A Class Of Perfect Fluids In General Relativity

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This thesis is concerned with exact solutions of Einstein's field equations of general relativity, in particular, when the source of the gravitational field is a perfect fluid with a purely electric Weyl tensor.

General relativity, cosmology and computer algebra are discussed briefly. A mathematical introduction to Riemannian geometry and the tetrad formalism is then given. This is followed by a review of some previous results and known solutions concerning purely electric perfect fluids. In addition, some orthonormal and null tetrad equations of the Ricci and Bianchi identities are displayed in a form suitable for investigating these space-times.

Conformally flat perfect fluids are characterised by the vanishing of the Weyl tensor and form a sub-class of the purely electric fields in which all solutions are known (Stephani 1967). The number of Killing vectors in these space-times is investigated and results presented for the non-expanding space-times. The existence of stationary fields that may also admit 0, 1 or 3 space-like Killing vectors is demonstrated.

Shear-free fluids in the class under consideration are shown to be either non-expanding or irrotational (Collins 1984) using both orthonormal and null tetrads. A discrepancy between Collins (1984) and Wolf (1986) is resolved by explicitly solving the field equations to prove that the only purely electric, shear-free, geodesic but rotating perfect fluid is the Gödel (1949) solution.

The irrotational fluids with shear are then studied and solutions due to Szafron (1977) and Allnutt (1982) are characterised. The metric is simplified in several cases where new solutions may be found. The geodesic space-times in this class and all Bianchi type 1 perfect fluid metrics are shown to have a metric expressible in a diagonal form. The position of spherically symmetric and Bianchi type 1 space-times in relation to the general case is also illustrated.

General Relativity
Exact Solutions
Perfect Fluids

Dedication

Edward George Herbert
Gertrude Agnes Herbert
Gertrude Esther Smith
Victor James Rowlingson
George Hipwood
Ivy Beatrice Rowlingson

They meant more to me than they ever knew
and will always be in my heart.

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CONTENTS

CHAPTER 1
Introduction to General Relativity

1.1 General Relativity, Cosmology and Exact Solutions 7
1.2 Experimental Tests of General Relativity 9
1.3 Computer Algebra 10

CHAPTER 2
Mathematical Introduction

2.1 The Field Equations of General Relativity 12
2.2 Perfect Fluids 15
2.3 The Ricci and Bianchi Identities 18
2.4 The Tetrad Formalism 21

CHAPTER 3
Space-times with a Purely Electric Weyl Tensor

3.1 Introduction 30
3.2 Petrov Types 32
3.3 Purely Electric Space-times in the Newman-Penrose Formalism 33
3.4 Orthonormal Tetrad Equations 38

CHAPTER 4
Killing Vectors in Conformally Flat Perfect Fluid Space-times

4.1 Introduction 46
4.2 Conformally Flat Perfect Fluid Space-times 47
4.3 Killing's Equations and Intrinsic Killing Vectors 49
4.4 Non-Trivial Tilted Killing Vectors 51
4.5 Spatial Killing Vectors 56
4.6 Examples of Stationary Non-Static Conformally Flat Perfect Fluids 63
4.7 Bianchi Types of the 3-Parameter Groups 65
4.8 Conclusion 67

CHAPTER 5
Shear-Free Perfect Fluids with a Purely Electric Weyl Tensor

5.1 Introduction 70
5.2 Applications of Null and Orthonormal Tetrads 72
5.3 The Gödel Solution 75

CHAPTER 6
Irrotational Perfect Fluids with a Purely Electric Weyl Tensor

6.1 Introduction 82
6.2 Petrov Type I Fields 84
6.3 Petrov Type D Fields 88
6.4 Petrov Type D Fields with $\sigma_1 = \sigma_2$ 89
6.5 Petrov Type D Fields with $\sigma_1 \neq \sigma_2$ 96
6.6 Concluding Remarks 106

References 108

Appendix
The Jacobi Identities 120

CHAPTER 1

Introduction to General Relativity

§1.1 General Relativity, Cosmology and Exact Solutions

Created in 1915 by Albert Einstein, general relativity has become an essential part of modern science. It came into being, historically, as an extension of special relativity which was prompted by the null results of the experiments to find an ether. This lack of an ether is paralleled by one of the essential differences between general relativity and Newtonian gravitation theory. In the Newtonian case, space and time are an 'absolute' Euclidean background to the physics of gravity and other processes. In general relativity the geometry of space-time itself is gravity, and the need for some absolute background is eliminated. The four-dimensional nature of space-time is also inherited from special relativity, but space-time is allowed to curve in response to the matter producing a gravitational field. The mathematical language of non-Euclidean space-time is Riemannian geometry (Eisenhart 1949) and this is described in more detail in chapter two.

Gravity is the weakest of the four fundamental forces. However for large masses it dominates over short range nuclear forces and so gravity is highly important in the formation and evolution of large scale structure in the universe (MacCallum 1979). Modern cosmology takes as its starting point the big bang– the primordial superdense state of the universe (see e.g. Narlikar 1986). However, general relativity is only valid at times greater than the Planck time $t \approx 10^{-43}$s. Prior to this, quantum effects are expected to be important. A full quantum theory of gravity has yet to be established. Current work aims to unify all four fundamental forces using supersymmetry and some hope is placed on superstring theory to achieve this (Bailin 1989). One of the basic tenets of modern cosmology is the Copernican or Cosmological principle. This states that man is in no special place in the universe implying that all points are essentially equivalent. The relegation of man to an 'average' part of the universe was only finally confirmed observationally in the 1950's from a revised estimate of the distance scale. This made it clear that the size of our galaxy was fairly typical and not significantly larger than every other observed galaxy as had previously been thought. Despite this rejection of man's centrality in the universe, anthropocentrism lives on in efforts to explain the existence of intelligent life in a universe which appears finely tuned to its needs (Barrow and Tipler 1987).

Large-scale isotropy is observed in galaxies, quasars and the microwave background and hence the cosmological principle implies that the universe is isotropic at every point and therefore homogeneous. This leads uniquely to the Robertson-Walker model of the universe as a description of an homogeneous expanding (or contracting) space-time (MacCallum 1979). This space-time is an exact solution of Einstein's field equations of general relativity, in that it satisfies the differential equations describing the gravitation field of a homogeneous universe in general relativity theory. A definition of an exact solution is not trivial. A possible definition might be those solutions that contain only analytic functions. However, these functions may possibly be defined only in terms of differential equations originally, so to what extent have the field equations been solved exactly? In their book, Kramer et al (1980) define an exact solution as such if it appears therein. The search for exact solutions is then the attempt to solve, as far as possible, all differential equations arising from the field equations in particular cases of the gravitational field. Whether or not the resulting space-time is actually considered to be an exact solution is then open to contention.

The Robertson-Walker solution is a very useful and widely used solution in cosmology. However, in the absence of a general solution to the field equations other solutions are required to describe different gravitational fields. For example, the gravitational field of stellar interiors or outside a massive object such as a black hole. In fact, the first solution found, the Schwarzschild solution, can describe the latter of these possibilities as well as the field surrounding a star or planet. It can therefore be used to calculate the motion of planets around the Sun and the behaviour of satellites and clocks near the Earth. Consequently it has been used as a basis for many of the experimental tests of general relativity which are, effectively, purely tests of the Schwarzschild metric.

§1.2 Experimental Tests of General Relativity

This thesis assumes the validity of general relativity as a description of the gravitational field. The acceptance of general relativity depends on its correspondence with experimental tests compared with alternative gravity theories.

Originally, general relativity had few links with observation: three 'classical tests' and relevance in cosmology. The evidence from these tests is surprisingly meagre (Will 1984). Mercury's perihelion shift is highly dependent on the internal structure and oblateness of the Sun. The observations of the deflection of starlight during solar eclipses are of low precision (Will 1984). In addition, they are tests only of the vacuum field equations and the Schwarzschild metric in particular. Any theory that agrees with general relativity to second order in $\frac{GM}{c^2 r}$ and obeys the geodesic hypothesis is equally valid under these tests. The third classical test, gravitational red-shift, is a test of the Einstein equivalence principle and not of general relativity specifically. However, astronomical discoveries such as pulsars, quasars and the possible detection of black holes and gravitational lenses have given added relevance to general relativity and has increased theoretical and experimental investigations. Technological progress has allowed new tests and more precise versions of the old tests. Experiments with atomic clocks in space and observations in astronomy, such as the binary pulsar, have been added to the repertoire. A framework for analysing tests of gravitation has been developed along with techniques for comparing the competing gravitation theories with each other through the results of those tests. In every case, general relativity has passed satisfactorily against rival theories (Will 1984). A great deal of work relevant to finding a new model of gravity is now being concentrated in unified field theories.

More recent tests of gravity have been concerned with the inverse square law, i.e. in the Newtonian limit. Deviations from this law have been reported including both attractive and repulsive corrections. The so-called 5^{th} and 6^{th} forces have been postulated to account for these deviations. However, firm conclusions cannot yet be drawn as there are conflicting results and in some cases the results are highly dependent on an uncertain knowledge of the internal structure of the Earth (Schwarzschild 1988).

§1.3 Computer Algebra

Several computer algebra systems are used in this thesis and with computer algebra gaining some acceptance it is worthwhile to consider it briefly here. Computer algebra is the ability of computers to perform algebraic calculations. This includes expansion of polynomials, differentiation and integration, substitutions, power series, infinite precision arithmetic and matrix calculations amongst others. In scientific applications computers are commonly seen as "number crunchers". In fact, they are equally suited to algebra as they are to other forms of non-numeric computation such as word processing.

The use of computer algebra has several advantages, one of the most obvious being the elimination of some typical human errors such as misplaced signs and factors of two. Of course this can only occur in output assuming an error-free input. Computer algebra can be used to eliminate the need to repeat routine, similar, but lengthy calculations by hand. It can reduce the time spent on problems with unwieldy algebraic expressions and make projects, too large to attempt by hand, into practical propositions.

In general relativity, computer algebra is used in the first of these modes to calculate tensors from exact solutions with SHEEP (Frick 1982). SHEEP can calculate useful tensors in general relativity, such as those required for the field equations, from various appropriate inputs. The program is distributed freely and the source code is documented and user extensible. STENSOR (Hornfeldt 1985) is based on SHEEP and allows new tensors and tensor relations to be defined and calculated. It can also perform indicial tensor manipulation including covariant differentiation and substitutions. It is not limited to relativity and has additional features for supergravity calculations and is useful in any field involving extensive tensor manipulations.

An example of a project made practical by computer algebra is the equivalence problem (Åman and Karlhede 1980). This concerns the need to determine whether two exact solutions that appear in different forms are in fact the same solution expressed in different coordinate systems. CLASSI (Åman 1983) is also built on SHEEP and can perform the equivalence calculations and determine invariant properties of solutions (Karlhede and MacCallum 1982). Consequently, computer algebra has become widely used in general relativity and several software systems written by relativists are available to perform relevant calculations.

In addition, software packages have been written for general relativity using more general purpose computer algebra systems (Czapor and McLenaghan 1986, Van den Bergh 1988). There are many such general purpose systems available, for example, REDUCE (Hearn 1983), one of the most widely used, has a high level programming language as well as the ability to perform interactive calculations. With increases in power and especially memory, computer algebra systems are also becoming available for microcomputers. A recent addition to the field is Mathematica (Wayner 1989) which is available for Macintosh computers. It has powerful graphics facilities and this may overcome the main drawback of many of the widely used mainframe systems which have limited graphics and one dimensional input of expressions.

CHAPTER 2

Mathematical Introduction

§2.1 The Field Equations of General Relativity

Einstein's field equations of general relativity are expressed using the techniques and assumptions of Riemannian geometry (see e.g. Eisenhart 1949). At each point x^i in 4-dimensional space-time there is a symmetric tensor g_{ij} of rank 4 and signature 2 that determines the space-time geometry at that point. This is known as the metric tensor. The metric or line-element is given by

$$ds^2 = g_{ij}\ dx^i\ dx^j \qquad (2.1.1)$$

where here and henceforward the Einstein summation convention is assumed over the range 1 to 4 for lower case Latin indices and 1 to 3 for Greek indices. Einstein suggested that g_{ij} should depend on the nature and distribution of everything in the space-time that can produce a gravitational field. It is therefore required that g_{ij} should be related to some tensor T_{ij}, known as the energy-momentum tensor, that describes the matter at that point. In order to determine g_{ij} it should be possible to solve a set of equations for g_{ij}, once T_{ij} has been specified.

To derive these gravitational field equations the following assumptions are made in the case of general relativity (see e.g. Stephani 1982b):

1) The field equations should be tensor equations so that they hold in all coordinate systems.
2) Like many other physical laws they should be second order partial differential equations in the functions to be determined (g_{ij}) and linear in the highest derivatives.

3) In some appropriate limit they should correspond with Newtonian gravitational theory:

$$\nabla^2\phi = 4\pi G\rho , \tag{2.1.2}$$

where ϕ is the gravitational potential, G is Newton's constant of gravitation and ρ the (Newtonian) mass density.

4) The source of the gravitational field should be the energy-momentum tensor.

5) If the space-time is flat, the energy-momentum tensor should vanish.

We require a tensor G_{ij} and using condition (4) write

$$G_{ij} = k\, T_{ij} , \tag{2.1.3}$$

where k is a constant that may be determined from condition (3) once the form of G_{ij} has been established. From considerations of angular momentum we find that the tensors are symmetric

$$T_{[ij]} = 0 \Rightarrow G_{[ij]} = 0 . \tag{2.1.4}$$

From energy-momentum conservation we obtain

$$T^{ij}{}_{;j} = 0 \Rightarrow G^{ij}{}_{;j} = 0 . \tag{2.1.5}$$

It can be shown that there is, in fact, only one possible form for G_{ij}, which satisfies condition (2). This is defined by

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} , \tag{2.1.6}$$

where Λ is an arbitrary constant known as the cosmological constant and R_{ij} and R are the Ricci tensor and Ricci scalar defined by

$$R_{ij} = R^k{}_{ikj} , \; R = R^i_i \tag{2.1.7}$$

and R_{ijkl} is the Riemann curvature tensor given by:

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\,\Gamma^m_{jl} - \Gamma^i_{ml}\,\Gamma^m_{jk} \,. \qquad (2.1.8)$$

Finally, we can see that the tensor (2.1.6) is second order in derivatives of the metric tensor from the definitions of the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(g_{lj,k} + g_{lk,j} - g_{jk,l}\right) \,. \qquad (2.1.9)$$

The following symmetries of the Riemann tensor may be deduced from (2.1.8) and (2.1.9)

$$R_{ijkl} = -R_{jikl} = -R_{jilk} \,, \quad R_{ijkl} = R_{klij} \qquad (2.1.10)$$

and furthermore

$$R_{i[jkl]} = \frac{1}{3}\left(R_{ijkl} + R_{iklj} + R_{iljk} \right) = 0 \,. \qquad (2.1.11)$$

The Riemann tensor is derived in such a way that if it vanishes, parallel transport is independent of path and the space-time is said to be flat. In this case the Ricci tensor and Ricci scalar clearly vanish, but T_{ij} is non-zero, unless $\Lambda = 0$, which violates requirement (5). The cosmological constant is sometimes assumed to be non-zero although there is no accurate observational evidence for any particular value. For example, the Einstein static universe had a non-zero Λ, introduced before the Hubble expansion of the universe was known. A non-zero Λ has been useful in cosmology more recently, for example, in inflation theory (Guth 1981).

It is worth noting that any g_{ij} is a solution to the field equations for some T_{ij}, since given g_{ij}, T_{ij} can be calculated. However, in general, this energy-momentum tensor will not be physically reasonable.

In attempting to find more relevant solutions to Einstein's field equations, (2.1.3) with G_{ij} given by (2.1.6), T_{ij} is assumed to take on a particular form depending on the nature of the matter. For example, in a vacuum T_{ij} vanishes. For tractability, the anisotropic pressure and heat flux of a general fluid approximation are assumed to be negligible and the energy-momentum tensor is then that of a perfect fluid (Weinberg 1972). This is given by

$$T_{ij} = \mu u_i u_j + ph_{ij} \tag{2.1.12}$$

where u_i is tangential to the world-lines determined by the 'average velocity' of the matter and is normalised so that

$$u_i u^i = -1 \ . \tag{2.1.13}$$

The quantities μ and p are the energy density and isotropic pressure of the fluid as defined by an observer co-moving along u_a. A 3+1 splitting of space-time is determined by u_i and the tensor h_{ij} defined by

$$h_{ij} = g_{ij} + u_i u_j \tag{2.1.14}$$

which is used to project tensors into the rest space of an observer moving with the fluid 4-velocity u_i .

§2.2 Perfect Fluids

From now on this thesis is concerned with the study of exact solutions of Einstein's field equations of general relativity. In particular, the motion of matter under self-gravitation and internal pressure forces is considered so that such solutions may be considered as models of the gravitational field in stellar interiors and as cosmological models. Henceforward the simplifying assumptions mentioned above are assumed so we are dealing with the energy-momentum tensor of a perfect fluid. The field equations (2.1.6) imply that

$$R_{ij} = \frac{1}{2}\left(\mu + 3p\right) u_i u_j + \frac{1}{2}\left(\mu - p\right) h_{ij} \qquad (2.2.1)$$

where the cosmological constant has been absorbed into the energy-momentum tensor by redefining μ and p as

$$\mu = \mu_{\text{fluid}} + \Lambda \ , \ p = p_{\text{fluid}} - \Lambda \ . \qquad (2.2.2)$$

The covariant derivative of the 4-velocity may be split up relative to u_i so that

$$u_{i;j} = \omega_{ij} + \sigma_{ij} + \frac{1}{3}\theta \ h_{ij} - \dot{u}_i u_j \ , \qquad (2.2.3)$$

where ω_{ij}, σ_{ij}, θ and $\dot{u}_i$ are respectively the rotation, shear, expansion and acceleration of the flow. The labels given to each of these 'kinematical' quantities are justified from consideration of the relative motion of neighbouring particles in the fluid over some small period of time (see e.g. Ellis 1971). The splitting arises naturally by decomposing the components of the derivative of u_i orthogonal to u_i into its antisymmetric part, symmetric trace-free part and the trace. The quantities have the following symmetries:

$$\sigma_{ij} = \sigma_{(ij)} \ , \quad \omega_{ij} = \omega_{[ij]}$$
$$\sigma_i{}^i = 0 \qquad (2.2.4)$$

with the definitions

$$\theta = u^i{}_{;i} \ , \ \dot{u}_i = u_{i;j} u^j \ ,$$

$$\sigma_{ij} = u_{(i;j)} - \frac{\theta}{3} h_{ij} - \dot{u}_{(i} u_{j)} \ ,$$

$$\omega_{ij} = u_{[i;j]} - \dot{u}_{[i} u_{j]} \ . \qquad (2.2.5)$$

Using (2.1.13) it is easily shown that the quantities are orthogonal to the 4-velocity, i.e.

$$\dot{u}_i u^i = \omega_{ij} u^i = \sigma_{ij} u^i = 0 \ . \qquad (2.2.6)$$

In addition, we define the shear and rotation scalars as

$$\sigma^2 = \frac{1}{2}\,\sigma_{ij}\,\sigma^{ij} \ , \quad \omega^2 = \frac{1}{2}\,\omega_{ij}\,\omega^{ij} \ , \qquad (2.2.7)$$

where in each case the vanishing of either the scalar or the tensor is sufficient to ensure the vanishing of the other. Finally, the rotation vector is defined as

$$\omega_i = \frac{1}{2}\,\eta_{ijkl}\;u^j\;\omega^{kl} \ , \qquad (2.2.8)$$

where η_{ijkl} is the completely anti-symmetric Levi-Cevita permutation tensor.

A natural classification scheme for perfect fluids is based on the vanishing of certain of the kinematic quantities. Except in a very few special cases, further restrictions on the space-time are necessary in order to solve the field equations.

At some stage in solving the field equations a coordinate system is introduced. Commonly, this takes the form of a co-moving coordinate system (i.e. the flow lines are given by x^α = constant) in which

$$u^a = -\frac{1}{V}\,\delta^a_4 \ . \qquad (2.2.9)$$

It is always possible to choose coordinates in this way (Eisenhart 1949). In general, u_a has more than one non-zero component. The line-element then takes the form (Trümper 1962)

$$ds^2 = e^{2M}\,\gamma_{\alpha\beta}\;dx^\alpha\;dx^\beta - V^2\left(dt + a_\alpha\;dx^\alpha\right)^2 \qquad (2.2.10)$$

where M, V, $\gamma_{\alpha\beta}$ and a_α are, in general, functions of all four coordinates and det $\left|\gamma_{\alpha\beta}\right|$ = 1. The kinematic quantities can be calculated in this coordinate system to obtain the following non-zero components:

$$\omega_{\alpha\beta} = -V\left(a_{[\alpha,\beta]} + \dot{a}_{[\alpha}\, a_{\beta]}\right)$$

$$\sigma_{\alpha\beta} = (2V)^{-1}\; e^{2M}\; \dot{\gamma}_{\alpha\beta}$$

$$\theta = \frac{3\dot{M}}{V}$$

$$\dot{u}_{\alpha} = V^{-1}\left(Va_{\alpha}\right)^{\cdot} + \frac{V_{,\alpha}}{V} \qquad (2.2.11)$$

where a dot denotes differentiation with respect to time.

In general, if the vorticity vanishes, Frobenius' theorem shows that

$$\omega = 0 \Leftrightarrow u_{[a}\, u_{b;c]} = 0$$

$$\Leftrightarrow \text{ locally } \exists \text{ functions } f,\ g:$$

$$u_a = f\, g_{,a} \qquad (2.2.12)$$

Thus there exist, locally, 3-surfaces in space-time orthogonal to the velocity vector field. Having chosen comoving coordinates as in (2.2.9) the allowable coordinate freedom still allows us to choose $t = g$ so that $u_a = V\delta^4_a$ from the normalisation (2.1.13). In this case $a_\alpha = 0$ in (2.2.10). If, in addition, the acceleration vanishes the fluid flow is said to be geodesic and it is possible to set $V = 1$ by rescaling the t coordinate. If the rotation is non-vanishing but $\sigma = \theta = 0$ then for an observer co-moving with u_i the distances to neighbouring matter are constant and we have a rigid rotation.

§2.3 The Ricci and Bianchi Identities

In this section the Ricci and Bianchi identities are written in a way that allows any restrictions on the kinematics of the fluid to be imposed directly. The resulting equations are useful in deriving theorems on exact solutions and in finding simplifications that may allow the field equations to be solved. The Ricci identity for the fluid 4-velocity is

$$u_{i;jk} - u_{i;kj} = u_l \, R^l{}_{ijk} \ . \tag{2.3.1}$$

Clearly the left hand side of this may be written in terms of the kinematic quantities using (2.2.3).

