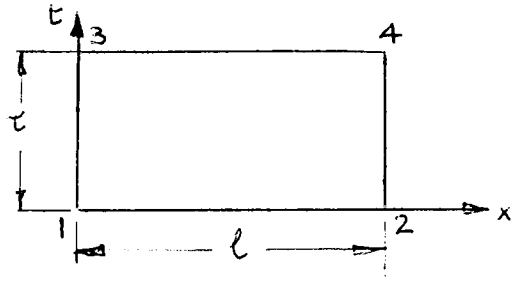
```
      'FOR' J:=1 'STEP' 1 'UNTIL' N-1 'DO'
      'FOR' I:=J+1 'STEP' 1 'UNTIL' N 'DO'
       FF[I]:=FF[I]-FF[J]*L[I,J];
       U[N]:=FF[N]/H[N,N];
      'FOR' I:=N-1 'STEP' -1 'UNTIL' 1 'DO'
      'BEGIN' SUM:=FF[I];

      'FOR' J:=N 'STEP' -1 'UNTIL' I+1 'DO'
       SUM:=SUM-H[I,J]*U[J];
       U[I]:=SUM/H[I,I];
      'END';
      'END';
      'END';
LAS:  'END';
```

B.3 Computer programs for [m*] and [k*] matrices for space-time elements

Program 3 gives [m*] and [k*] matrices for an 8-term shape function of x and t

i.e. $u = a_1 + a_2x + a_3t + a_4x^2 + a_5xt + ...$

Each term is of the form ax^rt^s where r and s may have any value. In the form given, the generalized displacements are the deflections and slopes at each corner of a rectangular space-time element.



The data required is

N = Number of terms in the shape function

R,S = Consecutive values of the indices of x and t respectively in each of the N terms.

X,T = Values of x/ℓ and t/τ respectively at each node in the order in which the matrices are required.

[x], $[x]^{-1}$, [k*] and [m*] matrices are printed in that order.

Velocity, acceleration, curvature etc. are readily incorporated as nodal displacements. One-dimensional elements in space or time are obtained by using zero indices for the dimension not to be included i.e. for a temporal element

$u = a_1x^0t^0 + a_2x^0t + a_3x^0t^2 + ...$

PROGRAM 3 SPACE - TIME ELEMENT

```
'BEGIN' 'REAL' FAC,SUM,TEM;
      'INTEGER' I,J,N,P,Q,X,T,FAIL;
      'ARRAY' A[1:20,1:20];
      'PROCEDURE' F01AAA(A,N,IFAIL);
      'VALUE' N; 'INTEGER' N,IFAIL; 'ARRAY' A;
      'ALGOL';
       N:=READ;
      'BEGIN' 'REAL'  'ARRAY'      U,V,K,W[1:N,1:N],R,S,UR,TS[1:N];
      'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN' R[J]:=READ; S[J]:=READ; 'END';
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO' A[I,J]:=1;
      'FOR' I:=1 'STEP' 2 'UNTIL' N-1 'DO'
      'BEGIN' X:=READ; T:=READ;
      'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN' A[I,J]:=A[I,J]*('IF' R[J]=0 'THEN' 1 'ELSE' X)
                            *('IF' S[J]=0 'THEN' 1 'ELSE' T);
       A[I+1,J]:=A[I+1,J]*R[J]*('IF' R[J]=1 'THEN' 1 'ELSE' X)
                            *('IF' S[J]=0 'THEN' 1 'ELSE' T);
      'END';  'END';
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN' NEWLINE(1);
      'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
       PRINT(A[I,J],2,0); 'END'; NEWLINE(3);
       FAIL:=1;
       F01AAA(A,N,FAIL);
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN' NEWLINE(1);
      'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
       PRINT(A[I,J],1,2);
      'END';
       NEWLINE(3);
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN'        U[I,J]:=R[J]*(R[J]-1)* A[J,I];
       V[I,J]:=S[J]* A[J,I];          K[I,J]:=0; W[I,J]:=0; 'END';
      'FOR' J:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN' UR[J]:= 'IF' R[J] 'GE' 3 'THEN' R[J]-2 'ELSE' 0;
       TS[J]:= 'IF' S[J] 'GE' 2 'THEN' S[J]-1 'ELSE' 0; 'END';
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'FOR' J:=I 'STEP' 1 'UNTIL' N 'DO'
      'FOR' P:=1 'STEP' 1 'UNTIL' N 'DO'
      'FOR' Q:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN' K[I,J]:=K[I,J]+U[I,P]*U[J,Q]/((UR[P]+UR[Q]+1)
                                          *(S[P]+S[Q]+1));
       W[I,J]:=W[I,J]+V[I,P]*V[J,Q]/((TS[P]+TS[Q]+1)*(R[P]+R[Q]+1));
      'END';
      'FOR' I:=2 'STEP' 1 'UNTIL' N 'DO'
      'FOR' J:=1 'STEP' 1 'UNTIL' I-1 'DO'
      'BEGIN' K[I,J]:=K[J,I]; W[I,J]:=W[J,I]; 'END';
```

```
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN'
      'FOR' J:=1 'STEP' 1 'UNTIL' N/2 'DO' PRINT(KLI,JJ*25200,6,5);
       NEWLINE(1); 'END';
       NEWLINE(3);
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN'
      'FOR' J:=N/2+1 'STEP' 1 'UNTIL' N 'DO' PRINT(KLI,JJ*25200,6,5);
       NEWLINE(1); 'END';
       NEWLINE(3);
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN'
      'FOR' J:=1 'STEP' 1 'UNTIL' N/2 'DO' PRINT(WLI,JJ*25200,5,5);
       NEWLINE(1); 'END';
       NEWLINE(3);
      'FOR' I:=1 'STEP' 1 'UNTIL' N 'DO'
      'BEGIN'
      'FOR' J:=N/2+1 'STEP' 1 'UNTIL' N 'DO' PRINT(WLI,JJ*25200,6,5);
       NEWLINE(1); 'END';
      'END';
'END';
```

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