The Riemann tensor may be written using the Weyl tensor, defined by

$$C^{ij}{}_{kl} = R^{ij}{}_{kl} + \frac{R}{3} \, \delta^{\,i}_{[k} \, \delta^{j}_{l]} - 2 \, \delta^{[i}_{[j} \, R^{k]}_{l]} \ . \tag{2.3.2}$$

The Weyl tensor has the same symmetry properties as the Riemann tensor and in addition satisfies $C^{ij}{}_{ik} = 0$. In what follows the Weyl tensor will be replaced by two symmetric trace-free tensors, E_{ij} and H_{ij}, the electric and magnetic parts of the Weyl tensor relative to u_i , so called by analogy with electromagnetism. These are defined by

$$E_{ij} = C_{ikjl} \, u^k \, u^l \ , \tag{2.3.3}$$

$$H_{ij} = \frac{1}{2} \, \eta_{ikmn} \, C^{mn}{}_{jl} \, u^k \, u^l \ . \tag{2.3.4}$$

The Weyl tensor is completely determined by E_{ij} and H_{ij} and can be expressed as

$$C_{ijkl} = \left(\eta_{ijmn} \, \eta_{klqp} + g_{ijmn} \, g_{klqp}\right) u^m \, u^q \, E^{np} - \left(\eta_{ijmn} \, g_{klqp} + g_{ijmn} \, \eta_{klqp}\right) u^n \, u^q \, H^{np} \tag{2.3.5}$$

where

$$g_{ijkl} = g_{ik}\, g_{jl} - g_{il}\, g_{jk} \ . \tag{2.3.6}$$

We now make use of the field equations (2.2.1) and the splitting of the covariant derivative of the 4-velocity (2.2.3) to write the Ricci identities in terms of the kinematical quantities. By examining appropriate symmetries and contractions with u_a the Ricci identities (2.3.1) can be put in the following form (Ellis 1971):

$$\dot{\theta} + \frac{1}{3}\theta^2 - \dot{u}^i{}_{;i} + 2(\sigma^2 - \omega^2) + \frac{1}{2}(\mu + 3p) = 0 \ , \tag{2.3.7}$$

$$h_i{}^k h_j{}^l \dot{\omega}_{kl} - h_i{}^k h_j{}^l \dot{u}_{[k;l]} + 2\sigma_{i[l}\omega^l{}_{j]} + \frac{2}{3}\theta\omega_{ij} = 0 \ , \tag{2.3.8}$$

$$h_i{}^k h_j{}^l \dot{\sigma}_{kl} - h_i{}^k h_j{}^l \dot{u}_{(k;l)} - \dot{u}_i \dot{u}_j + \omega_{il}\omega^l{}_j + \sigma_{il}\sigma^l{}_j + \frac{2}{3}\theta\sigma_{ij} + \frac{1}{3} h_{ij}(u^k{}_{;k} + 2(\omega^2 - \sigma^2)) + E_{ij} = 0 \ , \tag{2.3.9}$$

$$\omega_{[ij;k]} - \dot{u}_{[i;k}u_{j]} - \dot{u}_{[i}\omega_{jk]} = 0 \ , \tag{2.3.10}$$

$$h_i{}^j\,(\omega_j{}^k{}_{;k} - \sigma_j{}^k{}_{;k} + \frac{2}{3}\theta_{,j}) + (\omega_i{}^j + \sigma_i{}^j)\dot{u}_j = 0 \ , \tag{2.3.11}$$

$$2\,\dot{u}_{(i}\,\omega_{j)} + h_i{}^k h_j{}^l(\omega_{(k}{}^{m;n} + \sigma_{(k}{}^{m;n})\,\eta_{l)pmn}u^p = H_{ij} \ , \tag{2.3.12}$$

The Riemann tensor satisfies the Bianchi identities

$$R_{ij[kl;m]} = 0 \Leftrightarrow C^{ijkl}{}_{;l} = R^{k[i;j]} - \frac{1}{6} g^{k[i} R^{;j]} \ , \tag{2.3.13}$$

which imply the contracted Bianchi identities

$$R^{ij}{}_{;j} = \frac{1}{2} R^{;i} \qquad (2.3.14)$$

and hence $G^{ij}{}_{;j} = 0$ leading to the energy-momentum conservation equations for a perfect fluid:

$$\dot{\mu} + (\mu + p)\,\theta = 0 \ , \qquad (2.3.15)$$

$$h^k{}_i \; p_{,k} + (\mu + p)\; \dot{u}_i = 0 \ . \qquad (2.3.16)$$

Applying the 3+1 splitting to the Bianchi identities (2.3.7) we obtain the conservation equations above and the 16 equations:-

$$h_i{}^j E_{jk;l} h^{kl} + 3H_{ij}\omega^j + \eta_{ijkl} u^j \sigma^{km} H^l{}_m = \frac{1}{3}\mu_{,j} h_i{}^j \qquad (2.3.17)$$

$$h_i{}^j H_{jk;l} h^{kl} + \eta_{ijkl} u^j \sigma^{km} E^l{}_m - 3E_{ij}\omega^j = (\mu + p)\,\omega_i \qquad (2.3.18)$$

$$h_i{}^k h_j{}^l \dot{E}_{kl} + h^n{}_{(i}\eta_{j)klm} u^k H_n{}^{l;m} + E_{ij}\theta - 3E_{(i}{}^k\sigma_{j)k} + h_{ij}E_{kl}\sigma^{kl} - E^k{}_{(i}\omega_{j)k} + 2H_{(i}{}^l\eta_{j)klm}u^k u^m = -\frac{1}{2}(\mu + p)\sigma_{ij} \qquad (2.3.19)$$

$$h_i{}^k h_j{}^l \dot{H}_{kl} - h^n{}_{(i}\eta_{j)klm} u^k E_n{}^{l;m} + H_{ij}\theta - 3H_{(i}{}^k\sigma_{j)k} + h_{ij}H_{kl}\sigma^{kl} - H^k{}_{(i}\,\omega_{j)k} - 2E_{(i}{}^l\eta_{j)klm}u^k u^m = 0 \ . \qquad (2.3.20)$$

§2.4 The Tetrad Formalism

At each point in space-time we may set up a tetrad, i.e a basis of four independent vectors, with contravariant components e^i_A where $A = 1 \ldots 4$ identifies individual vectors of the tetrad and is known as a tetrad index (see e.g. Ellis and MacCallum 1969, Israel 1970). Associated with these vectors are the covariant vectors defined by

$$e_{Ai} = g_{ij} \, e^j_A \ . \tag{2.4.1}$$

In addition, we define the dual basis vectors e^A_i as the matrix inverse of e^i_A so that

$$e^i_A \, e^B_i = \delta^B_A \tag{2.4.2}$$

and

$$\sum_{A=1}^{4} e^i_A \, e^A_j = \delta^i_j \tag{2.4.3}$$

where now and henceforward the summation convention is not applied over repeated tetrad indices as the range of summation varies according to context. Note that

$$e^i_A \, e_{Bi} = e^i_A \, e^j_B \, g_{ij} \tag{2.4.4}$$

and writing

$$g_{AB} = e^i_A \, e_{Bi} \tag{2.4.5}$$

gives g_{AB} as the tetrad components of the metric tensor. We may define the inverse of g_{AB}

$$\sum_{C=1}^{4} g^{AC} g_{BC} = \delta^{A}_{B} \quad . \qquad (2.4.6)$$

Given a tensor we can project it into the tetrad frame to obtain its tetrad components, for example

$$a_A = a_i e^i_A \qquad (2.4.7)$$

with an obvious generalisation to higher rank tensors where each free tensor index is contracted with a tetrad vector. Clearly it is possible to change freely from tensor to tetrad components and vice-versa from (2.4.2) and (2.4.7). Also, it should be noted that tetrad indices are raised and lowered with g^{AB} and g_{AB} respectively in an analogous way to tensor indices with the metric tensor g_{ij}.

We can also define a tetrad component of a covariant derivative as follows

$$\phi_{\cdot A} = \phi_{;i} e^i_A$$

for some arbitrary scalar ϕ, this can be extended to higher order, for example

$$P_{A \cdot B} = P_{A;i} e^i_B \qquad (2.4.8)$$

where P_A are the tetrad components of some tensor P_i. Writing $P_A = P_j e^j_A$ on the right hand side of (2.4.8) and expanding gives

$$P_{A \cdot B} = P_{j;i} e^j_A e^i_B - \sum_{C} \gamma_{CAB} P^C \quad , \qquad (2.4.9)$$

where

$$\gamma_{ABC} = e_{Ai;j}\, e^{i}_{B}\, e^{j}_{C} \qquad (2.4.10)$$

are known as the Ricci rotation coefficients. In this thesis the tetrad will always be chosen so that g_{AB} is constant. The Ricci rotation coefficients are then antisymmetric in the first two indices:

$$\gamma_{ABC} = -\gamma_{BAC} \qquad (2.4.11)$$

which can be verified by considering

$$g_{AB;C} = \left(e_{Ai}\, e^{i}_{B}\right)_{;C} = 0 \ . \qquad (2.4.12)$$

Hence there are 24 independent rotation coefficients. We now have a prescription for rewriting tensor equations in terms of tetrad components. The covariant derivatives can be replaced by the tetrad derivatives and terms involving the Ricci rotation coefficients and the tetrad components. This is accomplished, along with obtaining tetrad components of tensors, by contracting all free tensor indices with a tetrad basis vector. This procedure can be applied immediately to the field equations, Ricci identities (2.3.1) and Bianchi identities (2.3.7). However, it will prove more useful to take tetrad components of the equations derived in §2.3 in terms of the kinematic quantities. It is worth noting that the tetrad approach decomposes the second order equations into first order equations of first order objects, the rotation coefficients.

The remaining element in the use of tetrads in general relativity is the choice of tetrad. For perfect fluids an orthonormal tetrad of basis vectors is usually chosen so that

$$g_{AB} = \mathrm{diag}\ (1, 1, 1, -1) \ . \qquad (2.4.13)$$

The tetrad is given by three space-like vectors e_x , e_y and e_z and a timelike vector e_t, usually chosen to be parallel to the 4-velocity.

Finally the commutation relations of the tetrad vectors are given by

$$[e_A , e_B] = \sum_{C=1}^{4} (\gamma_{BCA} - \gamma_{ACB}) e^C \qquad (2.4.14)$$

and these are calculated explicitly where needed in future chapters.

An alternative to an orthonormal tetrad is a complex null tetrad composed of two real null vectors k_a and l_a and two complex conjugate null vectors m_a and $\overline{m}_a$. The vectors are assumed to satisfy

$$k_a l^a = -1, \quad m^a \overline{m}_a = 1 ,$$

where all other contractions vanish. This is often called the Newman-Penrose (1962) null tetrad and an example can be related to an orthonormal tetrad as follows:

$$k^i = \frac{1}{\sqrt{2}} \left(e_t^i + e_z^i \right) ,$$

$$l^i = \frac{1}{\sqrt{2}} \left(e_t^i - e_z^i \right) ,$$

$$m^i = \frac{1}{\sqrt{2}} \left(e_x^i + i e_y^i \right) ,$$

$$\overline{m}^i = \frac{1}{\sqrt{2}} \left(e_x^i - i e_y^i \right) . \qquad (2.4.15)$$

The tetrad metric in this case is given by

$$g_{AB} = g^{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(2.4.16)

Since the basis is complex it is only necessary to write down half the number of equations of the orthonormal case. To ease in the legibility of these tetrad equations, the Ricci rotation coefficients are replaced by twelve complex quantities known as the Newman-Penrose spin coefficients:

$$\gamma_{311} = \kappa \qquad \gamma_{314} = \rho \qquad \frac{1}{2}\left(\gamma_{211} + \gamma_{341}\right) = \varepsilon \ ,$$

$$\gamma_{313} = \sigma \qquad \gamma_{243} = \mu \qquad \frac{1}{2}\left(\gamma_{212} + \gamma_{342}\right) = \gamma \ ,$$

$$\gamma_{244} = \lambda \qquad \gamma_{312} = \tau \qquad \frac{1}{2}\left(\gamma_{214} + \gamma_{344}\right) = \alpha \ ,$$

$$\gamma_{242} = \nu \qquad \gamma_{241} = \pi \qquad \frac{1}{2}\left(\gamma_{213} + \gamma_{343}\right) = \beta \ .$$

(2.4.17)

By interchanging the indices 3 and 4 we obtain the complex conjugates of the respective quantities. The tetrad derivatives along l, n, m and $\overline{m}$ are denoted by the operators D, Δ, δ and $\bar{\delta}$ respectively. The independent tetrad components of the Weyl tensor are written as complex scalars (Kramer et al 1980) given by

$$\Psi_0 = C_{ijkl}\, k^i\, m^j\, k^k\, m^l \ ,$$

$$\Psi_1 = C_{ijkl}\, k^i\, l^j\, k^k\, m^l \ ,$$

$$\Psi_2 = \frac{1}{2} C_{ijkl}\, k^i\, l^j\, (k^k\, l^l - m^k\, \overline{m}^l) \ ,$$

$$\Psi_3 = C_{ijkl}\, l^i\, k^j\, l^k\, \overline{m}^l \ ,$$

$$\Psi_4 = C_{ijkl}\, l^i\, \overline{m}^j\, l^k\, \overline{m}^l \qquad (2.4.18)$$

and the Ricci tensor is represented by complex scalars using tetrad components of the trace-free Ricci tensor S_{ij} defined by

$$S_{ij} = R_{ij} - \frac{Rg_{ij}}{4}$$

and by R itself. In this way we may produce spin coefficient versions of relevant tensor equations (Newman and Penrose (1962)). If the time-like tetrad vector of (2.4.15) is aligned with the fluid 4-velocity the kinematic quantities can be expressed as (Allnutt 1982):

$$\dot{u}_i = \frac{1}{\sqrt{2}}(\varepsilon + \overline{\varepsilon} + \gamma + \overline{\gamma})\, v_i + \frac{1}{\sqrt{2}}(\pi - \overline{\kappa} + \upsilon - \overline{\tau})\, m_i$$
$$+ \frac{1}{\sqrt{2}}(\overline{\pi} - \kappa + \overline{\upsilon} - \tau)\, \overline{m}_i \ , \qquad (2.4.19)$$

$$\sigma_{ij} = A1\,(v_i v_j - m_{(i}\overline{m}_{j)}) + A2\, v_{(i} m_{j)}$$
$$+ \overline{A2}\, v_{(i} m_{j)} + A3\, m_i m_j + A3\, \overline{m_i m_j} \ , \qquad (2.4.20)$$

where

$$v_i = \frac{1}{\sqrt{2}}(k_i - l_i) \ ,$$

$$A1 = \frac{-\sqrt{2}}{3}(\rho + \overline{\rho} - \mu - \overline{\mu} + 2(\varepsilon + \overline{\varepsilon}) - 2(\gamma + \overline{\gamma})) \ , \qquad (2.4.21)$$

$$A2 = \frac{-1}{2}(\overline{\tau} + \pi + 2(\alpha + \overline{\beta}) - \overline{\kappa} - \upsilon) \ , \qquad (2.4.22)$$

$$A3 = \frac{1}{\sqrt{2}}(\overline{\sigma} - \lambda) \ , \qquad (2.4.23)$$

and

$$\omega_{ij} = B1\, v_{[i} m_{j]} + \overline{B1}\, v_{[i} \overline{m}_{j]} + B2\, m_{[i} \overline{m}_{j]} \ , \qquad (2.4.24)$$

where

$$B1 = \frac{1}{2}(\overline{\tau} + \pi - 2(\alpha + \overline{\beta}) - \overline{\kappa} - \upsilon) \ , \qquad (2.4.25)$$

$$B2 = \frac{-1}{\sqrt{2}}(\rho - \bar{\rho} + \mu - \bar{\mu}) \tag{2.4.26}$$

and finally

$$\theta = \frac{1}{\sqrt{2}}(\varepsilon + \bar{\varepsilon} - \gamma - \bar{\gamma} - \rho - \bar{\rho} + \mu + \bar{\mu}) \quad . \tag{2.4.27}$$

Although the relevant equations can appear very different in the null and orthonormal tetrad formalisms the approaches are essentially equivalent. Each method is suited to particular problems in relativity depending on the nature of any vector fields under consideration. For example, orthonormal tetrads are commonly used to study perfect fluids (Ellis 1967) and null tetrads for gravitational radiation and electromagnetism (Newman and Penrose 1962). Null congruences are generally less relevant to perfect fluid space-times where the energy-momentum tensor defines a preferred time-like vector field. Null vectors are relevant to the Petrov classification which can be considered in terms of principal null directions (p.n.d's), n_i (Penrose 1960) defined by

$$n_{[m} C_{i]jk[l} n_{o]} n^j n^k = 0 \quad . \tag{2.4.28}$$

A fourth order equation can be derived for n_i and the multiplicities of the roots of this equation define the Petrov type (Kramer et al 1980).

Type	Multiplicities
I	(1,1,1,1)
D	(2,2)
II	(2,1,1)
III	(3,1)
N	(4)
O	0

The Petrov types D, II, III, N and O have a repeated p.n.d and are known as algebraically special. If the Newman-Penrose basis vector k_i is chosen to coincide with n_i then it can be shown that (Penrose 1960) $\Psi_0 = 0$. Similarly, if l_i is identified with a p.n.d then it can be shown that $\Psi_4 = 0$. This can be extended so that one can make use of the above characterisation of Petrov types to choose the tetrad vectors such that in each case certain of the Weyl scalars vanish. It is therefore easy to impose a specific Petrov type on spin coefficient equations. In the remainder of this thesis, fields with a purely electric Weyl tensor are considered and their possible Petrov types are given in the next chapter.

CHAPTER 3

Space-times with a Purely Electric Weyl Tensor

§3.1 Introduction

In the previous two chapters the mathematical and physical foundations of general relativity have been outlined. In addition, some background in computer algebra and exact solutions has been given as an introduction to the work presented here. The rest of this thesis is concerned with finding new, and classifying known, exact solutions. In particular, I will examine a class of perfect fluids in general relativity characterised by the vanishing of the magnetic part of the Weyl tensor- purely electric fields. Although this is a severe restriction on the Weyl tensor, it does not appear to be too restrictive on possible solutions as many known solutions have this property. Spherically symmetric, LRS, (Stewart and Ellis 1968) and static (Barnes 1972) perfect fluid metrics for example, necessarily have $H_{ij} = 0$. Space-times with $\sigma_{ij} = \omega_{ij} = 0$ are also purely electric as can be seen from (2.3.12). Only conformally flat fields also have a vanishing electric part of the Weyl tensor, from (2.3.3). Purely electric fields therefore contain, for example, the Robertson-Walker cosmologies. Conformally flat perfect fluids are also shear-free and irrotational. Putting $H_{ij} = E_{ij} = 0$ in (2.3.18) and (2.3.19) it follows that

$$(\mu + p)\omega_a = 0 \ ,$$

$$(\mu + p)\sigma_{ab} = 0 \ .$$

Thus when $\mu + p \neq 0$ the shear and vorticity vanish. If $\mu + p = 0$ the space is an Einstein space and being conformally flat has constant curvature (Eisenhart 1949). This result will be useful to rule out such classes as they are well-known De Sitter space-times or Minkowski flat space-time.

The Gödel solution (Gödel 1949) is also purely electric (Barnes 1984). This is a rotating shear-free metric and shows that $\sigma_{ij} = H_{ij} = 0 \Rightarrow \omega_{ij} = 0$ is not valid as claimed by Glass (1975). Several recently discovered perfect fluid solutions, found by a variety of methods, are also purely electric, for example:

1) The Szekeres (1975) and Szafron (1977) solutions derived by assuming a particular metric form with only two metric functions to be determined.

2) The Allnutt (1982) type D solutions found by imposing ad hoc restrictions on the Newman-Penrose coefficients and by assuming the 4-velocity and acceleration to be co-planar with the repeated principal null directions of the Weyl tensor.

3) The Stephani (1982a) rotating, expanding and shearing dust metric found by assuming a particularly simple metric form with only one metric function.

4) The Wolf (1986) rotating, expanding and shearing perfect fluid solution. This was found by assuming the metric to admit a family of flat three-dimensional slices and a tensor of exterior curvature covariantly constant within the slices.

5) The Senovilla (1986) stationary axisymmetric solution found by assuming the fluid 4-velocity to lie in the plane spanned by the repeated p.n.d's of the Weyl tensor in addition to the existence of a Killing vector parallel to the fluid 4-velocity.

In all of the above cases the resulting solutions had a purely electric Weyl tensor despite the fact that no restriction was placed on the Weyl tensor initially (apart from cases (2) and (5) where Petrov type D was assumed).

In this chapter the possible Petrov types of purely electric fields will be derived. A suitable mathematical approach to purely electric perfect fluid fields using both the Newman-Penrose formalism and the orthonormal tetrad formalism is then given. The fields are sub-divided by placing restrictions on the fluid 4-velocity. Space-times with either vanishing shear or vanishing rotation are examined as those where both quantities vanish have been analysed in a previous work (Barnes 1973). An extension to a class of rotating, shearing metrics is suggested as few such solutions are known. Ozsvath (1965) has found some homogeneous examples without expansion. The known rotating and expanding perfect fluids are necessarily shearing and hence are fairly general solutions of the Einstein field equations for a perfect fluid. A dust solution in this class is due to Stephani (1982) but as far as the author is aware, only one solution is known with shear, rotation, expansion and acceleration (Wolf 1986). These two solutions are both type D with a purely electric Weyl tensor. Furthermore they do not have high symmetry and may be relevant as inhomogeneous cosmologies. Homogeneous solutions with rotation, expansion and shear have been found by Demianski and Grishchuk (1972) and Rosquist (1983).

§3.2 Petrov Types

The vanishing of the magnetic part of the Weyl tensor simplifies the analysis of the Petrov type of the space-time and rules out three of the possible types. We follow the approach based on the symmetric tensor Q_{ij} defined by (Kramer et al 1980):-

$$Q_{ij} = E_{ij} + iH_{ij} = \overset{+}{C}_{ijkl} u^k u^l \qquad (3.2.1)$$

where $\overset{+}{C}_{abcd}$ is the self-dual Weyl tensor

$$\overset{+}{C}_{ijkl} = C_{ikjl} + \frac{1}{2} i \, \eta_{ikmn} C^{mn}{}_{jl} \ .$$

The Petrov classification is given by the normal form of Q_{ij} and the nature of its eigenvalues. This technique is consistent with the principal null directions approach outlined in chapter 2. The normal form of Q_{ij} for Petrov types 1, D and 0 is diagonal. For Petrov types 1, D and 0 an orthonormal tetrad may be chosen so that

$$E^{ij} + iH^{ij} = \lambda_1 e_1^i e_1^j + \lambda_2 e_2^i e_2^j + \lambda_3 e_3^i e_3^j \quad (3.2.2)$$

where for type 1 space-times the λ_A 's are distinct, for type D two of the λ_A are equal and for type O all the λ_A are zero. The tetrad vectors are clearly eigenvectors. We set

$$\lambda_A = \alpha_A + i\beta_A \quad (3.2.3)$$

so that α and β are the eigenvalues of E_{ij} and H_{ij} respectively. If the Weyl tensor is purely electric, then from the principal axes theorem for a real symmetric tensor, Q_{ij} may be diagonalised by a tetrad rotation that keeps u_i fixed. Similarly Q_{ij} can be diagonalised in the purely magnetic case or in fact whenever there exist scalars a and b such that $aE=b\mathbf{H}$ and a and b are not both zero. Consequently all such space-times, and all space-times to be considered in this thesis, are of Petrov types 1, D or 0.

§3.3 Purely Electric Space-times in the Newman-Penrose Formalism

Carminati and Wainwright (1985) have studied purely electric type D solutions using the Newman-Penrose formalism. In order to simplify the Bianchi identities they imposed the additional restriction that the 4-velocity lies in the 2-space defined by the principal null directions of the Weyl tensor. However it can be shown that all such solutions satisfy this restriction (Barnes 1987). In fact, it will be shown below that u_i is necessarily aligned with a null tetrad determined by the Weyl curvature even in the type 1 case.

Equation (3.58) of Kramer et al (1980) is

$$\frac{1}{2}\overset{+}{C}_{abcd} = \Psi_0 U_{ab} U_{cd} + \Psi_1 (U_{ab} W_{cd} + W_{ab} U_{cd}) + \Psi_2 (V_{ab} U_{cd} + U_{ab} V_{cd} + W_{ab} W_{cd}) + \Psi_3 (V_{ab} W_{cd} + W_{ab} V_{cd}) + \Psi_4 V_{ab} V_{cd} \ , \qquad (3.3.1)$$

where the bivectors U_{ab} , V_{ab} and W_{ab} are defined by

$$U_{ab} = -l_a \bar{m}_b + l_b \bar{m}_a \ ,$$

$$V_{ab} = k_a m_b - k_b m_a \ ,$$

$$W_{ab} = m_a \bar{m}_b - m_b \bar{m}_a - k_a l_b + k_b l_a \ , \qquad (3.3.2)$$

and $\Psi_0 \ldots \Psi_4$ are the Newman-Penrose components of the Weyl tensor defined with respect to the complex null tetrad (2.4.15) with e_t identified with the fluid 4-velocity. A straightforward calculation then shows that

$$E^{ab} = \mathrm{Re}(\Psi_2)\left(e_1^a e_1^b + e_2^a e_2^b - 2e_3^a e_3^b \right) + \frac{1}{2}\mathrm{Re}\,(\Psi_0 + \Psi_4)\left(e_2^a e_2^b - e_1^a e_1^b\right) - \mathrm{Im}\,(\Psi_0 - \Psi_4)\, e_1^{(a} e_2^{b)} + \mathrm{Re}(\Psi_1 - \Psi_3)\, e_1^{(a} e_3^{b)} + \mathrm{Im}(\Psi_1 + \Psi_3)\, e_2^{(a} e_3^{b)}$$

(3.3.3)

and

$$H^{ab} = \mathrm{Im}(\Psi_2)\left(e_1^a e_1^b + e_2^a e_2^b - 2e_3^a e_3^b \right) + \frac{1}{2}\mathrm{Im}\,(\Psi_0 + \Psi_4)\left(e_2^a e_2^b - e_1^a e_1^b\right) + \mathrm{Re}(\Psi_0 - \Psi_4)\, e_1^{(a} e_2^{b)} + \mathrm{Im}(\Psi_1 - \Psi_3)\, e_1^{(a} e_3^{b)} - \mathrm{Re}(\Psi_1 + \Psi_3)\, e_2^{(a} e_3^{b)} \ .$$

(3.3.4)

The conditions for $H_{ab} = 0$ are clearly

$$\Psi_2 = \bar{\Psi}_2 \ , \quad \Psi_0 = \bar{\Psi}_4 \ , \quad \Psi_1 = -\bar{\Psi}_3 \qquad (3.3.5)$$

whereas the conditions for $\{e_A^{\ a}, u^a\}$ A=1,2,3 to be a Weyl principal tetrad are

$$\Psi_1 = \Psi_3 = 0, \quad \Psi_0 = \Psi_4 \ . \qquad (3.3.6)$$

For the terads we are considering both (3.3.5) and (3.3.6) hold and hence all the Ψ's are real, in fact we have

$$\Psi_1 = \Psi_3 = 0, \quad \Psi_0 = \Psi_4 = \frac{1}{2}(\alpha_2 - \alpha_1), \quad \psi_2 = -\frac{1}{2}\alpha_3 \ . \qquad (3.3.7)$$

For type D fields we have, by renumbering space-like tetrad indices if necessary, $\alpha_1 = \alpha_2$. Hence $\Psi_0 = \Psi_4 = 0$ and l^a and k^a are principal null vectors. Consequently u_a lies in the 2–space defined by the repeated principal null directions of the Weyl tensor so that

$$u_{[a}k_b l_{c]} = 0 \ . \qquad (3.3.8)$$

For type I fields k^a and l^a are not principal null vectors as $\Psi_0 = \Psi_4 \neq 0$. However u_a is co-planar with the null vectors k^a and l^a of the canonical null tetrad used by Åman and Karlhede (1982) in their computer-aided classification of space-time metrics. We can also see that this canonical null tetrad is simply the null tetrad associated with the orthonormal tetrad of Weyl principal vectors.

In addition to the alignment condition (3.3.8) and the Weyl tensor being purely electric and of Petrov type D, the main restrictions imposed by Carminati and Wainwright (1985) were

1) The existence of an equation of state $p = p(\mu)$ such that $\mu + p > 0$ and $|dp/d\mu| \leq 1$.
2) The conditions $C_{abcd} = 0$ and $\theta \neq 0$ can be satisfied as a special case. This requires there to be a non-static Robertson-Walker space-time contained within the class of solutions i.e. the space-times generalise the Robertson-Walker models in some way and may be relevant as cosmological models.

Using (3.3.8) it can be shown that (Wainwright 1977a):

$$\varphi_{01} = \varphi_{12} = \varphi_{02} = 0 \ ,$$

$$\varphi_{00} = \varphi_{22} = 2\varphi_{11} = \frac{1}{4}(\mu + p) \ , \qquad (3.3.9)$$

for the Newman-Penrose components of the trace-free part of the Ricci tensor. The Bianchi identities subject to (3.3.7) and (3.3.8) are given in Wainwright (1977a) and can be simplified by taking suitable linear combinations. The Bianchi identities of Carminati and Wainwright (1985) are then obtained by further assuming $H_{ij} = 0$ and $p = p(\mu)$. They are:

$$\sigma + \bar{\lambda} = 0$$

$$\sigma(3\Psi_2 + 4\varphi_{11}) = 0$$

$$\rho - \bar{\rho} = \mu - \bar{\mu}$$

$$(\rho - \bar{\rho})(3\Psi_2 + 4\varphi_{11}) = 0$$

$$3 (X + Y)\Psi_2 - 4 X \varphi_{11} = 0$$

$$3Y \Psi_2 - 8(\alpha + \bar{\beta})\varphi_{11} = 0$$

$$4\delta \varphi_{11} = -3(\kappa - \bar{\nu})\Psi_2 + 2Z\varphi_{11}$$

$$2\delta \Psi_2 = (3\tau - 3 \bar{\pi} + \kappa - \bar{\nu})\Psi_2$$

$$\delta(\varphi_{11} - 3\Lambda) = 2Z\varphi_{11}$$

$$(D - \Delta)(\varphi_{11} - 3\Lambda) = -4 (\varepsilon + \bar{\varepsilon} + \gamma + \bar{\gamma})\varphi_{11}$$

$$2(D - \Delta)(3\Psi_2 - 4\varphi_{11}) = 9(\rho + \bar{\rho} + \mu + \bar{\mu})\Psi_2 + 8(\varepsilon + \bar{\varepsilon} + \gamma + \bar{\gamma})\varphi_{11}$$

$$3(D + \Delta)(\varphi_{11} + \Lambda) = 4(\rho + \bar{\rho} - \mu - \bar{\mu} - \varepsilon - \bar{\varepsilon} + \gamma + \bar{\gamma})\varphi_{11}$$

$$3(D + \Delta)\Psi_2 = 9(\rho - \mu)\Psi_2 + 2(\mu + \bar{\mu} - \rho - \bar{\rho} - 2\varepsilon - 2\bar{\varepsilon} + 2\gamma + 2\bar{\gamma})\varphi_{11}$$

(3.3.10)

where

$$X = \tau + \bar{\pi} - \kappa - \bar{\upsilon}$$

$$Y = \tau + \bar{\pi} + \kappa + \bar{\upsilon}$$

$$Z = \tau - \bar{\pi} + \kappa - \bar{\upsilon}$$

Using these equations, and the spin-coefficient form of the Ricci identities and commutation relations (2.4.14) the only possible equations of state were found to be $p'(\mu)=1$ and $p'(\mu)=0$ if the solutions were not to admit at least three Killing vectors (Carminati and Wainwright 1985). The non LRS solutions fall into three classes depending on whether the scalar

$$K = 9\Psi_2^2 - 16\varphi_{11}^2$$

vanishes or not. The three classes generalise the LRS solutions by introducing anisotropies in the 2-space orthogonal to the repeated p.n.d's of the Weyl tensor. Defining the following scalars

$$J_1 = \mu_{,a}m^a \ , \ J_2 = \sigma_{ab}m^a m^b \ , \ J_3 = \omega_a m^a \ , \qquad (3.3.11)$$

the non-vanishing of any of J_1, J_2, J_3 is sufficient to ensure that the space-time is not LRS. The following classification of the non LRS solutions is then possible as in classes I and II $J_3=0$ and in classes I and III, $J_2=0$.

I: $K \neq 0$, $\Rightarrow \omega = 0$ and $J_1 \neq 0$, $J_2=J_3=0$,
(a) $p'(\mu)=0$ (b) static, 3 Killing vectors

II: $3\Psi_2 + 4\varphi_{11} = 0$, $p'(\mu)=1$,

(a) $J_1 \neq 0$, $J_2 \neq 0$, $J_3 = 0$; (b) $J_1 \neq 0$, $J_2 = J_3 = 0$; (c) $J_1 = 0$, $J_2 \neq 0$, $J_3 = 0$;

III: $3\Psi_2 - 4\Phi_{11} = 0$, $p'(\mu) = 0$,

(a) $J_1 \neq 0$, $J_2 = 0$, $J_3 \neq 0$; (b) $J_1 \neq 0$, $J_2 = J_3 = 0$; (c) $J_1 = 0$, $J_2 = 0$, $J_3 \neq 0$;

The Newman-Penrose formalism has also been widely used in the exact solution of the vacuum field equations. The Goldberg-Sachs (1962) theorem states that for type D vacuum solutions the repeated p.n.d's of the Weyl tensor are tangential to geodesic shear-free null congruences. Consequently these space-times are amenable to study using this method (Kinnersley 1969).

In the case of purely electric fields, the orthonormal tetrad formalism has the advantage that the Weyl tensor defines a canonical orthonormal tetrad in addition to the preferred time-like vector field defined by the 4-velocity. However the Newman-Penrose formalism has been used to study perfect fluids, particularly in the Einstein-Petrov problem of finding solutions of a specific Petrov type. For example, Allnutt (1981) has found Petrov type III perfect fluid solutions and type N solutions have been given by Oleson (1971). Type II solutions have been presented by Wainwright (1974), Bonnor and Davidson (1985) and Martin-Pascual and Senovilla (1988), all of whom used the Newman-Penrose formalism.

§3.4 Orthonormal Tetrad Equations

The use of orthonormal tetrads in the study of perfect fluid space-times has the advantage that the energy-momentum tensor naturally picks out a time-like vector. Thus the time-like vector of an orthonormal tetrad can be chosen so that

$$e_4^{\ a} = u^a \ . \qquad (3.4.1)$$

This will always be the case in this thesis. Any tetrad freedom can be used to associate the tetrad with the kinematic properties of the fluid. It is therefore easy to impose kinematic restrictions on the resulting equations. In the Newman-Penrose formalism the relevant equations tend to be cumbersome as conditions such as vanishing shear are equivalent to the vanishing of the linear combinations of spin coefficients given in chapter 2. The null vectors do not determine a time-like vector, in general, unless an alignment condition of the form (3.3.8) is imposed. Clearly this is not a problem with solutions with a purely electric Weyl tensor as they are necessarily aligned. In fact it appears that very few known solutions do not have u_a aligned in this way (Wainwright 1977a).

As most of the work in this thesis will be carried out using orthonormal tetrads it will be useful to be able to compare the results with the above classification of Carminati and Wainwright (1985). Using the tetrad (2.4.15) and (3.3.11) we obtain

$$J_1 = \mu_{\cdot 1} + i\mu_{\cdot 2} \ , \ J_2 = \sigma_1 - \sigma_2 \ , \ J_3 = \omega_1 + i\omega_2 \ . \quad (3.4.2)$$

Furthermore as the non-vanishing of $\mu + p$ is not used in this work, most of the results will be valid for vacuum fields admitting a hypersurface orthogonal time-like vector field with respect to which $H_{ij}=0$.

Choosing the tetrad to be an eigenframe of the Weyl tensor, E_{ij} can be written in tetrad form from (3.2.2) as

$$E_{ij} = \sum_{A=1}^{3} \alpha_A e_{Ai} e_{Aj} \ . \quad (3.4.3)$$

It will be seen that the Bianchi identitiy (2.3.18) indicates how the kinematic quantities can be related to this tetrad. In the shear-free case (2.3.18) becomes

$$E_{ij}\omega^j = -\frac{1}{3}(\mu + p)\omega_i \quad (3.4.4)$$

and hence ω_i is an eigenvector of E_{ij}. Collins (1984) has investigated shear-free flows with $H_{ij}=0$ using this tetrad. Without loss of generality we align $\underset{3}{e}^{a}$ along ω^a and hence write

$$\omega^a = \omega \underset{3}{e}^{a} \ . \tag{3.4.5}$$

If the space-time is irrotational but shearing then equation (2.3.18) becomes

$$\eta_{ijkl}u^j\sigma^{km}E^l{}_m = 0$$

and has a simple interpretation: it implies that the matrices σ and **E** commute. This is easily seen by contracting with the tetrad vector $\underset{1}{e}^{a}$ obtaining

$$\sum_{C=1}^{3} \sigma^{C[2}E^{3]}{}_C = 0 \ ,$$

since $E_{A4}=\sigma_{A4}=0$. Here and below capital Latin indices take the values 1,2,3 (except where otherwise stated) and refer to tetrad components with respect to $\underset{A}{e}^{a}$. Similarly contracting with $\underset{2}{e}^{a}$ and $\underset{3}{e}^{a}$ one may deduce

$$\sum_{C=1}^{3} \sigma^{C[A}E^{B]}{}_C = 0$$

$$\Leftrightarrow \quad \sigma E - E\sigma = 0 \ . \tag{3.4.6}$$

Thus there exists an orthonormal tetrad $\{e_A^{\ a}, u^a\}$ which is simultaneously an eigentetrad of the shear and of the Weyl tensor. Using this tetrad we have

$$\sigma_{ij} = \sum_{A=1}^{3} \sigma_A e_{Ai} e_{Aj} \ , \tag{3.4.7}$$

where σ_A are the eigenvalues of σ_{ij} and the trace-free properties of E_{ij} and σ_{ij} give

$$\sum_{A=1}^{3} \alpha_A = \sum_{A=1}^{3} \sigma_A = 0 \ . \tag{3.4.8}$$

The tetrad is uniquely determined if the α_A are distinct or the σ_A are distinct or if (after suitable renumbering of the tetrad vectors) $\alpha_1 = \alpha_2 \neq \alpha_3$ and $\sigma_1 = \sigma_3 \neq \sigma_2$. If $\sigma_1 = \sigma_2 \neq \sigma_3$ and $\alpha_1 = \alpha_2 \neq \alpha_3$, the tetrad is only determined up to a rotation in the $\{x^1, x^2\}$ plane.

$$\tilde{e}_1^{\ a} = e_1^{\ a}\cos\phi + e_2^{\ a}\sin\phi \ ,$$

$$\tilde{e}_2^{\ a} = -\, e_1^{\ a}\sin\phi + e_2^{\ a}\cos\phi \ . \tag{3.4.9}$$

This approach may be extended to the case with non-zero shear and rotation. One possibility is to consider canonical forms of the matrices σ and $\mathbf{E}$ and evaluate ω_a. However, for simplicity, we assume that, as a generalisation of the above two classes, σ_{ij} and E_{ij} commute. This clearly implies that ω_a is an eigenvector of E_{ij}. Thus σ_{ij} and E_{ij} can be expressed as in the irrotational case above and ω_a can be expressed as for the shear-free case. We can now write the Ricci and Bianchi identities in tetrad form using (3.4.3), (3.4.5) and (3.4.7). From (2.3.8) we have

$$\frac{1}{2}\left(\gamma_{4A4.B} - \gamma_{4B4.A} + \sum_{C=1}^{3}\gamma_{4C4}(\gamma_B{}^C{}_A - \gamma_A{}^C{}_B)\right) +$$

$$\delta_{A1}\delta_{B2}(-\,\omega_{.4} - \omega(\sigma_A - \sigma_B) - \frac{2}{3}\omega\,\theta\,) - \delta_{A1}\delta_{B3}\,\omega\,\gamma_{234} +$$

$$\delta_{A2}\delta_{B3}\,\omega\,\gamma_{134} \quad = 0 \quad . \qquad (3.4.10)$$

From (2.3.9) we have

$$(\sigma_A - \sigma_B)\gamma_{AB4} - \gamma_{4A4}\gamma_{4B4} - \gamma_{4A4.B} - \gamma_{4B4.A} -$$

$$\sum_{C=1}^{3}\gamma_{4C4}(\gamma_B{}^C{}_A + \gamma_A{}^C{}_B) \quad = 0 \qquad \text{for } A\neq B \qquad (3.4.11)$$

and using the Raychaudhuri equation (2.3.7) we obtain from (2.3.9)

$$\sigma_{A\cdot 4} + 2/3\theta\sigma_A + \sigma_A^2 + \alpha_A + \frac{\theta_{\cdot 4}}{3} + \frac{\theta^2}{9} + \frac{(\mu\ +\ 3p)}{6} -$$

$$\gamma_{4A4.A} - \gamma_{4A4}^2 - \sum_{D=1}^{3}\gamma_{4D4}\gamma_{DAA} \; - \omega^2\delta_{A1}\delta_{B1} - \omega^2\delta_{A2}\delta_{B2} = 0 \quad .$$

(3.4.12)

From (2.3.10) we obtain

$$\omega(\,\gamma_{434} - \gamma_{311} - \gamma_{322}\,)\, -\omega_{.3} \;= 0 \qquad (3.4.13)$$

and

$$\frac{1}{2}\left(\gamma_{4A4.B} - \gamma_{4B4.A} + \sum_{C=1}^{3}\gamma_{4C4}(\gamma_B{}^C{}_A - \gamma_A{}^C{}_B)\right) +$$

$$\delta_{A1}\delta_{B2}(-\omega_{.4} - \omega(\gamma_{411} + \gamma_{422})) - \delta_{A1}\delta_{B3}\,\omega(\gamma_{234} - \gamma_{243}) +$$

$$\delta_{A3}\delta_{B2}\,\omega(\gamma_{143} - \gamma_{134}) = 0 \quad . \qquad (3.4.14)$$

From (2.3.11):-

$$\sigma_{A\cdot A} + \sum_{D=1}^{3}(\sigma_A - \sigma_D)\gamma_{ADD} - \frac{2\theta_{\cdot A}}{3} -$$

$$\delta_{A1}(\omega_{\cdot 2} + \omega\gamma_{233} + 2\omega\gamma_{424}) - \delta_{A2}(\omega_{\cdot 1} + \omega\,\gamma_{133} + 2\omega\,\gamma_{414}) -$$

$$\delta_{A3}\,\omega(\gamma_{132} + \gamma_{231}) = 0 \quad . \qquad (3.4.15)$$

From (2.3.12) we obtain

$$(\sigma_B - \sigma_C)_{\cdot A} + (\sigma_A - \sigma_C)\gamma_{ACC} - (\sigma_A - \sigma_B)\gamma_{ABB} +$$

$$\delta_{B3}\delta_{C2}(-\omega_{\cdot 2} + \omega\gamma_{233} - 2\omega\,\gamma_{424}) - \delta_{B1}\delta_{C2}\omega(\gamma_{231} + \gamma_{132})$$

$$+ \delta_{B1}\delta_{C3}(-\omega_{\cdot 1} + \omega\,\gamma_{133} - 2\omega\,\gamma_{414}) = 0 \;,$$

(3.4.16)

$$(\sigma_1 - \sigma_2)\gamma_{123} - (\sigma_3 - \sigma_1)\gamma_{312} + (\omega_{.3} + \omega\,\gamma_{322}) = 0 \;,$$

$$(\sigma_2 - \sigma_1)\gamma_{123} + (\sigma_2 - \sigma_3)\gamma_{231} + (\omega_{.3} + \omega\,\gamma_{311}) = 0 \;,$$

$$(\sigma_3 - \sigma_2)\gamma_{231} + (\sigma_3 - \sigma_1)\gamma_{312} + \omega(2\gamma_{434} - \gamma_{311} - \gamma_{322}) = 0$$

(3.4.17)

and from (2.3.17)

$$\alpha_{A\cdot A} + \sum_{D=1}^{3}(\alpha_A - \alpha_D)\gamma_{ADD} - \frac{\mu_{\cdot A}}{3} = 0 \quad . \qquad (3.4.18)$$

From (2.3.19) we obtain

$$
\begin{aligned}
&(\alpha_1 - \alpha_2)(\gamma_{124} + \frac{\omega}{2}) = 0 \ , \\
&(\alpha_2 - \alpha_3)\gamma_{234} = 0 \ , \qquad (3.4.19) \\
&(\alpha_3 - \alpha_1)\gamma_{314} = 0 \ .
\end{aligned}
$$

Note that the corresponding Ricci identity (3.4.11) is complicated by the presence of terms involving the acceleration and its derivatives. From (2.3.19) we also have

$$
\alpha_{A \cdot 4} + \theta\alpha_A + \{(\mu + p)/2 - 3\alpha_A\}\sigma_A + \sum_{B=1}^{3}\alpha_B\sigma_B = 0 \ , \qquad (3.4.20)
$$

$$
(\alpha_1 - \alpha_2)\gamma_{123} = (\alpha_3 - \alpha_1)\gamma_{312} = (\alpha_2 - \alpha_3)\gamma_{231} \ , \qquad (3.4.21)
$$

$$
\begin{aligned}
&(\alpha_B - \alpha_C)_{\cdot A} + (\alpha_A - \alpha_C)\gamma_{ACC} - (\alpha_A - \alpha_B)\gamma_{ABB} + \\
&+ 2(\alpha_B - \alpha_C)\gamma_{4A4} = 0 \ . \qquad (3.4.22)
\end{aligned}
$$

Clearly, we can easily specialise to the shear-free and irrotational cases by putting $\sigma = 0$ and $\omega = 0$ respectively. The space-times with both $\omega_a = 0$ and $\sigma_{ab} = 0$ have been considered in an earlier work (Barnes 1973). The type 1 solutions are static and few exact solutions are known. The type D solutions were found explicitly except for the cases with spherical or related symmetry where one ordinary differential equation remains to be integrated. The conformally flat fields (type O) are all known explicitly (Stephani 1967). However, in the next chapter, Killing vectors in conformally flat space-times will be considered. Ensuing chapters will deal with the cases of shear-free rotation and vanishing rotation using the derived tetrad equations.

Finally, the Jacobi identities are given by

$$[\,[\,e_A\,,e_B\,]\,,\,e_C\,]+[\,[\,e_B\,,e_C\,]\,,\,e_A\,]+[\,[\,e_C\,,e_A\,]\,,\,e_B\,]=0$$

Contracting with $e_D^{\;a}$ we may write this in terms of Ricci rotation coefficients as $J_{D[ABC]} = 0$, where

$$J_{DABC} = \gamma_{DBA.C} - \gamma_{DAB.C} + \sum_{E=1}^{3}(\gamma_B{}^E{}_A - \gamma_A{}^E{}_B)(\gamma_{DEC} - \gamma_{DCE})$$

(3.4.23)

The general Jacobi identities in terms of the Ricci rotation coefficients were calculated using an STENSOR program (Hornfeldt 1986) and are given in the appendix. The identities are also given subject to certain simplifications that will be used in chapter 6.

CHAPTER 4

Killing Vectors in Conformally Flat Perfect Fluid Space-times

§4.1 Introduction

Analyses of isometry groups of space-times can often be accomplished automatically with the computer program CLASSI (Åman 1983). However, difficulties can arise due to the limited power of SHEEP (Frick 1982) for performing automatic simplifications. If the metric has, for example, a polynomial denominator then this may lead to extremely large terms in the fourth order expressions needed for classification, preventing the completion of the program. Conformally flat space-times also form one of the few known cases when third order derivatives are needed in the classification algorithm (Bradley 1986).

A conformally flat space-time is defined by the vanishing of the Weyl tensor or equivalently there is a coordinate system in which the metric takes the form

$$g_{ij} = S^2 \eta_{ij} \tag{4.1.1}$$

where η_{ij} is the flat space-time metric and S is the conformal factor. Killing vectors (KVs) in conformally flat space-times in general have been considered by Levine (1936, 1939) before the general solution for a perfect fluid was discovered. He derived restrictions on the form of S for Killing vectors to exist. However, the conformally flat solutions were discovered in a coordinate system in which the metrics are not manifestly conformally flat (Stephani 1967, Barnes 1973) and the coordinate transformations required to put the metrics into the form (4.1.1) are not known. Consequently a new analysis of Killing vectors in conformally flat perfect fluid space-times is needed.

This chapter considers the number and nature of Killing vectors admitted by conformally flat perfect fluid metrics. The 6 spatial Killing vectors of the Robertson-Walker metric are intrinsic KVs of the conformally flat space-times that is they satisfy Killing's equations on a hypersurface. They also satsify 6 of the 10 Killing's equations. Restrictions on arbitrary functions appearing in the conformally flat metrics are imposed in order to find sub-classes of the general case in which these intrinsic KVs become KVs of the full space-time. In the expanding case the only possible KV apart from these 6 is either hypersurface orthogonal or the metric is De Sitter space-time. In the non-expanding case non-trivial tilted KVs are shown to exist without the requirement of high symmetry. Thus a class of stationary conformally flat perfect fluids is established. These are shown to admit 0, 1 or 3 additional spatial Killing vectors. A theorem in the literature (Collinson (1976), Garcia Diaz (1988)) claiming that the only stationary axisymmetric conformally flat perfect fluid is the interior Schwarzschild solution and its counterparts with hyperbolic and planar symmetry is therefore shown to be false. A counter-example is given in §4.6.

§4.2 Conformally Flat Perfect Fluid Space-times

All conformally flat perfect fluids in general relativity are known (Stephani 1967). There are two metric forms corresponding to zero and non-zero expansion of the fluid (Barnes 1973). In the following case, θ, the expansion, is non-zero and is a function of time only. The metric is given by

$$ds^2 = P^{-2}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) - V^2 dt^2 \qquad (4.2.1)$$

where

$$P = a(t) + b(t)r^2 - 2\mathbf{r.c} \quad \text{and} \quad V = \frac{-3\dot{P}}{\theta(t)P} \qquad (4.2.2)$$

where c_1, c_2, c_3, a and b are five arbitrary functions of time. The notation differs slightly from that of Barnes (1973). A dot denotes differentiation w.r.t time, three dimensional vector notation has been used for conciseness, and r, θ, and ϕ, are related to x, y and z in the usual way. Note that θ has been used both for the expansion and as a coordinate but no confusion should arise. It is also worth noting for future reference that the energy density, μ, is given by

$$\mu = 12(ab - c^2) + \frac{\theta^2(t)}{3} \,. \tag{4.2.3}$$

For θ = 0 the metric is given by

$$ds^2 = (1 + kr^2/4)^{-2}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) - V^2 dt^2, \tag{4.2.4}$$

where k=0, ±1 and V is given by

$$V = \frac{(a + br^2 + \mathbf{r}.\mathbf{c})}{(1 + kr^2/4)} \,. \tag{4.2.5}$$

The energy density μ, and the pressure p are related by

$$\mu + p = \frac{(ak + 4b)}{V} \,. \tag{4.2.6}$$

If $\mu + p = 0$ then the space is an Einstein space and being conformally flat it must be of constant curvature i.e. it is a De Sitter space-time (Eisenhart 1949) admitting 10 Killing vectors or Minkowski flat space-time. As these space-times are well known they will not be considered here and so henceforth we assume $\mu + p \neq 0$.

In both conformally flat metrics there are five arbitrary functions of time. Several important space-times including the Robertson-Walker and interior Schwarzschild metrics are contained within this class of solutions. These solutions admit 6 and 4 KVs respectively and so I investigate the conditions the arbitrary metric functions must fulfil for KVs to exist.

§4.3 Killing's Equations and Intrinsic Killing Vectors

Killing's equations arise from the vanishing of the Lie derivative of the metric tensor g_{ij}

$$\pounds_X g_{ij} = g_{ij,k} X^k + X^k{}_{,j} g_{ik} + X^k{}_{,j} g_{ki} = 0 \qquad (4.3.1)$$

A solution X^i of this equation is known as a Killing vector and we require the number of independent solutions to these equations given the metric tensor of the conformally flat space-time.

We will now look at the 3+1 splitting of these equations relative to u_a. Equation (4.3.1) implies for i, j=1, 2, 3 (noting $g_{\alpha 4}=0$)

$$g_{\alpha\beta,4} X^4 + g_{\alpha\beta,\gamma} X^\gamma + X^\gamma{}_{,\beta} g_{\alpha\gamma} + X^\gamma{}_{,\alpha} g_{\gamma\beta} = 0 \qquad (4.3.2)$$

where here and henceforward Greek indices take the values 1, 2 and 3 only. The remaining equations are

$$g_{44,\alpha} X^\alpha + g_{44,4} X^4 + 2X^4{}_{,4} g_{44} = 0 , \qquad (4.3.3)$$

$$X^\beta{}_{,4} g_{\beta\alpha} + X^4{}_{,\alpha} g_{44} = 0 \quad . \qquad (4.3.4)$$

In the case when the KVs are tangential to the space-like hypersurface i.e. $X^4=0$ we obtain the simpler forms

$$g_{\alpha\beta,\gamma} X^\gamma + X^\gamma{}_{,\beta} g_{\alpha\gamma} + X^\gamma{}_{,\beta} g_{\gamma\beta} = 0 , \qquad (4.3.5)$$

$$g_{44,\alpha} X^\alpha = 0 , \qquad (4.3.6)$$

$$X^\beta{}_{,4} g_{\beta\alpha} = 0 . \qquad (4.3.7)$$

Equation (4.3.6) clearly implies

$$V_{,\alpha} X^\alpha = 0 , \qquad (4.3.8)$$

and from (4.3.7)

$$X^{\beta}{}_{,4} = 0 \,. \tag{4.3.9}$$

as $g_{\alpha\beta}$ is non-singular. The 6 equations (4.3.5) are satisfied identically by the spatial KVs of the Robertson-Walker metric. Consider the three dimensional metric of constant curvature

$$ds^2 = \frac{(dx^2 + dy^2 + dz^2)}{(1+\frac{kr^2}{4})^2} \,, \tag{4.3.10}$$

where k=0, ±1. These admit a six dimensional group of isometries generated by (Maartens and Maharaj 1986)

$$\begin{aligned}
X_1 &= (1- kr^2/4+kx^2/2)\partial_x + kxy/2\partial_y + kxz/2\partial_z \\
X_2 &= kxy/2\partial_x + (1- kr^2/4+ky^2/2)\partial_y + kyz/2\partial_z \\
X_3 &= kxz/2\partial_x + kyz/2\partial_y + (1- kr^2/4+kz^2/2)\partial_z \\
X_4 &= y\partial_z - z\partial_y \\
X_5 &= z\partial_x - x\partial_z \\
X_6 &= x\partial_y - y\partial_x \,.
\end{aligned} \tag{4.3.11}$$

The X_α generate spatial translations (when k=0) and the $X_{\alpha+3}$ generate spatial rotations, in the homogeneous hypersurface t=constant.

Comparing the metric (4.3.10) with the non-expanding conformally flat metric (4.2.4) it is clear that the non-expanding conformally flat space-times admit a family of hypersurfaces t=constant of constant curvature. The 6 spatial Killing vectors satisfy (4.3.5) with $g_{\alpha\beta}$ given by (4.3.10) and so will also satisfy it for the conformally flat spatial metric. If the KVs X_1 ... X_6 also satisfy (4.3.6) and (4.3.7) then they are Killing vectors of the full space-time. However, this will not be true in general for the conformally flat metric. We refer to X_1 ... X_6 as intrinsic Killing vectors (Collins and Szafron 1977) of the conformally flat space-time since they satisfy Killings equations in each spatial hypersurface. In section 4.5(ii) it will be shown that the expanding metrics also admit 6 intrinsic KVs. In fact any linear combination of the Killing vectors

$$X = \sum_{A=1}^{6} f_A(t)\, X_A \ , \tag{4.3.12}$$

where f_A are six arbitrary functions of time, clearly also satisfies (4.3.5) and is thus also an intrinsic KV of the space-time. This result will be useful in determining the number of spatial KVs admitted by the conformally flat metric. Although this KV is not the general solution of the $i=\alpha$, $j=\beta$ Killing equation (4.3.2) this equation can always be replaced by (4.3.5) for the conformally flat metrics. In the non-expanding case this is because $g_{\alpha\beta,4} = 0$ as can be seen from the metric. In the expanding case it will be shown that $X^4 = 0$ except in one special case. Consequently the intrinsic Killing vectors satisfy 6 of Killing's equations.

We shall see what conditions need to be imposed on the metric functions for 1 or more of these intrinsic KVs to become KVs of the full space-time by solving the remaining equations (4.3.3) and (4.3.4) or the simpler forms (4.3.6) and (4.3.7) for purely spatial KVs (SKVs). Firstly it must be determined whether these metrics admit Killing vectors which are neither tangential or orthogonal to the space-like hypersurfaces t=constant. We refer to such KVs as tilted KVs.

§4.4 Non-Trivial Tilted Killing Vectors

Any constant linear combination of Killing vectors is also a Killing vector. In the expanding case the only tilted KVs are such trivial combinations of KVs orthogonal and tangential to the spatial hypersurfaces t=constant except in the De Sitter space-time ($\mu + p = 0$) as will be seen below. However in the non-expanding case a non-trivial tilted KV is possible in more cases. In fact it will be shown that the f_A's are constant and given any set of constants f_A it is always possible to find metric functions a, b and c such that

$$X = \partial_t + \sum_{A=1}^{6} f_A(t)\, X_A \qquad (4.4.1)$$

is a Killing vector. In general this KV is not hypersurface orthogonal and in a region surrounding the world-line r=0 it will be time-like.

(i) $\theta = \theta(t) \neq 0$

Since the energy density is a metric invariant its Lie derivative vanishes along any Killing vector. Since μ is a function of time only, from (4.2.3), we obtain

$$\pounds_X \mu = 0 \;\Rightarrow\; \frac{d\mu}{dt} X^4 = 0, \qquad (4.4.2)$$

and either X^4 vanishes and there are no non-spatial KVs or $\frac{d\mu}{dt} = 0$. In the latter case using the conservation equation

$$\dot{\mu} + (\mu + p)\theta = 0 \qquad (4.4.3)$$

it follows that $\mu + p = 0$ as θ is non-zero. This is one of the well known De Sitter space-times or, of course, Minkowski space-time admitting 10 KVs and will not be considered further.

(ii) $\theta = 0$

Since $g_{\alpha\beta,4} = 0$ the spatial components of possible KVs are given by (4.3.12) and only the 4 remaining equations (4.3.3) and (4.3.4) need to be solved. The analysis uses the Lie derivative of the pressure plus energy density given by (4.2.6)

$$\pounds_X (\mu + p) = 0 \;\Rightarrow\; (ak+4b)_{,4}\, X^4 - V_{,i} X^i \frac{(ak+4b)}{V} = 0. \qquad (4.4.4)$$

Now using the i=4 j=4 Killings equation (4.3.3) to eliminate $V_{,i}X^i$ we obtain

$$(\dot{a}k+4\dot{b})X^4 + (ak+4b)X^4{}_{,4} = 0 . \qquad (4.4.5)$$

Integrating the above gives

$$X^4 = \frac{h(x^\alpha)}{(ak+4b)} \qquad (4.4.6)$$

where $h(x^\alpha)$ is an arbitrary function of the spatial coordinates only. This is the only allowable form (apart from zero) of the time component of any Killing vector. We now determine $h(x^\alpha)$ and the spatial components of possible Killing vectors. The intrinsic Killing vectors are independent of time so when the time derivative of (4.3.12) is taken we obtain

$$X^\beta{}_{,4} = \sum_{A=1}^{6} \dot{f}_A(t)\, X^\beta{}_A \qquad (4.4.7)$$

From eqs. (4.3.4) and (4.4.7) we can deduce

$$h_{,\alpha} = \frac{(ak + 4b)\sum_{A=1}^{6} \dot{f}_A(t)\, X^\alpha{}_A}{(a + br^2+\mathbf{r}.\mathbf{c})^2} \qquad (4.4.8)$$

Calculating $h_{,\alpha\beta}$ and using the fact that partial derivatives commute we can eliminate h from the above equation obtaining three compatibility conditions. These conditions take the form of polynomials in x, y and z with time-dependent coefficients. Equating coefficients of the various powers of x, y and z in each of the 3 equations gives

$$(ak + 4b)\dot{f}_A = 0 . \qquad (4.4.9)$$

Assuming $(ak + 4b) \neq 0$ (i.e. excluding the De Sitter cosmological model) we have that the f_A are constants. Hence $X^{\beta}{}_{,4} = 0$ and from (4.3.4) we obtain

$$X^4{}_{,\alpha} = 0 \tag{4.4.10}$$

and therefore $h(x^\alpha)$ = constant. Since any linear combination of KVs is also a KV the constant h may be set to 1 without loss of generality. Returning to equation (4.3.3) with these new conditions yields an expression for $V_{,\alpha}X^\alpha$ equal to a second order polynomial. $V_{,\alpha}X^\alpha$ was expanded using a simple REDUCE (Hearn 1983) program. On equating coefficients from the resulting expression with the polynomial we obtain a system of differential equations. The coordinate t may be rescaled such that $ak + 4b = 1$ and hence from (4.4.6) and (4.3.12) we are considering KVs of the form (4.4.1). For $k \neq 0$ we write the remaining equations in matrix form as

$$\frac{d}{dt}\begin{pmatrix} \frac{g}{2} \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 & kf_1 & kf_2 & kf_3 \\ -f_1 & 0 & -f_6 & f_5 \\ -f_2 & f_6 & 0 & -f_4 \\ -f_3 & -f_5 & f_4 & 0 \end{pmatrix} \begin{pmatrix} \frac{g}{2} \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{4.4.11}$$

where

$$g = ak - 4b \; . \tag{4.4.12}$$

If $k = 0$ we obtain a system of differential equations which can be represented by

$$\dot{a} + f_1 c_1 + f_2 c_2 + f_3 c_3 = 0 \tag{4.4.13}$$

and

$$\frac{d}{dt}\begin{pmatrix} c1 \\ c2 \\ c3 \end{pmatrix} = \begin{pmatrix} 0 & -f6 & f5 \\ f6 & 0 & -f4 \\ -f5 & f4 & 0 \end{pmatrix} \begin{pmatrix} c1 \\ c2 \\ c3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} f1 \\ f2 \\ f3 \end{pmatrix} \tag{4.4.14}$$

The solutions to these matrix o.d.e's can be found from the exponential matrix e^{At}. This may be calculated by finding B the Jordan normal form of the matrices in (4.4.12) and (4.4.14) where $B = S^{-1}AS$ and then evaluating $Se^{Bt}S^{-1}$ which is now easy because of the form of B.

When k = 1 the matrix in (4.4.12) is antisymmetric and does not possess non-simple elementary divisors. The eigenvalues are given by:

$$\lambda = \pm\frac{i}{2}\{\,|\underline{f}+\underline{F}| \pm |\underline{f}-\underline{F}|\,\}, \tag{4.4.15}$$

where $\underline{f} = (f_1 , f_2 , f_3) , \underline{F} = (f_4 , f_5 , f_6)$. (4.4.16)

Thus g(t) and **c** are linear combinations of sin μt , cos μt , sin νt and cos νt with the eigenvalues labelled $\pm i\mu$ and $\pm i\nu$ where μ and ν are real. In three special cases g(t) and **c** are given by linear combinations of

a)1 , sin μt , cos μt ,when $\nu = 0$,
b)1 , sin νt , cos νt , when $\mu = 0$,
c) sin νt , cos νt , when $\mu = \nu$.

When k = -1 the eigenvalues of the matrix in (4.4.12) are

$$\lambda = \pm\frac{i}{2}\{\,|\underline{F}-i\underline{f}| \pm |\underline{F}+i\underline{f}|\,\}, \tag{4.4.17}$$

where $|\underline{f}|$ denotes the real Euclidean 'norm' of $\underline{f}$. The eigenvalues may be represented by $\pm i\mu$ and $\pm \nu$. Thus g(t) and **c** are linear combinations of sin μt , cos μt , sinh νt and cosh νt . In two special cases we have that g(t) and **c** are given by linear combinations of 1 , sin μt , cos μt, when $\nu = 0$ and by 1 , sinh νt , cosh νt , when $\mu = 0$. In the case $\underline{f} \cdot \underline{F} = 0$, $|\underline{f}| = |\underline{F}|$ we have $\mu = \nu = 0$ and the matrix has a non-simple elementary divisor. The minimal equation is then $A^3 = 0$ and the Jordan normal form of A is therefore

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (4.4.18)$$

consequently the solutions for g(t) and **c** involve linear combinations of 1, t and t^2. When k = 0 the eigenvalues of the matrix in (4.4.14) are

$$\lambda = \pm\ i\,|\underline{F}|\ \text{ and }\ \lambda = 0\ . \qquad (4.4.19)$$

In this case **c** is a linear combinations of 1, t, sin μt and cos μt and a(t) is a linear combination of 1, t, t^2, sin μt and cos μt. In the case $\underline{F} = 0$, **c** is just a linear combination of 1 and t but a(t) may also include t^2.

§4.5 Spatial Killing Vectors

(i) $\theta = 0$

In the preceeding section the existence of a class of stationary conformally flat perfect fluid metrics was demonstrated. If a metric admits two such vectors of the form (4.4.14) e.g.

$$X_1 = h_1\,\partial_t + \sum_{A=1}^{6} f_A\, X_A\ ,$$

$$X_2 = h_2\,\partial_t + \sum_{A=1}^{6} g_A\, X_A\ ,$$

where h_1 and h_2 are constants, it is always possible to eliminate the time component from one of the KVs by taking suitable linear combinations. For KVs with no time component, spatial KVs (SKVs), we need only consider equation (4.3.6) as (4.3.7) is identically satisfied by the fact that the f_A are constants. We can therefore take $h = 0$ in the equations of the preceeding section which may then be written in a unified form as

$$\begin{pmatrix} c_1 & c_2 & c_3 & 0 & 0 & 0 \\ 0 & 0 & g & -2c_2 & 2c_1 & 0 \\ 0 & g & 0 & 2c_3 & 0 & -2c_1 \\ g & 0 & 0 & 0 & -2c_3 & 2c_2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix} = 0 \ . \qquad (4.5.1)$$

These equations will now be examined by assuming the rank of the four time dependent functions c_1, c_2, c_3, and g to be of a certain order and then determining the number of independent solutions for the f_A in each case.

a) Rank zero: $c_i = 0$, $g = 0$

This condition implies that the column matrix of the f_A is completely arbitrary and hence there are 6 independent solutions for constant f_A and therefore 6 independent spatial Killing vectors i.e. $X_1 \ldots X_6$.

b) Rank one: $c_i = 0$, $g \neq 0$

In this case we have

$$gf_1 = 0, \ gf_2 = 0, \ gf_3 = 0$$

and so the remaining f_4, f_5, f_6 are arbitrary constants and 3 SKVs exist. These are the generators of spatial rotations and hence the metric is spherically symmetric.

c) Rank one: $c_i \neq 0$, $g = 0$

Without loss of generality we assume c_1 is non-zero, then c_2 and c_3 are related to it via

$$c_2 = Ac_1 \ , c_3 = Bc_1 \qquad (4.5.2)$$

where A and B are constants.These conditions imply that the rank of the matrix of coefficients of the f_A is 3. Hence there are 3 independent solutions for these equations for which the f_A are constants and therefore the metric admits exactly 3 independent SKVs. For example, 3 independent KVs are given by

$$X_4+AX_5+BX_6 \ , -AX_1+X_2 \ , -BX_1+X_3 \ . \qquad (4.5.3)$$

where A and B are arbitrary constants.

d) Rank one: $c_i \neq 0$, $g \neq 0$

In this case we take g to be the independent function and thus there exist constants A, B and C such that

$$c_1 = Ag \ , c_2 = Bg \ , c_3 = Cg$$

The rank of these equations is again 3 and 3 independent SKVs are given by

$$-2CX_2 + 2BX_3 + X_4 \ , \quad 2CX_1 - 2AX_3 + X_5,$$
$$-2BX_1 + 2AX_2 + X_6 \ . \qquad (4.5.4)$$

where A , B and C are constants.

e) Rank two: $c_i \neq 0$, $g = 0$

Without loss of generality we take c_1 and c_2 to be the independent functions

$$c_3 = Ac_1 + Bc_2 \ , \qquad (4.5.5)$$

where A and B are constants. Row 1 of the matrix is now

$$f_3c_3 = -f_2c_2 - f_1c_1 \qquad (4.5.6)$$

and f_3 cannot be zero or c_1 and c_2 would be dependent. Comparing (4.5.5) and (4.5.6)

$$f_2 = -Af_3 \ , f_1 = -Bf_3$$

and using the remaining rows $f_4 = f_5 = f_6 = 0$ for c_1 and c_2 to be independent. Clearly there is only one independent solution and hence one spatial Killing vector.

f) Rank two: $c_i \neq 0 \ , g \neq 0$

Without loss of generality we take c_1 and g to be the independent functions,

$$c_2 = Ag + Bc_1 \ ,$$

$$c_3 = Cg + Dc_1 \ ,$$

where A, B, C, and D are constants. Row 2 of the matrix is now

$$f_4 c_2 = -f_3 \, g/2 + f_5 c_1 \ ,$$

Now f_4 cannot be zero or g and c_1 would be dependent. Comparing the two expressions for c_2

$$f_3 = -2Af_4 \ , f_5 = Bf_4$$

similarly, using other rows

$$f_2 = -2Cf_4 \ , \ f_6 = Df_4 \ , \ f_1 = (-2AD + 2BC)f_4$$

and so there is only one independent solution and one SKV.

g) Rank ≥ 3

There are no spatial KVs in this case and this can be shown by considering two separate cases. If $g = 0$ the first row implies that either the **c** are dependent and hence the rank must be less than three or $f_\alpha = 0$. In the latter case the remaining rows imply dependence of the **c** and hence the rank must be less than three for KVs to exist. If g is non-zero then without loss of generality we take g, c_1 and c_2 to be independent. Then from row two, f_3, f_4 and f_5 must be zero in which case rows 3 and 4 imply dependence between the three functions and we have a contradiction. Hence no SKVs can exist when the rank is three or greater.

(ii)$\underline{\theta = \theta(t) \neq 0}$

The coordinate transformation

$$\tilde{x} = x + \frac{c1}{b}, \quad \tilde{y} = y + \frac{c2}{b}, \quad \tilde{z} = z + \frac{c3}{b} \qquad (4.5.7)$$

applied to the metric (4.3.10) yields the t=constant 3-space of the expanding metric (4.2.1) up to a time-dependent conformal factor and where k is given by

$$k = \frac{4b}{a - \mathbf{c}^2/b} \qquad (4.5.8)$$

The transformations (4.5.7) are clearly not valid in the case b = 0. However by performing the inversion $\tilde{r} = \frac{1}{r}$ on the expanding metric it is possible to interchange a and b so the analysis is only invalid in the case a = b = 0. In this case we can use a coordinate translation followed by an inversion to set b ≠ 0. A similar procedure can be applied in the exceptional case $a - \mathbf{c}^2/b = 0$

The conformal factor does not affect Killing's equations in the hypersurface and so the expanding metric admits 6 intrinsic KVs given by

$$X_1 = \left(1 - \frac{k}{4}\left(r^2 + \frac{\mathbf{c}^2}{b^2} - \frac{2\mathbf{c}.\mathbf{r}}{b}\right) + \frac{k}{2}\left(x^2 + \frac{c_1{}^2}{b^2} - \frac{2c_1x}{\beta}\right)\right)\partial_x +$$

$$\frac{k}{2}\left(xy + \frac{c_1c_2}{b^2} - \frac{c_1y}{b} - \frac{c_2x}{b}\right)\partial_y + \frac{k}{2}\left(xz + \frac{c_1c_3}{b^2} - \frac{c_1z}{b} - \frac{c_3x}{b}\right)\partial_z$$

$$X_2 = \left(1 - \frac{k}{4}(r^2 + \frac{\mathbf{c}^2}{b^2} - \frac{2\mathbf{c}.\mathbf{r}}{b}) + \frac{k}{2}(y^2 + \frac{c_2{}^2}{b^2} - \frac{2c_2y}{b})\right)\partial_y +$$

$$\frac{k}{2}\left(xy + \frac{c_1c_2}{b^2} - \frac{c_1y}{b} - \frac{c_2x}{b}\right)\partial_x + \frac{k}{2}\left(yz + \frac{c_2c_3}{b^2} - \frac{c_2z}{b} - \frac{c_3y}{b}\right)\partial_z$$

$$X_3 = \left(1 - \frac{k}{4}(r^2 + \frac{\mathbf{c}^2}{b^2} - \frac{2\mathbf{c}.\mathbf{r}}{b}) + \frac{k}{2}(z^2 + \frac{c_3{}^2}{b^2} - \frac{2c_3z}{b})\right)\partial_z +$$

$$\frac{k}{2}\left(xz + \frac{c_1c_3}{b^2} - \frac{c_1z}{b} - \frac{c_3x}{b}\right)\partial_x + \frac{k}{2}\left(yz + \frac{c_2c_3}{b^2} - \frac{c_2z}{b} - \frac{c_3y}{b}\right)\partial_y$$

$$X_4 = \left(y - \frac{c_2}{b}\right)\partial_z - \left(z - \frac{c_3}{b}\right)\partial_y$$

$$X_5 = \left(z - \frac{c_3}{b}\right)\partial_x - \left(x - \frac{c_1}{b}\right)\partial_z$$

$$X_6 = \left(x - \frac{c_1}{b}\right)\partial_y - \left(y - \frac{c_2}{b}\right)\partial_x \qquad (4.5.9)$$

in the coordinate system of (4.2.1), i.e. tildes have been dropped. These are obtained by applying the inverse of the coordinate transformation (4.5.7) to the Killing vectors of the metric (4.3.10). The REDUCE program to evaluate $V_{,\alpha}X^\alpha$ was modified to take into account the transformed coordinate system, the non-constant k and the different g_{44} in this metric. The number of terms in the resulting expression can be reduced by writing

$$V_{,\alpha}X^\alpha = \{x_\alpha(w_{45} + 2w_{5i}x_i) - w_{4\alpha} - r^2w_{5\alpha} + 2x^iw_{i\alpha}\}X^\alpha = 0 \qquad (4.5.10)$$

where w_{AB} are Wronskians of the five metric functions c_1, c_2, c_3, a and b labelled 1 - 5 respectively i.e. a is written temporarily as c_4 and b as c_5. For example

$$w_{34} = \dot{a}\, c_3 - \dot{c}_3\, a$$

Equation (4.5.10) was expanded using a REDUCE program and on equating coefficients of the various powers of x, y, and z the following was obtained:-

$$w_{51}f_{1k} + w_{52}f_{2k} + w_{53}f_{3k} = 0\ ,$$

$$-w_{54}f_{1k} + 4w_{53}f_5 - 4w_{52}f_6 + 2w_{31}f_{3k} + 2w_{21}f_{2k} = 0\ ,$$
$$-w_{54}f_{2k} - 4w_{53}f_4 + 4w_{51}f_6 - 2w_{21}f_{1k} + 2w_{32}f_{3k} = 0\ ,$$
$$-w_{54}f_{3k} + 4w_{52}f_4 - 4w_{51}f_5 - 2w_{31}f_{1k} - 2w_{32}f_{2k} = 0\ ,$$

$$-w_{54}f_1 + w_{43}f_5 - w_{42}f_6 - 2w_{31}f_3 - 2w_{21}f_2 = 0\ ,$$
$$-w_{54}f_2 - w_{43}f_4 + w_{41}f_6 - 2w_{32}f_3 + 2w_{21}f_1 = 0\ ,$$
$$-w_{54}f_3 + w_{42}f_4 - w_{41}f_5 + 2w_{32}f_2 + 2w_{31}f_1 = 0\ ,$$

$$4w_{53}f_1 + 4w_{51}f_3 - w_{43}f_{1k} - w_{41}f_{3k} + 4w_{32}f_6 + 4w_{21}f_4 = 0\ ,$$
$$4w_{53}f_2 + 4w_{52}f_3 - w_{43}f_{2k} - w_{42}f_{3k} - 4w_{31}f_6 + 4w_{21}f_5 = 0\ ,$$
$$4w_{52}f_1 + 4w_{51}f_2 - w_{42}f_{1k} - w_{41}f_{2k} + 4w_{32}f_5 - 4w_{31}f_4 = 0\ ,$$

$$4w_{51}f_1 - 4w_{52}f_2 - 4w_{53}f_3 - w_{41}f_{1k} + w_{42}f_{2k} + w_{43}f_{3k} + 8w_{31}f_5 - 8w_{21}f_6 = 0\ ,$$

$$-4w_{51}f_1 + 4w_{52}f_2 - 4w_{53}f_3 + w_{41}f_{1k} - w_{42}f_{2k} + w_{43}f_{3k} - 8w_{32}f_4 - 8w_{21}f_6 = 0\ ,$$

$$-4w_{51}f_1 - 4w_{52}f_2 + 4w_{53}f_3 + w_{41}f_{1k} + w_{42}f_{2k} - w_{43}f_{3k} - 8w_{32}f_4 + 8w_{31}f_5 = 0\ ,$$

$$w_{41}f_1 + w_{42}f_2 + w_{43}f_3 = 0\ . \qquad (4.5.11)$$

In addition the $i=4$ $j=\alpha$ Killing's equations provide further equations since they do not imply as before that the f_A are constants. This is due to the fact that the 6 intrinsic Killing vectors are not independent of time, as in the non-expanding case, because the coordinate transformation (4.5.7) has introduced time dependence. However from (4.3.9) the linear combination of KVs

(4.3.12) must be independent of time. Consequently it does not simply imply that the f_A are constants. In fact we obtain from (4.3.9)

$$f_1\left(1+\frac{k}{4b^2}(c_1^2-c_2^2-c_3^2)\right) + f_2\frac{kc_2c_3}{2b^2} + f_3\frac{kc_1c_3}{2b^2} + f_5\frac{c_3}{b} - f_6\frac{c_2}{b} = k_1 ,$$

$$f_1\frac{kc_1c_2}{2b^2} + f_2\left(1+\frac{k}{4b^2}(c_2^2-c_1^2-c_3^2)\right) + f_3\frac{kc_2c_3}{2b^2} - f_4\frac{c_3}{b} + f_6\frac{c_1}{b} = k_2 ,$$

$$f_1\frac{kc_1c_3}{2b^2} + f_2\frac{kc_2c_3}{2b^2} + f_3\left(1+\frac{k}{4b^2}(c_3^2-c_2^2-c_1^2)\right) + f_4\frac{c_2}{b} - f_5\frac{c_1}{b} = k_3 ,$$

$$\frac{k}{b}(f_1c_1 + f_2c_2 + f_3c_3) = k_4 ,$$

$$\frac{k}{b}(f_2c_1 - f_1c_2) - 2f_6 = k_5 ,$$

$$\frac{k}{b}(f_1c_3 - f_3c_1) - 2f_5 = k_6 ,$$

$$\frac{k}{b}(f_3c_2 - f_2c_3) - 2f_4 = k_7 \qquad (4.5.12)$$

where $k_1 \ldots k_7$ are constants. It has proven difficult, due to the size of these equations, to extend the analysis of Killing vectors to the expanding metrics.

§4.6 Examples of Stationary Non-Static Conformally Flat Perfect Fluids

Using the preceding results it is simple to find stationary conformally flat metrics which admit 0, 1 or even 3 spatial Killing vectors. In the latter case if we choose $f_6 = 0$ and $f_A = 0$ otherwise, the metric admits the KV

$$X = \partial_t + m X_6 . \qquad (4.6.1)$$

If the metric is to admit no SKVs then the rank of g(t) and **c** must be 3. In each of the three cases considered above it is possible to find stationary metrics fulfilling this condition, for example

a) $k = \pm 1$, the metric functions are given by

$$g = A , c_1 = B\sin mt + C\cos mt, c_2 = C\sin mt - B\cos mt, c_3 = D$$

where A, B, C and D are arbitrary constants.

b) $k = 0$, the metric functions are given by

$$a = A, c_1 = B\sin mt + C\cos mt, c_2 = C\sin mt - B\cos mt , c_3 = D$$

For stationary axisymmetric solutions we simply choose g(t) and **c** to have rank two such that they also satisfy the conditions for a tilted KV. If we put A=D=0 in the above case we obtain a metric admitting two KVs given by

$$Y_1 = \partial_t + m X_6 ,$$
$$Y_2 = X_3 . \qquad (4.6.2)$$

To find a stationary solution with three space-like vectors the rank of the 4 metric functions g, **c** must be one. However, when the rank of the metric functions a, b, **c** is one the KV is hypersurface orthogonal and this is the case when k=1 or 0. When k=-1 we choose

$$g = 2Ae^{mt}, \ c_1 = c_2 = 0, c_3 = -Ae^{mt}$$

and the metric admits 4 KVs given by:

$$Y_0 = \partial_t + m X_3 , \qquad Y_1 = X_1 + X_5 ,$$
$$Y_2 = X_2 - X_4 , \qquad Y_3 = X_6 . \qquad (4.6.3)$$

The vectors Y_1, Y_2, Y_3 from a group of Bianchi type $VII_{q=0}$.

§4.7 Bianchi Types of the 3-Parameter Groups

Since the Killing vectors of the conformally flat space-time are linear combinations of the Robertson-Walker spatial KVs we consider their commutation relations

$$[X_4,X_6] = X_5$$
$$[X_5,X_4] = X_6$$
$$[X_6,X_5] = X_4$$

$$[X_1,X_4] = [X_2,X_5] = [X_3,X_6] = 0$$

$$[X_1,X_5] = [X_4,X_2] = -X_3$$
$$[X_1,X_6] = [X_4,X_3] = X_2$$
$$[X_3,X_5] = [X_6,X_2] = X_1$$

$$[X_1,X_2] = -kX_6$$
$$[X_1,X_3] = kX_5$$
$$[X_3,X_2] = kX_4$$

So for the Killing vectors of section 4.5 the Bianchi types can be found by finding the commutators in terms of the above and making appropriate changes of bases to get them into standard forms as for example in Stephani (1982a) .

(i) $c_i=0$, $g\neq0$

The KVs already form a standard basis of a Bianchi type IX algebra.

(ii) $c_i \neq 0$, $g=0$

Three Killing vectors are given by (4.5.3) and their commutators are:

$$[Y_2,Y_3] = -kY_1$$
$$[Y_1,Y_2] = -(1 + A^2)\, Y_3 + ABY_2$$

$$[Y_3, Y_1] = -(1 + B^2)\, Y_2 + ABY_3 \quad .$$

If we now transform these vectors according to $\tilde{Y}_1 = Y_1$, $\tilde{Y}_2 = Y_2$, $\tilde{Y}_3 = Y_3 + \delta Y_2$, and identify $\delta = \dfrac{-AB}{1 + A^2}$ we obtain

$$[\tilde{Y}_2, \tilde{Y}_3] = -k\tilde{Y}_1 \ ,$$
$$[\tilde{Y}_1, \tilde{Y}_2] = -(1 + A^2)\, \tilde{Y}_3 \ ,$$
$$[\tilde{Y}_3, \tilde{Y}_1] = -\frac{(1 + A^2 + B^2)}{1 + A^2}\, \tilde{Y}_2 \ . \quad .$$

Now redefining the vectors according to $\tilde{\tilde{Y}}_1 = \alpha \tilde{Y}_1$, $\tilde{\tilde{Y}}_2 = \beta \tilde{Y}_2$, $\tilde{\tilde{Y}}_3 = \gamma \tilde{Y}_3$, and suitably defining α, β and γ as

$$\alpha^2 = 1 + A^2 + B^2 \quad , \quad \beta^2 = 1 + A^2 \ ,$$
$$\gamma = -\frac{\alpha}{\beta} \ ,$$

we obtain

$$[Y_2, Y_3] = kY_1$$
$$[Y_1, Y_2] = Y_3$$
$$[Y_3, Y_1] = Y_2$$

where tildes have been dropped. The Bianchi type of the group depends on the sign of k as follows:

$k > 0$	Bianchi type IX
$k < 0$	Bianchi type VIII
$k = 0$	Bianchi type $VII_{q=0}$

(iii) $c_i \neq 0$, $g \neq 0$

The three Killing vectors are given by (4.5.4) and by a similar procedure to the above case we may deduce that

$\lvert G \rvert < 1$	Bianchi type IX
$\lvert G \rvert = 1$	Bianchi type $VII_{q=0}$
$\lvert G \rvert > 1$	Bianchi type VIII

where

$$G = \frac{4BCk}{1+4C^2k^2+4A^2k^2}$$

and $1+4A^2k \neq 0$, $BCk \neq 0$. In the special cases we have:

$1+4A^2k=0$	Bianchi type VIII
$k=0$	Bianchi type IX

$C=0$

$1+4A^2k>0$	$1+4B^2k+4A^2k>0$	Type IX
$1+4A^2k>0$	$1+4B^2k+4A^2k=0$	Type $VII_{q=0}$
$1+4A^2k>0$	$1+4B^2k+4A^2k<0$	Type VIII
$1+4A^2k<0$		Type VIII

$B=0$

$1+4A^2k>0$	$1+4C^2k+4A^2k>0$	Type IX
$1+4A^2k>0$	$1+4C^2k+4A^2k=0$	Type $VII_{q=0}$
$1+4A^2k>0$	$1+4C^2k+4A^2k<0$	Type VIII
$1+4A^2k<0$		Type VIII

Note that the three possible Bianchi types correspond to spherical, hyperbolic and planar symmetry as do the three branches of the interior Schwarzschild solution.

§4.8 Conclusion

The existence of a class of stationary conformally flat perfect fluid space-times has been demonstrated including a sub-class that is stationary and axisymmetric. This contradicts a theorem due to Collinson (1976) and Garcia Diaz (1988) who stated that the only stationary axisymmetric conformally flat perfect fluid space-time is the interior Schwarzschild solution and its counterparts with planar and hyperbolic symmetry. Their work relies on the assumption of a metric form which is not that of the most general stationary axisymmetric space-time (Kramer et al 1980). Barnes (1973) suggests that the number of Killing vectors is dependent on the rank of the 5 metric functions appearing in g_{44}. However we have found that the number of spatial Killing vectors actually depends on the rank of the 4 functions **c** and g. The rank of these functions can differ from the rank of the 5 by, at most, one. The existence of a tilted KV is dependent on the form of the metric. For this KV to be hypersurface orthogonal it may be deduced from (4.3.2) - (4.3.4) that V must be separable in t which is only possible when the rank of the 5 metric functions is one. Hence a summary of the results:

Rank of a, b, **c**	Rank of **c**, g	Number of KVs
1	0	7 or 10
1 or 2	1	3 or 4
2 or 3	2	1 or 2
≥ 3	≥ 3	0 or 1

The results presented here seem to suggest a special role for the function g. We may see this in the following example. If $p + \mu$ = constant $g \neq 0$ the solution is the Einstein universe and admits a 7 dimensional group. Using equation (4.2.6) we have

$$\mu + p = \frac{ak + 4b}{V} = C$$

where C is an arbitrary constant. Writing V from the $\theta = 0$ case and equating coefficients gives

$$c_i = 0 \ ,$$

$$a(k-C) + 4b = 0$$

$$(ak + 4b)k - 4Cb = 0$$

This implies that

$$ak - 4b = g = 0.$$

This agrees with the analysis presented earlier and points out the significance of g in one special case.

It would be interesting to extend the method presented here to determine the number of Killing vectors in other space-times with intrinsic Killing vectors for example Szekeres' solutions (Szekeres 1975). These admit conformally flat hypersurfaces and are intrinsically spherically symmetric (Berger et al 1977). It has been shown that in general they admit no KVs but have 4 or 5 arbitrary functions of a single variable (Bonnor et al 1977). As with the conformally flat space-times they include specialisations that admit KVs and so restrictions on these functions could be examined and Killing's equations solved. However the method presented above may be of limited use because of the 3-parameter intrinsic symmetry.

CHAPTER 5

Shear-Free Perfect Fluids with a Purely Electric Weyl Tensor

§5.1 Introduction

The assumption of vanishing shear has been used by several authors as an aid in the investigation of exact solutions of Einstein's field equations, see for example Ellis (1967), Barnes (1973), Kramer et al (1980), White (1981). Some of the most well known solutions such as the Robertson-Walker and interior Schwarzschild solutions have this property. An interesting conjecture arising from a consideration of the known solutions is the following:

For any shear-free perfect fluid in general relativity obeying an equation of state $p = p(\mu)$, such that $p + \mu \neq 0$, the fluid 4-velocity is either irrotational or non-expanding.

The importance of this conjecture, were it to be valid in general, is that it may highlight certain essential differences between Newtonian and Einsteinian theory. This is because Newtonian self gravitating shear-free fluids with both rotation and expansion are known (Ellis 1971). The conjecture has not been proven in general, although no counter-example has been found and proofs in several special cases have been published. Clearly it is valid in conformally flat perfect fluids as from section 3.1, $\sigma = \omega = 0$. It is also valid in the case of dust (Ellis 1967), spatial homogeneity (King and Ellis 1973) and when the acceleration and rotation vectors are parallel (White and Collins 1984). Recently, the conjecture has been shown to be true for Petrov type N space-times (Carminati 1987) and in the class to be considered here, purely electric perfect fluids (Collins 1984) i.e.

$$H_{ij} = \sigma_{ij} = 0 \text{ and } p=p(\mu) \Rightarrow \omega\theta = 0 \ . \tag{5.1.1}$$

Collins went on to consider solutions with $\theta = 0$ and $\omega_{ij} \neq 0$ as those with $\omega_{ij} = 0$ are contained within the class considered by Barnes (1973), (Collins and Wainwright 1983). The rotating solutions with an equation of state were found to be of Petrov type D and could be sub-divided into two classes. Class I space-times have an acceleration vector which is not aligned with the plane spanned by the repeated principal null directions and admit at least two Killing vectors. In class II space-times the acceleration lies in this plane and the fields are necessarily LRS.

Two further shear-free rotating space-times of note are those of Kramer (1984) and Wahlquist (1968). These are stationary (non-static) axisymmetric space-times without a higher symmetry. Solutions of this kind may be relevant as stellar models if a suitable equation of state is allowed and appropriate boundary conditions satisfied. Few stationary axisymmetric perfect fluid space-times are known and, as far as the author is aware, none appear to meet the required conditions for a realistic stellar model (Herlt 1988). The solutions of Kramer and Wahlquist are type D with the 4-velocity not aligned to the plane spanned by the repeated p.n.d's. Apart from some solutions found recently (Martin Pascual and Senovilla 1988) these are the only solutions known with this property. Senovilla (1986) has subsequently found all stationary axisymmetric space-times with u_a aligned and in fact, these all have a purely electric Weyl tensor and belong to class I of Collins (1984).

The 'most symmetric' member of this class of solutions is the Gödel (1949) solution which admits five Killing vectors and is space-time homogeneous. Collins (1984) claimed that the only geodesic and shear-free purely electric solution is the Gödel metric. However, Wolf (1986) has presented a solution which is geodesic and shear-free with non-vanishing rotation and equation of state $p = \mu$. He claims that the magnetic part of the Weyl tensor vanishes and that it admits only two Killing vectors. The terms in the metric are fairly lengthy and the solution is displayed in a non-co-moving coordinate system. Killing's equations are rather complicated and difficult to solve. Wolf used a computer program (Wolf 1985) to simplify these differential equations.

In this chapter, I will resolve the apparent anomaly over geodesic solutions in this class by solving the field equations explicitly under the assumptions

$$H_{ij} = \dot{u}_i = \sigma_{ij} = 0, \quad \omega_{ij} \neq 0 \ . \qquad (5.1.2)$$

It will be shown that the only solution in this class is the Gödel solution (1949).

Firstly, however, accelerating metrics in this class are studied. Collins' theorem regarding the conjecture mentioned above is confirmed using null and orthonormal tetrads. The Bianchi identities (3.3.10) given by Carminati and Wainwright (1985) are simplified and the similarity between the general solution of Collins' class I and Senovilla's solutions is discussed. A few simple results using the orthonormal tetrad formalism are also presented.

§5.2 Applications of Null and Orthonormal Tetrads

A useful starting point for an analysis using null tetrads is clearly the Bianchi identities (3.3.10). These equations are equivalent to the general Bianchi identities for perfect fluid subject to $H_{ij} = 0$ and $p = p(\mu)$ from the discussion of chapter 3. The kinematic restrictions of this chapter can now be applied in order to simplify the identities in each of the three possible classes considered by Carminati and Wainwright (1985). Class I is necessarily irrotational and so we restrict ourselves to classes II and III. In class III the following simplifications were found:

$$\sigma = \lambda = \tau + \kappa = \pi + \nu = 0 \ , \qquad (5.2.1)$$

$$\rho - \bar{\rho} = \mu - \bar{\mu} = \alpha + \bar{\beta} = 0 \ , \qquad (5.2.2)$$

$$3(\rho + \mu) + \varepsilon + \bar{\varepsilon} + \gamma + \bar{\gamma} = 0 \ . \qquad (5.2.3)$$

Using the expressions for shear in the Newman-Penrose formalism (2.4.20) - (2.4.23) we find that vanishing shear is equivalent to $A_1 = A_2 = A_3 = 0$ and hence we obtain

$$k = -\bar{\nu} = -\tau = \bar{\pi} \ . \qquad (5.2.4)$$

From (2.4.24) - (2.4.26) clearly the rotation now vanishes and so shear-free solutions of class III are necessarily irrotational. All rotating non LRS solutions with $H_{ij} = \sigma_{ij} = 0$ and $p=p(\mu)$ must therefore belong to class II. Under these assumptions, the Bianchi identities for class II include:

$$3\Psi_2 + 4\varphi_{11} = 0 \tag{5.2.5}$$

$$\sigma = \lambda = \alpha + \bar{\beta} = 0 \tag{5.2.6}$$

$$\varepsilon + \bar{\varepsilon} = \gamma + \bar{\gamma} \quad , \quad \rho = \mu \quad , \quad \rho + \bar{\rho} + 2(\varepsilon + \bar{\varepsilon}) = 0 \quad . \tag{5.2.7}$$

It is now easy to see from (2.4.27) that the expansion vanishes and Collins (1984) theorem has been confirmed. A further point of interest is that the relations (5.2.5) - (5.2.7) were found by Senovilla (1986) in his study of stationary axisymmetric space-times. As mentioned earlier, the solutions were found to have a purely electric Weyl tensor. Using the assumption of the two Killing vectors he was able to derive additional relations to (5.2.5) - (5.2.7) and solve the field equations. Collins (1984) has shown that space-times with $H_{ij} = \sigma_{ij} = 0$ and $p=p(\mu)$ and with the acceleration not aligned to the repeated p.n.d's admit a G_2 of Killing vectors with time-like orbits. The question remains as to whether Senovilla's solutions complete this class.

Using orthonormal tetrads we make use of the fact that for vanishing shear the vorticity is an eigenvector of E_{ij} when H_{ij} vanishes. Without loss of generality we may align $\underset{3}{e}^a$ along ω^a so that

$$\omega^a = \omega \underset{3}{e}^a \quad . \tag{5.2.8}$$

From (3.4.3) and (3.4.4) we have

$$\alpha_3 = -\frac{1}{3}(\mu + \rho) \tag{5.2.9}$$

and Collins' proof of (5.1.1) can be followed easily. Equation (3.4.20) gives

$$\alpha_{3.4} + \alpha_3 \theta = 0 \tag{5.2.10}$$

and using this in the conservation equation (2.3.15) yields

$$\mu_{.4} = 3\alpha_{3.4} \tag{5.2.11}$$

and hence

$$p_{.4} = 0 \quad . \tag{5.2.12}$$

Now if $p = p(\mu)$ such that $^{\partial p}/_{\partial\mu} \neq 0$ then clearly $\mu_{.4} = 0$. From (2.3.16) the expansion vanishes providing $\mu + p \neq 0$. The vanishing of shear and expansion implies that

$$\gamma_{411} = \gamma_{422} = \gamma_{433} = 0 \quad . \tag{5.2.13}$$

It then follows from (3.4.15) and (3.4.16) that

$$\gamma_{231} = \gamma_{132} = 0 \tag{5.2.14}$$

and hence from (3.4.21) either $\alpha_1 = \alpha_2$ or $\gamma_{123} = 0$. In the latter case the space-time may be of Petrov type 1. It is clear that no type 1 geodesic solutions exist as (3.4.12) implies $\alpha_1 = \alpha_2$. Collins (1984) has shown that those space-times with $p=p(\mu)$ are all type D. Using (3.4.15) and (3.4.16) we find

$$\gamma_{133} = \gamma_{233} = 0 \tag{5.2.15}$$

which corresponds to Collins' $d_2 = d_3 = 0$. In the case when the acceleration is not aligned with the repeated p.n.d's of the Weyl tensor, Collins has shown that (in our notation) $\alpha_1 = \alpha_2$. If the acceleration is aligned with this plane then either $\alpha_1 = \alpha_2$, $p_{.1} = p_{.2} = 0$ or $\alpha_1 = \alpha_3$ and $p_{.1} = p_{.3} = 0$. The latter case is impossible however as (3.4.12) then requires $\omega = 0$. Hence all solutions in this class with the acceleration aligned with the repeated p.n.d's of the Weyl tensor or are geodesic, necessarily have $\alpha_1 = \alpha_2$.

Note that in all shear-free rotating purely electric perfect fluids we have

$$\gamma_{434} = \gamma_{311} = \gamma_{322} \tag{5.2.16}$$

from (3.4.17). Apart from the conjecture (5.1.1) the results derived above are not dependent on the existence of an equation of state.

§5.3 The Gödel Solution

The Gödel (1949) solution is a space-time homogeneous dust solution of Einstein's field equations. The line-element can be given by (Reboucas and Tiomno 1983)

$$ds^2 = D^2(y)dx^2 + dy^2 + dz^2 - (dt + H(y)dx)^2 \tag{5.3.1}$$

where

$$H = \sqrt{2}\, D = e^{my} \quad . \tag{5.3.2}$$

where m i a constant and the fluid 4-velocity is given by

$$u^i = \delta^i_4 \quad . \tag{5.3.3}$$

By defining an obvious orthonormal tetrad it is straightforward to derive the claimed properties: the solution is shear-free non-expanding and geodesic, of Petrov type D with $H_{ij} = 0$. I will now solve the field equations using the conditions (5.1.2) as simplifying assumptions (the assumption of vanishing expansion is unnecessary provided that the rotation is taken to be non-zero from the comments of § 5.2 .)

Firstly, the pressure must be constant otherwise $\omega = 0$ (Ellis 1967). This is consistent with the dust solution of Gödel as we may introduce the Λ-term. As shown in Chapter 3 dust solutions with a cosmological constant are geometrically equivalent to the cases with constant pressure and zero cosmological constant. Clearly $\gamma_{4A4} = 0$ as the space-time is geodesic. From Raychaudhuri's equation (2.3.7)

$$\mu + 3p = 4\omega^2 \qquad (5.3.4)$$

and from (3.4.12) with $c = 3$

$$\alpha_3 + \frac{1}{6}(\mu + 3p) = 0 \quad . \qquad (5.3.5)$$

Now we know $\alpha_1 = \alpha_2 = \alpha$ say and hence $\alpha_3 = -2\alpha$ and comparing (5.3.4), (5.3.5) and (5.2.8) we obtain

$$\mu = p = \omega^2 = 3\alpha \ . \qquad (5.3.6)$$

Further Ricci rotation coefficients vanish: from (5.2.16) clearly

$$\gamma_{311} = \gamma_{322} = 0 \qquad (5.3.7)$$

and from (3.4.14), (3.4.10) and (3.4.21) we have

$$\gamma_{423} = \gamma_{413} = 0 \qquad (5.3.8)$$

$$\gamma_{234} = \gamma_{134} = 0 \qquad (5.3.9)$$

$$\gamma_{312} = \gamma_{231} = 0 \quad . \qquad (5.3.10)$$

The Jacobi identities are now given by simplifying the general identities given in the appendix yielding:

$$\gamma_{124.3} - \gamma_{123.4} = 0 \qquad (5.3.11)$$

$$\gamma_{121.4} - \gamma_{412}\ \gamma_{122} = 0 \tag{5.3.12}$$

$$\gamma_{122.4} - \gamma_{421}\ \gamma_{121} = 0 \tag{5.3.13}$$

$$\gamma_{121.3} = \gamma_{122.3} = 0 \ . \tag{5.3.14}$$

Equations (5.3.12) and (5.3.13) imply that

$$\left(\gamma_{121}^{2} + \gamma_{122}^{2}\right)_{.4} = 0 \ . \tag{5.3.15}$$

The non-vanishing commutation relations are

$$\left[e_1 \ , e_2\right] = \gamma_{211}\ e_{\mu}^{1} - \gamma_{122}\ e_{\mu}^{2} + 2\omega\ e_{\mu}^{4} \ , \tag{5.3.16}$$

$$\left[e_1 \ , e_4\right] = \omega\ e_{\mu}^{2} \ , \tag{5.3.17}$$

$$\left[e_1 \ , e_4\right] = -\omega\ e_{\mu}^{1} \ . \tag{5.3.18}$$

It is clear that e_3^{a} is hypersurface orthogonal and constant since $\gamma_{3AB} = 0$ where A and B range from 1 to 4, and thus we may choose

$$e_{3\ a} = (0,\ 0,\ 1,\ 0) \tag{5.3.19}$$

and since the tetrad vector e_3^{a} commutes with all other tetrad vectors it is also a Killing vector of the space-time. Hence $g_{33} = 1$, $g_{13} = g_{23} = g_{43} = 0$ and all metric functions are independent of the coordinate $x^3 = z$. This does not affect e_4^{a} , since it commutes with e_3^{a} , and hence it is still possible to set

$$e_4^{a} = (0,\ 0,\ 0,\ 1) \tag{5.3.20}$$

and we shall choose $x^4 = t$. Consequently (5.3.15) implies

$$\gamma_{121}^{2} + \gamma_{122}^{2} = F^2 (x, y) \tag{5.3.21}$$

where F(x,y) is an arbitrary function of $x^1 = x$ and $x^2 = y$. Thus there exist functions $\theta(x, y, t)$ such that

$$\gamma_{121} = F \cos\theta , \tag{5.3.22}$$

$$\gamma_{122} = F \sin\theta . \tag{5.3.23}$$

Also from (5.3.12) and (5.3.13) we obtain

$$\begin{aligned} \gamma_{121.44} + \omega^2 \gamma_{121} &= 0 \\ \gamma_{122.44} + \omega^2 \gamma_{122} &= 0 \end{aligned} \tag{5.3.24}$$

Since ω is a constant (5.3.24) can be integrated and we may set $\theta = \omega t + \phi(x, y)$. The covariant components of $\overset{a}{e}_1$ and $\overset{a}{e}_2$ are labeled so that

$$\begin{aligned} \overset{a}{e}_1 &= (A, B, 0, C), \\ \overset{a}{e}_2 &= (D, E, 0, F) \end{aligned} \tag{5.3.25}$$

and using (5.3.16) it can be shown that

$$A_t = \omega D , \quad D_t = - \omega A . \tag{5.3.26}$$

Similar relations exist for B and E, and C and F so that $A = A_0 \cos\theta_1$, $B = B_0 \cos \theta_2$, $C = C_0 \cos \theta_3$ where $\theta_i = \omega t + \phi_i (x, y)$ and A_0, B_0, and C_0 are constants of integration. From (2.4.4) and (2.4.5) we may calculate the contravariant metric:-

$$g^{44} = C_0{}^2 - 1 , \quad g^{11} = A_0{}^2 , \quad g^{22} = B_0{}^2 ,$$

$$g^{12} = A_0\, B_0 \quad \cos(\theta_1 - \theta_2)\,,$$
$$g^{14} = A_0\, C_0 \quad \cos(\theta_1 - \theta_3)\,,$$
$$g^{24} = B_0\, C_0 \quad \cos(\theta_1 - \theta_3)\,. \qquad (5.3.27)$$

Clearly the metric is stationary and $\overset{a}{e}_4$ is parallel to a Killing vector. We may use the remaining coordinate freedom

$$\tilde{t} = t + f(x, y)\,, \qquad (5.3.28)$$
$$\tilde{x} = \tilde{x}(x, y)\,, \qquad (5.3.29)$$
$$\tilde{y} = \tilde{y}(x, y)\,, \qquad (5.3.30)$$

to simplify the metric. Now g^{24} transforms as follows

$$\tilde{g}^{24} = \frac{\partial \tilde{y}}{\partial x}\left(g^{11} f_{,1} + g^{14}\right) + \frac{\partial \tilde{y}}{\partial y}\left(g^{22} f_{,2} + g^{24}\right) \qquad (5.3.31)$$

consequently, we may set $\tilde{g}^{24} = 0$ by an obvious choice of the function f(x,y). Any 2-space can be put into diagonal form. Consequently with the remaining coordinate freedom we may set $g^{12} = 0$. However, such coordinate transformations (5.3.29 - 30) will introduce cross terms such that $g^{24} \neq 0$, but a further application of (5.3.28) can then be used to set $g^{24} = 0$ as in (5.3.31). We may now determine g_{ij} and the metric can be put into the following form

$$ds^2 = A^2(x,y)dx^2 + dy^2 + dz^2 - (dt + C(x,y)dy)^2 \qquad (5.3.32)$$

From the the field equations for this metric we obtain

$$\frac{C_{yy}}{C_y} = \frac{A_y}{A} \qquad \frac{C_{yx}}{C_y} = \frac{A_x}{A} \qquad (5.3.33)$$

$$A_{yy} = \frac{1}{2}\,\frac{C_y^2}{A} \qquad (5.3.34)$$

where here and henceforward a subscripted coordinate denotes partial differentiation. Integrating (5.3.33) we obtain

$$C_y = kA \tag{5.3.35}$$

where k is a constant and substituting for C_y in (5.3.34) we obtain

$$A_{yy} = \frac{1}{2} k^2 A \ . \tag{5.3.36}$$

The curvature $\mathbf{K}$ of a two dimensional metric of the form

$$ds^2 = (W\ dx)^2 + (V\ dy)^2 \tag{5.3.37}$$

is given by (Ehlers 1961):

$$\mathbf{K} = -\frac{1}{VW}((V_{,x}W^{-1})_{,x} + (W_{,y}V^{-1})_{,y}) \tag{5.3.38}$$

and hence for the $\{x,y\}$ 2-space of (5.3.32) we obtain

$$\mathbf{K} = -\frac{A_{yy}}{A} \ . \tag{5.3.39}$$

Hence the 2-space is of constant negative curvature from (5.3.36). Consequently we may choose coordinates to put the two dimensional metric into standard form so that

$$A = e^{\frac{ky}{\sqrt{2}}} \ . \tag{5.3.40}$$

Integrating (5.3.35) we obtain

$$C = \sqrt{2}\, A + G(x) \tag{5.3.41}$$

and the function of integration G can be removed by a simple coordinate transformation.

The metric form derived above can be put into the form of Gödel's (5.3.1) and (5.3.2) with the transformation

$$\tilde{x} = \frac{x}{\sqrt{2}}$$

where $m = \frac{k}{\sqrt{2}}$. This completes the proof that the only shear-free, rotating and geodesic, purely electric perfect fluid solution of the Einstein field equations is the Gödel metric.

CHAPTER 6

Irrotational Perfect Fluids with a Purely Electric Weyl Tensor

§6.1 Introduction

Relatively few solutions to the field equations with shear are known, especially those without symmetries which could represent inhomogeneous cosmologies (Wainwright 1981). Shear-free fluids have been studied by several authors and many of the most well known solutions belong to this category (see for e.g. Kramer et al 1980). The shear-free condition has proven useful in finding and classifying solutions but appears to be too restrictive in a cosmological context. For example, Collins (1986) has shown that the only shear-free models with matter reasonable on a global scale are the Robertson-Walker cosmologies. The conjecture considered in chapter 5 may limit the generality of shear-free solutions. Also, in the study of stellar models, it appears extremely difficult to extract physical information from the known shear-free spherically symmetric solutions (Stephani 1983). The lack of such shearing solutions with an equation of state has led to a search for new solutions in this class (Van Den Bergh and Wils 1985). Bianchi type 1 space-times are anisotropic and have shear and expansion. Consequently, they have been widely used in cosmology (MacCallum 1985). However, the universe is inhomogeneous and although Friedmann Robertson-Walker and Bianchi type 1 models tend to describe general properties of the universe quite well, they are particularly deficient in, for example, explaining galaxy formation and other processes where inhomogeneity may be important.

Observational estimates of the shear (and rotation) of the universe place very low upper limits on their magnitudes at the present time and consequently the universe is Robertson-Walker to a good approximation. This does not stop us from considering situations where shear and rotation may be more important. Chaotic cosmology (Misner 1969) postulated an early epoch of turbulence whereby viscous processes 'damped' the shear to produce a homogenous universe. This was an attempt to explain how such a 'special' universe could have arisen without assuming any particular initial conditions. This theory is now generally thought to be untenable (Barrow and Matzner 1977).

More recently the initial conditions problem has been addressed by inflation theory (Guth 1981) which postulates a rapid exponential expansion very soon after the big bang. This expansion smooths out any inhomogeneities. Quiescent cosmology (Barrow 1978) has suggested that the universe is, in fact, evolving away from this homogeneity and it has been postulated that the gravitational 'clumping' taking place defines an arrow of time (Penrose 1985).

Furthermore, the universe is known to have very large scale structure in the form of clusters of superclusters of galaxies separated by large voids. The dynamics of superclusters has been considered and shear has been shown to have an important effect on the formation of structures from collapsing objects (Hoffman 1986).

All these observational and theoretical reasons coupled with the lack of relevant solutions gives considerable importance to the search for new inhomogenous cosmologies.

In this chapter it is assumed that u_i is irrotational ($\omega_{ij}=0$) and that the Weyl tensor is purely electric ($H_{ij}=0$).

This work generalises earlier work of Barnes (1973) on shear-free irrotational flows. These conditions imply $H_{ij}=0$ (Trumper 1962) but the converse $\omega_{ij}=H_{ij}=0 \Rightarrow \sigma_{ij}=0$ is not valid. Furthermore as the non-vanishing of $\mu + p$ is not used in this work, most of the results will be valid for vacuum fields admitting a hypersurface orthogonal time-like vector field with respect to which $H_{ij}=0$.

Firstly, Petrov type 1 space-times are considered using the orthonormal tetrad formalism outlined in chapters 2 and 3. For non-vanishing shear the metric is shown to take on a diagonal form. Certain solutions are shown not to exist and some of the problems in finding new solutions are demonstrated.

Petrov type D space-times are considered and sub-divided into classes depending on whether or not the shear is degenerate. If the shear is degenerate in the same plane as the Weyl tensor then the metric is shown to be diagonal. When the shear is not degenerate in this plane the proof is only extended to the case when the acceleration is aligned to the plane of the repeated principal null directions of the Weyl tensor.

A characterisation of the Szafron (1977) solutions and spherically symmetric space-times is given. The field-equations for one sub-class are solved completely but the resulting solutions have been previously found by Allnutt (1982) from a different set of assumptions.

Although no new solutions are obtained, the metric is simplified in two classes where, according to Carminati and Wainwright (1985), and to the author's knowledge, no solutions are known. It is hoped that further analysis may yield solutions in these classes.

§6.2 Petrov Type I Fields

Petrov type I fields are characterised as having distinct eigenvalues which in our notation is $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$. Clearly not more than one of the eigenvalues may be zero because of the vanishing trace of E_{ij}. The Petrov classification is concerned only with the eigenvalues of the Weyl tensor and so the shear may or may not be degenerate. If the shear does have repeated eigenvalues then, since no direction is singled out, without loss of generality we may set $\sigma_1 = \sigma_2$.

Few Petrov type I perfect fluid solutions are in fact known. As far as the author is aware the only irrotational solutions with non-zero shear and a purely electric Weyl tensor that have been found are homogeneous Bianchi type 1 space-times. An inhomogeneous Petrov type 1 space-time with shear and an equation of state $p = \gamma\mu$ has been presented (Wainwright and Goode 1980) but it appears that H_{ij} is non-zero. If we are to find solutions with realistic equations of state then these will be Petrov type I as the possible equations of state for type D solutions are limited to the cases of constant pressure and stiff matter (Carminati and Wainwright 1985).

For Petrov type I fields we have immediately from (3.4.18) that

$$\gamma_{AB4} = 0\,. \tag{6.2.1}$$

Hence the spatial triad is Fermi propagated along u_i. If, say, $\gamma_{123}=0$ then from (3.4.20), $\gamma_{312} = \gamma_{231} = 0$ and all the tetrad vectors are hypersurface orthogonal in this case.

Conversely if $\gamma_{123} \neq 0$, then $\gamma_{312} \neq 0$ and $\gamma_{231} \neq 0$ and none of the 3 spatial vectors are hypersurface orthogonal. However in this case we can show that the shear vanishes. It follows from (3.4.20) and (3.4.16) that, for some scalar λ:

$$\frac{(\sigma_1 - \sigma_2)}{(\alpha_1 - \alpha_2)} = \frac{(\sigma_2 - \sigma_3)}{(\alpha_2 - \alpha_3)} = \frac{(\sigma_3 - \sigma_1)}{(\alpha_3 - \alpha_1)} = \lambda\,, \tag{6.2.2}$$

from which we may deduce

$$\sigma_A = \lambda\,\alpha_A\,. \tag{6.2.3}$$

If we now take the directional derivative of equation (3.4.20) along u^a and use equation (3.4.19) and the Jacobi identities {0123}, {0132} and {0231} to eliminate the derivatives $\alpha_{A\cdot 4}$, $\gamma_{123\cdot 4}$ etc we obtain, with the aid of (3.4.16) and (6.2.3)

$$\lambda(\alpha_1^2-\alpha_2^2)\ \gamma_{123} = \lambda\,(\alpha_2^2-\alpha_3^2)\ \gamma_{231} = \lambda(\alpha_3^2-\alpha_1^2)\ \gamma_{312} \tag{6.2.4}$$

Thus if λ is non-zero it follows immediately that

$$\alpha_1 + \alpha_2 = \alpha_2 + \alpha_3 = \alpha_3 + \alpha_1$$

which implies that $\alpha_A = 0$ which is a contradiction. Hence we have proved that for Petrov Type I fields that either the Weyl principal vectors are hypersurface orthogonal or the fluid is shear-free. In the latter case it follows that the space-time is static (Barnes 1972).

When the principal vectors are hypersurface orthogonal, coordinates x^A exist such that

$$e_{Ai} \propto x^A{}_{,i} \,.$$

In this coordinate system we have a line-element of the form.

$$ds^2 = A^2dx^2 + B^2dy^2 + C^2dz^2 - V^2dt^2 \qquad (6.2.5)$$

where A, B, C, V are functions of the space-time variables and we label $x^1=x$, $x^2=y$, $x^3=z$, $x^4=t$. We note the following relations found from the definitions of chapter 2:

$$u_a = V\delta^4_a \,,$$

$$\dot{u}_\alpha = V^{-1}V_{,\alpha} \qquad \dot{u}_4 = 0 \,. \qquad (6.2.6)$$

An orthonormal tetrad for metric (6.2.5) is given by u_a and

$$e_{1a} = A\,\delta^1_a, \quad e_{2a} = B\,\delta^2_a, \quad e_{3a} = C\,\delta^3_a \quad .$$

Calculating the Ricci rotation coefficients for this tetrad we obtain

$$\gamma_{4A4} = \frac{V_{.A}}{V}\,, \quad \gamma_{ABB} = \frac{g_{B.A}}{g_B}\,,$$

$$\gamma_{4AA} = \sigma_A + \frac{\theta}{3} = \frac{g_{A.4}}{g_A} \qquad (6.2.7)$$

where $g_A = \sqrt{g_{AA}}$ and $A \neq B$. Clearly γ_{4A4} correspond to the acceleration and γ_{4AA} determine the shear and expansion of the flow. The vanishing of rotation is equivalent to $\gamma_{4AB} = 0$ for $A \neq B$.

The tetrad equations (3.4.10) to (3.4.21) with $\omega_{ij}=0$ are the equations to be considered. However without further simplifications these appear intractable in fact it will prove difficult to solve them even in sub-classes of the type D case.

Allowing one of the Weyl eigenvalues to vanish e.g. $\alpha_3 = 0$ then $\alpha_1 = -\alpha_2 = \alpha$ say and from (3.4.19) we obtain

$$\alpha(\sigma_1 - \sigma_2) + \sigma_3(\mu + p)/2 = 0 \ .$$

The case $\sigma_3 = 0$ is unhelpful as the solutions are either shear-free and hence static, or conformally flat. If we now assume $\sigma_1 = \sigma_2$ then either the space-time is shear-free or an Einstein space. However Brans (1975) has shown using the NP formalism that there are no Petrov type I Einstein spaces with the following conditions imposed on the Weyl tensor:

$$\Psi_0 = \Psi_4 , \quad \Psi_1 = \Psi_2 = \Psi_3 = 0 \ .$$

From the tetrad (3.2.7) defined in chapter 3 it is clear that these conditions are equivalent to $\alpha_3 = 0$. The only possible type 1 solution in this case is therefore static.

The remaining equation of (3.4.20) is

$$\alpha_{.4} + \theta\alpha + \frac{3\alpha\sigma_3}{2} = 0 \ .$$

Note that there are only two independent equations (3.4.20) because of the vanishing trace of the shear and Weyl tensors. This equation may be integrated, furthermore (3.4.22) are also easily integrated but this still appears to give a complicated metric form even in the geodesic case.

If we now drop the assumption $\alpha_3 = 0$ but keep $\sigma_1 = \sigma_2 = \sigma$, say then we have $\gamma_{411} = \gamma_{422}$ and from (3.4.16) $\gamma_{311} = \gamma_{322}$ and hence

$$B = A\,k(x,y) \ . \tag{6.2.8}$$

Using (3.4.15) and (3.4.16) we obtain

$$\gamma_{411} = f(z,t) \ , \tag{6.2.9}$$

$$C^2 = \sigma^{-2} g^2(z,t) \ , \tag{6.2.10}$$

where f, g, and k are arbitrary functions of their respective variables.

We might attempt to make progress by assuming vanishing acceleration. However it can easily be seen that there exist no type 1 geodesic solutions with repeated shear eigenvalues as (3.4.12) implies that the Weyl tensor is degenerate in the same plane as the shear i.e. $\sigma_1 = \sigma_2 \Rightarrow \alpha_1 = \alpha_2$ etc. In the case of non-degenerate shear further assumptions still need to be made to simplify the metric but the absence of any 'special' directions (apart from the 4-velocity) limits any possible simplifications.

§6.3 Petrov Type D Fields

Although perfect fluid solutions are known for all the Petrov types only a very few are known that are not type D or O. The type D solutions include a wide range of interesting space-times, homogeneous and inhomogeneous cosmologies, rotating stellar models (Wainwright 1977a), all solutions with spherical symmetry and all LRS solutions (Kramer et al 1980).

All Petrov type D vacuum solutions have been found explicitly (Kinnersley 1969), using the Newman-Penrose formalism. The solutions satisfy the following conditions:

1) The metrics depend on arbitrary parameters and not arbitrary functions.
2) The metrics admit at least two Killing vectors.
3) The repeated principal null directions of the Weyl tensor are tangent to shear-free geodesics.

The perfect fluid solutions, in general, satisfy none of these restrictions as is obvious from the Szekeres solutions (Szekeres 1975). Conditions (1) and (2) are clearly not satisfied from the comments made in chapter 4 and (3) is not valid from the results of Wainwright (1977b).

Wainwright (1977a) has classified the type D solutions according to the alignment of the kinematic quantities with the repeated p.n.d.'s of the Weyl tensor. As has already been shown, all solutions with a purely electric Weyl tensor necessarily have u_a aligned and hence belong to Wainwright's class 1.

For type D fields we may assume without loss of generality that $\alpha_1 = \alpha_2 = \alpha$, say. In this case from (3.4.21) and (3.4.19)

$$\gamma_{312} = \gamma_{231} = \gamma_{314} = \gamma_{234} = 0 \qquad (6.3.1)$$

and from (3.4.17)

$$(\sigma_1 - \sigma_2)\gamma_{123} = 0. \qquad (6.3.2)$$

This leads us to consider the following two classes:

a) $\sigma_1 = \sigma_2$ b) $\sigma_1 \neq \sigma_2$

In both cases from (3.4.22) we have $\gamma_{311} = \gamma_{322}$.

§6.4 Petrov Type D Fields with $\sigma_1 = \sigma_2$

In this section we assume $\sigma_1 = \sigma_2 = \sigma$, say. The shear has a repeated eigenvalue and is degenerate in the same plane as the Weyl tensor. Under the available tetrad freedom (3.4.9) we may rotate in the {x,y} plane. The Ricci rotation coefficients transform according to

$$\tilde{\gamma}_{123} = \gamma_{123} + \phi_{.3}, \quad \tilde{\gamma}_{124} = \gamma_{124} + \phi_{.4} .$$

We may choose ϕ so that $\tilde{\gamma}_{123} = \tilde{\gamma}_{124} = 0$, as the integrability conditions of $\phi.3 = -\gamma_{123}$ and $\phi.4 = -\gamma_{124}$:

$$-\phi.43 + \phi.34 - \gamma_{434}\phi.4 + \gamma_{433}\phi.3 = 0 ,$$

follow from the Jacobi identity {0132}. Hence the tetrad vectors are hypersurface orthogonal and the metric tensor is diagonal. This result was proven for the shear-free case by an alternative method.

The results (6.2.8) - (6.2.10) for type 1 fields only depend on the shear degeneracy and so are also valid here. We may also integrate (3.4.22) to obtain

$$V^2C\alpha = h(z,t) , \qquad (6.4.1)$$

where h is an arbitrary function of integration.

Further simplifications can be made to the metric functions by, for example, assuming vanishing acceleration or limiting the functional dependencies of the kinematic quantities. This latter step can be carried out in an invariant fashion by effectively assuming some alignment condition with the Weyl tensor. Wainwright (1979) has given a classification of inhomogeneous cosmologies using this approach.

(i) Geodesic solutions

Using (6.2.8) and (6.2.10) the metric takes the form

$$ds^2 = A^2(dx^2 + k^2(x,y)dy^2) + \sigma^{-2}\, g^2(z,t)dz^2 - dt^2 \qquad (6.4.2)$$

where V has been set equal to 1 by a suitable rescaling of the t coordinate. Any 2-space is conformally flat (Eisenhart 1949) and hence we may put the {x,y} two dimensional metric into explicitly conformally flat form. By absorbing the conformal factor of this 2-space into A we have a line element identical to that assumed by

Szekeres (1975). Assuming this metric form a priori for irrotational dust he was able to solve the field equations completely. The metrics comprise two classes, 'quasi-spherical collapsing space-times' and 'inhomogeneous cosmologies' (Bonnor et al 1977). They possess a number of interesting properties notably conformally flat slices and a lack of Killing vectors. They also contain the Robertson-Walker space-time as a limiting case as well as the inhomogeneous cosmology of Kantowski and Sachs (1966). It has also been shown that they generalise in a natural way the LRS space-times (Wainwright 1977b). Szafron (1977) has attempted to extend Szekeres work to geodesic perfect fluids. However the system of field equations is indeterminate and further ad hoc assumptions are needed to find new solutions. Barnes (1974) found some members of this class in a study of space-times of embedding class 1 before the dust solutions were recognised as inhomogeneous cosmologies.

In the course of deriving the metric form (6.4.2) it has been shown that the solutions considered by Szafron (1977) as perfect fluid generalisations of Szekeres' dust solutions are uniquely characterised by the conditions

1) The Weyl tensor is purely electric and of Petrov type D.
2) The fluid flow is irrotational and geodesic.
3) The shear tensor has two equal (non-zero) eigenvalues and its degenerate eigenblade coincides with that of the Weyl tensor.

Szekeres' original solutions have a similar characterisation as dust. It is also known that if the space-time is not LRS and an equation of state of the form $p = p(\mu)$ holds, then $p'(\mu)=0$ and the pressure is equal to a constant (Carminati and Wainwright 1985). Since this is equivalent to zero pressure with a cosmological constant these solutions are just the generalisations of Szekeres' with a non-vanishing cosmological constant. These have been found explicitly (Barrow and Stein-Schabes 1984). The results of this section and the notes from Carminati and Wainwright (1985) in chapter 3 show that these solutions complete their class 1. Since classes 2 and 3 have a condition that prevent there being any non-empty Robertson-Walker metrics as a limiting case the following theorem is valid:

If the magnetic part of the Weyl tensor vanishes and the fluid flow is irrotational with an equation of state of the form $p=p(\mu)$ then the only solutions with a non-empty Robertson-Walker metric as a limiting case are the generalisations of Szekeres solutions with cosmological constant or the metric is static or LRS.

Note that in the space-times of this section the density gradient does not, in general, lie in the plane spanned by the repeated principal null directions (p.n.d's) of the Weyl tensor. If in fact $\mu=\mu(z,t)$, the density gradient is aligned with this plane and these space-times are contained within the next section where $p=p(z,t)$.

(ii) Solutions with $p=p(z,t)$

The acceleration vector has a component in the z direction only and is contained within the plane spanned by the repeated p.n.d's and may be referred to as the aligned case. In general, $\mu=\mu(z,t)$ and the energy density is also aligned with this plane. It may straightforwardly be deduced from equations (3.4.15), (3.4.16), (3.4.18) and (3.4.22) that C is also a function of z and t only. Thus (3.4.16) then implies $\gamma_{311} = f(z,t)$, which may be integrated along with (6.2.9) to give a metric of the form:

$$ds^2 = A^2(z,t)(dx^2+ k^2(x,y)dy^2) + C^2(z,t)dz^2 - V^2(z,t)dt^2 \tag{6.4.3}$$

and the field equations for this metric give

$$\frac{k_{xx}}{k} = \text{constant} \ , \tag{6.4.4}$$

as all other terms are dependent on z and t only. The curvature **K** of a 2-d metric is given by (5.3.38) and for the {x,y} space in (6.4.3) we obtain

$$\mathbf{K} = \frac{k_{xx}}{k} \; . \qquad (6.4.5)$$

This implies that the curvature of the {x,y} two dimensional metric is constant and hence the metric is necessarily that of spherical or related symmetry. Compared to shear-free spherically symmetric solutions, few shearing solutions are known (Kramer et al 1980). This led Van den Bergh and Wils (1985) to consider shearing solutions with an equation of state. They found that many assumptions still reduced to Friedmann cosmologies but three new solutions as well as a generalisation of Wesson's stiff fluid solution were found (Wesson 1978). Another approach to these space-times was taken by Hajj-Boutros (1985) who took A to be separable in radial coordinate (z in the above notation) and t, and C and V to be functions of z only. Finally it is noted that the general exact solutions for static spherical symmetry have been obtained (Berger et al 1987) in terms of one arbitrary function of the radial coordinate. The result may possibly be extended to non-static fields.

(iii) Solutions with p=p(x,y,t)

In this case we assume that the acceleration vector is orthogonal to the plane spanned by the repeated p.n.d's. It has components in the x and y directions only and hence from the conservation equations, we may deduce that the pressure p, the energy density μ and the volume expansion θ are functions of x, y and t only. We may immediately integrate (3.4.15) and (3.4.18) to give $A^3\sigma = i(x, y, t)$, & $\alpha = j(x, y, t)\sigma$ where i and j are arbitrary functions of integration. Then using this along with (6.2.8) - (6.2.10) and (6.4.1) we may deduce that $V^2 j = h / g$. As the left and right hand sides of this equation are functions of x, y & t and z & t respectively, then each is a function l(t), say, of t only. After a coordinate transformation to make the metric of the {x, y} space explicitly conformally flat and after a suitable renaming of the function i(x, y, t), the metric may be written in the form:

$$ds^2 = i^2(x,y,t)\sigma^{-2/3}(dx^2 + dy^2) + g^2(z,t)\sigma^{-2}dz^2 - V^2(x,y,t)dt^2 .$$

Now, using (6.2.9) for γ_{411}, we may deduce, by repeatedly differentiating w.r.t. z and using a 'separation of variables' argument, that $f_{,z} = 0$ thus

$$\sigma = f(t) - 1/3\theta(x,y,t) . \qquad (6.4.6)$$

A similar argument applied to γ_{433} shows that g(z,t) is a separable function of z and t. Hence, by a suitable rescaling of the z coordinate, we can make g a function of t only. The metric is independent of z and hence admits the Killing vector ∂_z. As far as the author is aware no solutions are known in this class. A similar classification to the above can be applied to the shear scalar σ.

a) $\underline{\sigma = \sigma(z,t)}$

From (6.2.10) we obtain immediately that C = C(z,t) and then from (3.4.18)

$$\alpha = \frac{\mu}{3} + f(z,t) \qquad (6.4.7)$$

since $\gamma_{E33} = 0$ for E = 1,2 and where f is an arbitrary function of integration . We can also deduce that $\theta = \theta(z,t)$ so $\gamma_{433.E} = 0$ and from the Jacobi identities {0133} and {0233}

$$\gamma_{4E4}\,\gamma_{433} = 0 . \qquad (6.4.8)$$

Clearly if $\gamma_{4E4} = 0$ the analysis is as above in the case $p = p(z,t)$ and the metric is spherically symmetric. If $\gamma_{433} = 0$ then $\sigma = \frac{\theta}{6}$ and using this in (3.4.20) gives

$$\alpha._4 + \frac{3\alpha\theta}{2} + (\mu + p)\frac{\theta}{12} = 0 \quad .$$

Using the conservation equations $\alpha._4$ can be eliminated giving

$$p - \mu = g(z,t)$$

where g is arbitrary. Taking tetrad derivatives of this equation and using the remaining conservation equation we obtain

$$(2\alpha + G(z,t))V^2 = H(z,t)$$

where G and H are arbitrary functions. From (6.4.1) $V^2\alpha$ is a function of z and t only so $\gamma_{4E4} = 0$ for E=1,2 and the metric is again of the form (6.4.3) and therefore has spherical or related symmetry.

(b) $\sigma = \sigma(x,y,t)$

From (6.2.9) we have $\sigma = -\frac{\theta}{3} + f(z,t)$ and using this in (3.4.15) we obtain

$$3\sigma\,\gamma_{311} + f._3 = 0$$

and using (6.2.10) this can be integrated to give

$$A = e^{(h(z\ t) + l(x\ y\ t))/\ \sigma^2} \ .$$

As far as the author is aware no solutions are known in this class.

§6.5 Petrov Type D Fields with $\sigma_1 \neq \sigma_2$

The shear may be degenerate in this case as we have not ruled out the the possibility of $\sigma_1 = \sigma_3$. However, without loss of generality we consider the case $\sigma_1 \neq \sigma_2$.

We have from (6.3.2) that $\gamma_{123} = 0$, and for geodesic flows (3.4.11) implies $\gamma_{124} = 0$ but it has not been possible to prove this in the general case. Now (3.4.20) gives

$$\mu + p = 6\alpha \tag{6.5.1}$$

and so the only Einstein spaces in this class are conformally flat and hence of constant curvature. This relationship has been found using the Newman-Penrose formalism (Carminati and Wainwright 1985) and shows that no non-empty Robertson-Walker metrics exist as a limiting case in this class of solutions. The conservation equations become

$$p_{.A} = - 6\alpha \gamma_{4A4} \ , \tag{6.5.2}$$

$$\mu_{.4} = - 6\alpha\theta \ . \tag{6.5.3}$$

On subtracting the Jacobi identity {0131} from {0232} and comparing with (3.4.16) we obtain

$$\gamma_{434} = \gamma_{311} = \gamma_{322} = - \frac{\Sigma_{.3}}{\Sigma} \tag{6.5.4}$$

where $\Sigma = \sigma_1 - \sigma_2$. Using (6.5.1) ... (6.5.4) in (3.4.18) and (3.4.22) yields

$$\gamma_{133} = \gamma_{233} = 0 \ , \tag{6.5.5}$$

$$p_{.A} = \mu_{.A} = 3\alpha_{.A} = - 6\alpha \gamma_{4A4} \ . \tag{6.5.6}$$

Hence we may write $p = \mu + c(t)$. Writing (3.4.20) with A=3 gives

$$\alpha_{\cdot 4} - \alpha(3\sigma_3 - \theta) = 0 \tag{6.5.7}$$

and using this and (6.5.3) in the derivative of (6.5.1) along u_a we obtain

$$p_{\cdot 4} = 18\alpha\sigma_3 , \tag{6.5.8}$$

which allows us to derive

$$c_{\cdot 4} = 3\alpha\gamma_{433} , \tag{6.5.9}$$

and hence if $\gamma_{433} = 0$ there is an equation of state $p = p(\mu)$. Also, using (3.4.12) with (6.5.1) and (6.5.5) we obtain

$$\frac{c}{6} = \gamma_{433\cdot 4} + \gamma_{433}^2 - \gamma_{434\cdot 3} - \gamma_{434}^2 \quad . \tag{6.5.10}$$

The remaining equations are (3.4.10) - (3.4.12), the Jacobi identities and the shear derivatives (3.4.15) and (3.4.16) which yield

$$\gamma_{433.C} = 0 , \tag{6.5.11}$$

$$\sigma_{3.3} + 3\sigma_3 \gamma_{311} - \frac{2\theta_{.3}}{3} = 0 , \tag{6.5.12}$$

$$\sigma_{2.1} + (\sigma_2 - \sigma_1)\gamma_{122} + \frac{\theta_{.1}}{3} = 0 , \tag{6.5.13}$$

$$\sigma_{1.2} + (\sigma_1 - \sigma_2)\gamma_{211} + \frac{\theta_{.2}}{3} = 0 . \tag{6.5.14}$$

If we now put (6.5.5) and (6.5.11) in the Jacobi identities {0133} and {0233} we obtain

$$\gamma_{4E4} \; \gamma_{433} = 0 \quad E=1,2 \; . \qquad (6.5.15)$$

It will be seen that in fact, all solutions have $\gamma_{433} = 0$ and hence have an equation of state $p=p(\mu)$ from (6.5.9).

(i) Geodesic Solutions

From (3.4.11) we have immediately that $\gamma_{124}=0$ and the metric takes the diagonal form (6.2.5) with V=1. This simplifies the above equations a great deal: firstly we note that p, μ and α are all functions of time only and using (6.5.7) and (6.5.8) so are σ_3 and θ consequently

$$\gamma_{433} = f(t) \;\Rightarrow\; C = C(t)\, D(z)$$

and the function D(z) can be removed by a coordinate transformation. The shear propagation equation becomes

$$\gamma_{411} + \gamma_{422} + \frac{\Sigma_{.4}}{\Sigma} = 0 \; . \qquad (6.5.16)$$

Using (6.5.11) and the fact that σ_3 and θ are independent of x and y we can rewrite (6.5.13) and (6.5.14) as

$$2\gamma_{211} + \frac{\Sigma_{.2}}{\Sigma} = 0 \; , \qquad (6.5.17)$$

$$2\gamma_{122} + \frac{\Sigma_{.1}}{\Sigma} = 0 \; . \qquad (6.5.18)$$

On integrating these equations we obtain

$$AB\Sigma = f_3\,(x^{\alpha}) \ , \tag{6.5.19}$$

$$A^2\Sigma = f_2\,(x^1,\, x^3,\, t) \ , \tag{6.5.20}$$

$$B^2\Sigma = f_1\,(x^2,\, x^3,\, t) \ , \tag{6.5.21}$$

where f_A, now and henceforth, are functions of integration. Noting that $A=A(x,y,t)$ and $B=B(x,y,t)$ from (6.5.4) these equations imply that the f_A must be separable functions of their respective coordinates. So B may be written as $Af(t)$ with appropriate coordinate rescaling and an obvious relabelling of A. Using the definition of expansion $\theta = \gamma_{411} + \gamma_{422} + \gamma_{433}$ we can now show that A may be written as $F(x,y)g(t)$ where f, g and F are functions of their respective coordinates. The field equations $G_{01} = G_{02} = 0$, where G_{ij} is the Einstein tensor, imply $F_{,x} = F_{,y} = 0$ (or else $\sigma_1 = \sigma_2$) and so with suitable relabelling the metric takes the form:

$$ds^2 = A^2(t)\ dx^2 + B^2(t)dy^2 + C^2(t)\ dz^2 - dt^2 \tag{6.5.22}$$

Hence we have shown that the only solutions in this class are Bianchi type 1 space-times.

In fact it is possible to integrate the field equations completely for the Bianchi type 1 metrics in this class. The tetrad components of the Einstein tensor can be calculated with SHEEP and henceforward the computer output will be given:

$$G_{00} = B^{-1}C^{-1}B_T C_T + A^{-1}C^{-1}A_T C_T + A^{-1}B^{-1}A_T B_T$$

$$G_{11} = -C^{-1}C_{TT} - B^{-1}B_{TT} - B^{-1}C^{-1}B_T C_T$$

$$G_{22} = -C^{-1}C_{TT} - A^{-1}A_{TT} - A^{-1}C^{-1}A_T C_T$$

$$G_{33} = -B^{-1}B_{TT} - A^{-1}A_{TT} - A^{-1}B^{-1}A_T B_T \qquad (6.5.23)$$

Here and henceforward the partial derivative is implied by a subscripted coordinate. The shear propagation equations (3.4.11) will also be used and subtracting the A=2 equation from the A=1 and writing in terms of partial derivatives gives:

$$\frac{A_{TT}}{A} - \frac{B_{TT}}{B} = 0 \ . \qquad (6.5.24)$$

Now $G_{11} - G_{22}$ gives

$$\frac{C_T}{C}\left(\frac{A_T}{A} - \frac{B_T}{B}\right) = 0$$

and since $\gamma_{411} \neq \gamma_{422}$ in this class then this implies $C_T = 0$ and we may set $C = 1$. Clearly γ_{433} is now zero and there is an equation of state $p = \mu$. Bianchi type 1 models with this equation of state have been studied by Jacobs (1969) who found all the solutions. However, we will find those in this class.Now

$$G_{22} - G_{33} = AB_{TT} + A_T B_T = 0 \ , \qquad (6.5.25)$$

which is easily integrated to give

$$AB_T = \text{constant} \ . \qquad (6.5.26)$$

Using this to replace A in (6.5.24) gives the following third order differential equation

$$\frac{B_{TT}}{B_T^2}\left(\frac{B_T^2}{B_{TT}}\right)_T - \frac{B_T}{B} = 0 \ . \qquad (6.5.27)$$

The cases $B_{TT} = 0$ and $B_T = 0$ need also be considered. In the former case it is possible to set A = B = t and this is excluded from this class. With $B_T = 0$ we obtain an exact solution as we may set B=1 and A=t. Clearly this is a vacuum solution as the field equations (6.5.23) now imply $T_{ij}=0$

Equation (6.5.27) can be integrated to yield

$$B_T = kB^{1/d}$$

where k and d are constants and this is integrated to give two classes

$$\text{i) } d = 1, \quad B = e^{kt+to}$$

$$\text{ii) } d \neq 1, \quad B = (1 - \frac{1}{d})\,(kt + t_o)^{\,d/(d-1)} \ . \qquad (6.5.28)$$

In case i) we may set $B=e^{kt}$ by rescaling y. From (6.5.26) $A=e^{\pm kt}$ but the positive root is not a member of this class since $\gamma_{411} \neq \gamma_{422}$. The metric is therefore given by

$$ds^2 = e^{-2kt}\, dx^2 + e^{2kt}\, dy^2 + dz^2 - dt^2$$

In case ii) we may set $B = t^n$, by suitable rescalings of t and y, where n is a constant, and using (6.5.26) we may set $A = t^{1-n}$. This is the Oleson-Tupper solution mentioned by Wainwright (1977). Both solutions were found by Allnutt (1980).

It would be interesting to see what restrictions $H_{ij} = 0$ is on the general Bianchi type 1 metric

$$ds^2 = - dt^2 + g_{\alpha\beta}(t)\, dx^\alpha\, dx^\beta \qquad (6.5.29)$$

In fact it will be shown that all Bianchi type 1 metrics necessarily have a purely electric Weyl tensor and hence perfect fluid Bianchi type 1 space-times have a diagonal metric form. The metric (6.5.29) has $\omega_{ij} = \dot{u}_{ij} = 0$ and hence the identity (2.3.12) becomes

$$H_{ij} = h_i{}^k h_j{}^l \sigma_{(k}{}^{m;n} \eta_{l)pmn} u^p \ . \qquad (6.5.30)$$

Now from the definitions of the kinematical quantities we have

$$u_{i;j} = \sigma_{ij} + \frac{1}{3}\theta h_{ij}$$

and from the metric form (6.5.29)

$$u_{i;j} = -\Gamma^4{}_{ij} = \frac{g_{ij\cdot 4}}{2} \ .$$

Since u_a is hypersurface orthogonal then p = 4 in (6.5.30) and in fact it can be proved that $\sigma_{ij;\alpha} = 0$ using

$$\sigma_{ij;\alpha} = \frac{1}{2} g_{ij,4\alpha} - \Gamma^k{}_{i\alpha}\,\sigma_{kj} - \Gamma^k{}_{j\alpha}\,\sigma_{ki} \ .$$

The only non-zero Christoffel symbols are $\Gamma^4{}_{\alpha\beta}$ and $\Gamma^\alpha{}_{\beta 4}$, but $\sigma_{i4} = 0$ and $g_{i4,4} = 0$ and so immediately $\sigma_{ij;\alpha} = 0$. Hence $H_{ij} = 0$ and all Bianchi type 1 perfect fluids can be put into the diagonal metric form (6.5.22).

(ii) Solutions with p=p(z,t)

Equation (3.4.11) again gives $\gamma_{124} = 0$ and therefore the metric is diagonal. Since $\gamma_{414} = \gamma_{424} = 0$ the acceleration is aligned with the repeated p.n.d's and in a similar fashion as to the geodesic case we may deduce that V, C, μ, p, α, σ, and θ are functions of z and t only. Hence (6.5.16) - (6.5.18) are again applicable and can be integrated, furthermore we have

$$\gamma_{311} = f_4(z,t) \qquad (6.5.31)$$

and from the definition of θ

$$\gamma_{411} + \gamma_{422} = f_5(z,t) \ . \qquad (6.5.32)$$

The metric then takes the form

$$ds^2 = A^2(z)\ D^2(x,y)\ (F^2(t)dx^2 + G^2(t)dy^2) + C^2(z,t)dz^2 - A^2(z)dt^2 \qquad (6.5.33)$$

As the metric is diagonal so is T_{ij} and the off-diagonal terms of the Einstein tensor $G_{0\alpha} = 0$ give

$$D_{,1} = D_{,2} = C_{,4}A_{,3} = 0$$

Now if $A_{,3} = 0$ then $\gamma_{434} = 0$ and we are in the geodesic case and so we have $\gamma_{433} = 0$ and the metric can be put into the following form

$$ds^2 = A^2(z)\ (F(t)dx^2 + G(t)dy^2 - dt^2) + dz^2 \qquad (6.5.34)$$

The tetrad components of G_{ij} for this metric are:-

$$G_{00} = -2A^{-1}A_{ZZ} - A^{-2}(A_Z)^2 + A^{-2}B^{-1}G^{-1}B_T G_T$$

$$G_{11} = 2A^{-1}A_{ZZ} + A^{-2}(A_Z)^2 - A^{-2}G^{-1}G_{TT}$$

$$G_{22} = 2A^{-1}A_{ZZ} + A^{-2}(A_Z)^2 - A^{-2}B^{-1}B_{TT}$$

$$G_{33} = 3A^{-2}(A_Z)^2 - A^{-2}G^{-1}G_{TT} - A^{-1}B^{-1}B_{TT} - A^{-2}B^{-1}G^{-1}B_T G_T$$

We also have

$$A_{zz} - cA = 0 \,, \tag{6.5.35}$$

from (6.5.10) and $p=\mu+6c$. Now $G_{11} - G_{22}$ gives

$$\frac{G_{TT}}{G} - \frac{B_{TT}}{B} = 0 \tag{6.5.36}$$

and from $2 \times G_{33} - G_{11} - G_{22} = 0$ we obtain the following two relations

$$A_z^2 - cA^2 = k \,, \tag{6.5.37}$$

$$\frac{B_{TT}}{B} + \frac{G_{TT}}{G} + \frac{2B_T G_T}{BG} = 4k \,. \tag{6.5.38}$$

Equation (6.5.37) is, in fact, a first integral of (6.5.35). Making the coordinate transformation $\tilde{z} = A(z)$ produces the same metric form as obtained by Allnutt (1980). In fact, all solutions to the above equations have previously been given by him and were derived by placing certain ad hoc assumptions on the Newman-Penrose spin coefficients.

If $k = 0$ the equation for B and G are the same as in the geodesic case and the metrics correspond to Allnutts 1a(i) and 1a(ii).

For $k \neq 0$ the remaining four solutions are obtained. Putting $b = \log B$ and $g = \log G$, (6.5.37) and (6.5.38) become

$$\dot{y} + \dot{y}^2 = k \,, \tag{6.5.39}$$

$$(g-b)^{\cdot} = a\, e^{-y} \,, \tag{6.5.40}$$

where $y = b + g$ and a is constant. Equation (6.5.39) can be integrated to yield

$$\dot{y}^2 = k + d\, e^{-2y} \quad .$$

The solutions to these equations correspond to Allnutts' as follows:

a)$K<0$ $d=0$, 2b(i)
b)$K>0$ $d<0$, 2b(ii) $\varepsilon = 1$
c)$K<0$ $d>0$, 2b(ii) $\varepsilon = -1$
d)$k>0$ $d>0$, 2b(iii)

It can be shown that all the above solutions (including the geodesic ones) have non-degenerate shear. If the shear is to have a repeated eigenvalue then either $G_T=0$ or $B_T=0$. An inspection of the metrics shows that the only possibility is the Oleson-Tupper solution with $n=0$ (or $n=1$). However this is the vacuum solution derived earlier. The following theorem is therefore valid:

The Allnutt (1980, pp39-44) solutions are all solutions to Einstein's field equations for perfect fluid subject to the following restrictions:

1) The Weyl tensor is purely electric and of Petrov type D.
2) The fluid flow is irrotational with a non-degenerate shear tensor.
3) The acceleration vector lies in the plane, spanned by the repeated p.n.d's of the Weyl tensor.

(iii) <u>Solutions with $p=p(x,y)$</u>

For this case in general $\gamma_{124} \neq 0$ but for simplicity we only consider solutions with $\gamma_{124} = 0$ in this section. From (6.5.4) and (6.5.15) we have

$$\gamma_{434} = \gamma_{311} = \gamma_{322} = \gamma_{433} = 0 \tag{6.5.41}$$

and hence the equation of state is $p=\mu$ from (6.5.9) and (6.5.11). Since $p_{.4} = 0$ we must have $\sigma_3=0$ from (6.5.8) and so $\theta=0$ and we have

$$\gamma_{411} + \gamma_{422} = 0\,. \tag{6.5.42}$$

As $\alpha = \alpha(x,y)$ and integrating (6.5.6) yields $V^2\alpha = f(t)$ we may set $V=V(x,y)$ by a coordinate transformation. Equations (6.5.16) and (6.5.17) are also applicable here and integrating (6.5.42) we obtain a metric of the form

$$ds^2 = A^2(x,y,t)dx^2 + \frac{k^2(x,y)}{A^2(x,y,t)}\, dy^2 + dz^2 - V^2(x,y)dt^2 \tag{6.5.43}$$

where $A=\dfrac{f(x,t)}{g(y,t)}$.

As with the case $p=p(x,y)$ and $\sigma_1=\sigma_2$ the author is unaware of any solutions with p_x and p_y non-zero. These metrics belong to case II(a) of Carminati and Wainwright (1985) for which no solutions have yet been found.

§6.6 Concluding Remarks

It has been shown elsewhere that the assumption of a purely electric Weyl tensor means that the 4-velocity is necessarily aligned with a null tetrad determined by the Weyl tensor. The consequent characterisation of type 1 space-times may lead to simpler forms for the Newman-Penrose Bianchi identities. Further work on these fields with an equation of state is desirable in light of the results of Carminati and Wainwright (1985) for type D fields. It would also be useful to see if a proof of hypersurface orthogonality of the tetrad vectors could be found in the two remaining special cases: the type 1 static fields and the type D fields with non-degenerate shear with the acceleration vector not coplanar with the principal null vectors.

For type D fields, the 4-velocity lies in the 2-space defined by the repeated principal null directions of the Weyl tensor. With the additional assumption of vanishing vorticity the shear tensor and Weyl tensor commute and this leads to a natural choice of tetrad which is simultaneously an eigentetrad of the shear and of the Weyl tensor. In all but one class all vectors in the eigentetrad are hypersurface orthogonal and the metric takes on a particular diagonal form. In what may be termed the 'aligned' case, when the acceleration vector also lies in the 2-space spanned by the repeated principal null directions of the Weyl tensor we have shown that all solutions are, in principle, known unless they have at least three Killing vectors. Consequently the search for new inhomogeneous cosmological models in the class considered here must be restricted to the case where the acceleration does not lie in this plane or to the type 1 solutions.

Where simplified metric forms have been presented in this chapter the field equations have been calculated using SHEEP. Unfortuneately no significant progress in integrating them has yet been achieved.

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Appendix

The Jacobi Identities

The Jacobi identities are calculated in terms of Ricci rotation coefficients using (3.4.23). An STENSOR program to do this for orthonormal tetrads is given below. For ease of legibility and comparison with the text, the output from the program has been transferred to a word processor and the symbol V, used in the program to refer to the Ricci rotation cefficients, has been replaced by γ.

```
(PDEF VMI A23)
<V A B C>+<V C A B>$
(DECLT (V A12))
(TCOMP VMI)
(LOAD CORD)
(SYMBOLIC V)
(TCOMP VMI)
(MAKE VMI)
(RPL  GDD)-1$0$0$0$1$0$0$1$0$1$
(PDEF JAC A12)
<V A D B ,C>+<V D B A ,C>+<VMI %E A B><VMI D E C>$
(TCOMP JAC)
(FUNS (V ALL))
(DC MINUS EFUN (SXP) -1)
(MAKE JAC)
(PDEF JACID)
<JAC (A (B (C D>$
(DECLT (JACID (A 1 TO 3)))
(TCOMP JACID)
(WMAKE JACID)
```

1) General Jacobi identies (Orthonormal tetrad)

$$JACID_{0120} = \gamma_{010,2} - \gamma_{012,0} - \gamma_{020,1} + \gamma_{021,0} - \gamma_{010}\gamma_{121} - \gamma_{011}\gamma_{012} + \gamma_{011}\gamma_{021} -$$

$$- \gamma_{012}\gamma_{022} - \gamma_{013}\gamma_{032} + \gamma_{013}\gamma_{230} - \gamma_{020}\gamma_{122} + \gamma_{021}\gamma_{022} + \gamma_{023}\gamma_{031} - \gamma_{023}\gamma_{130} -$$

$$- \gamma_{030}\gamma_{132} + \gamma_{030}\gamma_{231} - \gamma_{031}\gamma_{230} + \gamma_{032}\gamma_{130}$$

$$JACID_{0121} = \gamma_{011,2} - \gamma_{012,1} - \gamma_{120,1} + \gamma_{121,0} - \gamma_{010}\gamma_{012} - \gamma_{010}\gamma_{120} + \gamma_{011}\gamma_{020} -$$

$$- \gamma_{012}\gamma_{122} - \gamma_{013}\gamma_{132} + \gamma_{013}\gamma_{231} + \gamma_{022}\gamma_{121} + \gamma_{031}\gamma_{123} - \gamma_{031}\gamma_{132} + \gamma_{032}\gamma_{131} -$$

$$- \gamma_{120}\gamma_{122} - \gamma_{123}\gamma_{130} + \gamma_{130}\gamma_{231} - \gamma_{131}\gamma_{230}$$

$$JACID_{0122} = \gamma_{021,2} - \gamma_{022,1} - \gamma_{120,2} + \gamma_{122,0} - \gamma_{010}\gamma_{022} + \gamma_{011}\gamma_{122} + \gamma_{020}\gamma_{021} -$$

$$- \gamma_{020}\gamma_{120} - \gamma_{021}\gamma_{121} - \gamma_{023}\gamma_{132} + \gamma_{023}\gamma_{231} - \gamma_{031}\gamma_{232} + \gamma_{032}\gamma_{123} + \gamma_{032}\gamma_{231} +$$

$$+ \gamma_{120}\gamma_{121} - \gamma_{123}\gamma_{230} + \gamma_{130}\gamma_{232} - \gamma_{132}\gamma_{230}$$

$$JACID_{0123} = \gamma_{031,2} - \gamma_{032,1} - \gamma_{130,2} + \gamma_{132,0} + \gamma_{230,1} - \gamma_{231,0} - \gamma_{010}\gamma_{032} +$$

$$+ \gamma_{010}\gamma_{230} + \gamma_{011}\gamma_{132} - \gamma_{011}\gamma_{231} + \gamma_{020}\gamma_{031} - \gamma_{020}\gamma_{130} + \gamma_{022}\gamma_{132} - \gamma_{022}\gamma_{231} -$$

$$- \gamma_{031}\gamma_{121} - \gamma_{031}\gamma_{233} - \gamma_{032}\gamma_{122} + \gamma_{032}\gamma_{133} - \gamma_{033}\gamma_{132} + \gamma_{033}\gamma_{231} + \gamma_{121}\gamma_{130} +$$

$+\gamma_{122}\gamma_{230}+\gamma_{130}\gamma_{233}-\gamma_{133}\gamma_{230}$

$JACID_{0130}=\gamma_{010,3}-\gamma_{013,0}-\gamma_{030,1}+\gamma_{031,0}-\gamma_{010}\gamma_{131}-\gamma_{011}\gamma_{013}+\gamma_{011}\gamma_{031}-$

$-\gamma_{012}\gamma_{023}-\gamma_{012}\gamma_{230}-\gamma_{013}\gamma_{033}-\gamma_{020}\gamma_{123}-\gamma_{020}\gamma_{231}+\gamma_{021}\gamma_{032}+\gamma_{021}\gamma_{230}+$

$+\gamma_{023}\gamma_{120}-\gamma_{030}\gamma_{133}+\gamma_{031}\gamma_{033}-\gamma_{032}\gamma_{120}$

$JACID_{0131}=\gamma_{011,3}-\gamma_{013,1}-\gamma_{130,1}+\gamma_{131,0}-\gamma_{010}\gamma_{013}-\gamma_{010}\gamma_{130}+\gamma_{011}\gamma_{030}-$

$-\gamma_{012}\gamma_{123}-\gamma_{012}\gamma_{231}-\gamma_{013}\gamma_{133}-\gamma_{021}\gamma_{123}+\gamma_{021}\gamma_{132}+\gamma_{023}\gamma_{121}+\gamma_{033}\gamma_{131}-$

$-\gamma_{120}\gamma_{132}-\gamma_{120}\gamma_{231}+\gamma_{121}\gamma_{230}-\gamma_{130}\gamma_{133}$

$JACID_{0132}=\gamma_{021,3}-\gamma_{023,1}-\gamma_{120,3}+\gamma_{123,0}-\gamma_{230,1}+\gamma_{231,0}-\gamma_{010}\gamma_{023}-$

$-\gamma_{010}\gamma_{230}+\gamma_{011}\gamma_{123}+\gamma_{011}\gamma_{231}+\gamma_{021}\gamma_{030}-\gamma_{021}\gamma_{131}+\gamma_{021}\gamma_{232}-\gamma_{022}\gamma_{123}-$

$-\gamma_{022}\gamma_{231}+\gamma_{023}\gamma_{122}-\gamma_{023}\gamma_{133}-\gamma_{030}\gamma_{120}+\gamma_{033}\gamma_{123}+\gamma_{033}\gamma_{231}+\gamma_{120}\gamma_{131}-$

$-\gamma_{120}\gamma_{232}+\gamma_{122}\gamma_{230}-\gamma_{133}\gamma_{230}$

$JACID_{0133}=\gamma_{031,3}-\gamma_{033,1}-\gamma_{130,3}+\gamma_{133,0}-\gamma_{010}\gamma_{033}+\gamma_{011}\gamma_{133}+\gamma_{021}\gamma_{233}+$

$+\gamma_{023}\gamma_{132}-\gamma_{023}\gamma_{231}+\gamma_{030}\gamma_{031}-\gamma_{030}\gamma_{130}-\gamma_{031}\gamma_{131}-\gamma_{032}\gamma_{123}-\gamma_{032}\gamma_{231}-$

$$-\gamma_{120}\gamma_{233} + \gamma_{123}\gamma_{230} + \gamma_{130}\gamma_{131} + \gamma_{132}\gamma_{230}$$

$$JACID_{0230} = \gamma_{020,3} - \gamma_{023,0} - \gamma_{030,2} + \gamma_{032,0} + \gamma_{010}\gamma_{123} - \gamma_{010}\gamma_{132} + \gamma_{012}\gamma_{031} +$$

$$+ \gamma_{012}\gamma_{130} - \gamma_{013}\gamma_{021} - \gamma_{013}\gamma_{120} - \gamma_{020}\gamma_{232} - \gamma_{021}\gamma_{130} - \gamma_{022}\gamma_{023} + \gamma_{022}\gamma_{032} -$$

$$- \gamma_{023}\gamma_{033} - \gamma_{030}\gamma_{233} + \gamma_{031}\gamma_{120} + \gamma_{032}\gamma_{033}$$

$$JACID_{0231} = \gamma_{012,3} - \gamma_{013,2} + \gamma_{120,3} - \gamma_{123,0} - \gamma_{130,2} + \gamma_{132,0} + \gamma_{011}\gamma_{123} -$$

$$- \gamma_{011}\gamma_{132} + \gamma_{012}\gamma_{030} + \gamma_{012}\gamma_{131} - \gamma_{012}\gamma_{232} - \gamma_{013}\gamma_{020} - \gamma_{013}\gamma_{121} - \gamma_{013}\gamma_{233} -$$

$$- \gamma_{020}\gamma_{130} - \gamma_{022}\gamma_{123} + \gamma_{022}\gamma_{132} + \gamma_{030}\gamma_{120} - \gamma_{033}\gamma_{123} + \gamma_{033}\gamma_{132} + \gamma_{120}\gamma_{131} -$$

$$- \gamma_{120}\gamma_{232} - \gamma_{121}\gamma_{130} - \gamma_{130}\gamma_{233}$$

$$JACID_{0232} = \gamma_{022,3} - \gamma_{023,2} - \gamma_{230,2} + \gamma_{232,0} + \gamma_{012}\gamma_{123} + \gamma_{012}\gamma_{231} - \gamma_{013}\gamma_{122} -$$

$$- \gamma_{020}\gamma_{023} - \gamma_{020}\gamma_{230} + \gamma_{021}\gamma_{123} - \gamma_{021}\gamma_{132} + \gamma_{022}\gamma_{030} - \gamma_{023}\gamma_{233} + \gamma_{033}\gamma_{232} +$$

$$+ \gamma_{120}\gamma_{132} + \gamma_{120}\gamma_{231} - \gamma_{122}\gamma_{130} - \gamma_{230}\gamma_{233}$$

$$JACID_{0233} = \gamma_{032,3} - \gamma_{033,2} - \gamma_{230,3} + \gamma_{233,0} + \gamma_{012}\gamma_{133} - \gamma_{013}\gamma_{132} + \gamma_{013}\gamma_{231} -$$

$$- \gamma_{020}\gamma_{033} + \gamma_{022}\gamma_{233} + \gamma_{030}\gamma_{032} - \gamma_{030}\gamma_{230} + \gamma_{031}\gamma_{123} - \gamma_{031}\gamma_{132} - \gamma_{032}\gamma_{232} +$$

$$+ \gamma_{120}\gamma_{133} - \gamma_{123}\gamma_{130} + \gamma_{130}\gamma_{231} + \gamma_{230}\gamma_{232}$$

$$JACID_{1230} = - \gamma_{012,3} + \gamma_{013,2} + \gamma_{021,3} - \gamma_{023,1} - \gamma_{031,2} + \gamma_{032,1} + \gamma_{010}\gamma_{023} -$$

$$- \gamma_{010}\gamma_{032} + \gamma_{012}\gamma_{030} + \gamma_{012}\gamma_{131} + \gamma_{012}\gamma_{232} - \gamma_{013}\gamma_{020} - \gamma_{013}\gamma_{121} + \gamma_{013}\gamma_{233} +$$

$$+ \gamma_{020}\gamma_{031} - \gamma_{021}\gamma_{030} - \gamma_{021}\gamma_{131} - \gamma_{021}\gamma_{232} - \gamma_{023}\gamma_{122} - \gamma_{023}\gamma_{133} + \gamma_{031}\gamma_{121} -$$

$$- \gamma_{031}\gamma_{233} + \gamma_{032}\gamma_{122} + \gamma_{032}\gamma_{133}$$

$$JACID_{1231} = \gamma_{121,3} - \gamma_{123,1} - \gamma_{131,2} + \gamma_{132,1} + \gamma_{011}\gamma_{023} - \gamma_{011}\gamma_{032} + \gamma_{012}\gamma_{031} +$$

$$+ \gamma_{012}\gamma_{130} - \gamma_{013}\gamma_{021} - \gamma_{013}\gamma_{120} - \gamma_{021}\gamma_{130} + \gamma_{031}\gamma_{120} - \gamma_{121}\gamma_{232} - \gamma_{122}\gamma_{123} +$$

$$+ \gamma_{122}\gamma_{132} - \gamma_{123}\gamma_{133} - \gamma_{131}\gamma_{233} + \gamma_{132}\gamma_{133}$$

$$JACID_{1232} = \gamma_{122,3} - \gamma_{123,2} - \gamma_{231,2} + \gamma_{232,1} + \gamma_{012}\gamma_{023} + \gamma_{012}\gamma_{230} - \gamma_{013}\gamma_{022} -$$

$$- \gamma_{021}\gamma_{032} - \gamma_{021}\gamma_{230} + \gamma_{022}\gamma_{031} - \gamma_{023}\gamma_{120} + \gamma_{032}\gamma_{120} + \gamma_{121}\gamma_{123} + \gamma_{121}\gamma_{231} -$$

$$- \gamma_{122}\gamma_{131} - \gamma_{123}\gamma_{233} + \gamma_{133}\gamma_{232} - \gamma_{231}\gamma_{233}$$

$$JACID_{1233} = \gamma_{132,3} - \gamma_{133,2} - \gamma_{231,3} + \gamma_{233,1} + \gamma_{012}\gamma_{033} - \gamma_{013}\gamma_{032} + \gamma_{013}\gamma_{230} -$$

$$-\gamma_{021}\gamma_{033} + \gamma_{023}\gamma_{031} - \gamma_{023}\gamma_{130} - \gamma_{031}\gamma_{230} + \gamma_{032}\gamma_{130} + \gamma_{121}\gamma_{133} + \gamma_{122}\gamma_{233} -$$

$$-\gamma_{131}\gamma_{132} + \gamma_{131}\gamma_{231} - \gamma_{132}\gamma_{232} + \gamma_{231}\gamma_{232}$$

2) Jacobi Identities for Irrotational Fields with a Purely Electric Weyl Tensor

For all the cases considered in chapter six, the tetrad is a shear eigenframe and the vector e_3 is Fermi-propagated along u^a which is irrotational. The minimal set of restrictions on the Ricci rotation coefficients is thus

$$\gamma_{4AB} = \gamma_{314} = \gamma_{234} = 0, \qquad \text{where } A \neq B.$$

We may apply these restrictions to the general Jacobi identities given above with the following STENSOR program. Note that there is an error in (A5) given in Barnes and Rowlingson (1989) which should be $Jacid_{0231}$ below.

```
% Specialisations of the rotation coefficients for irrotational purely electric
% fields

(SETNSUB 6 ESUL (DIFF 1))
V(0,1,2)$0$
V(0,2,1)$0$
V(0,2,3)$0$
V(0,3,2)$0$
V(0,1,3)$0$
V(0,3,1)$0$

(SETNSUB 2 (DIFF 1))
V(1,3,0)$0$
V(2,3,0)$0$

(ON ESUBS)

(WLSIMP JACID)
```

$$JACID_{0120} = \gamma_{010,2} - \gamma_{020,1} - \gamma_{010}\gamma_{121} - \gamma_{020}\gamma_{122} - \gamma_{030}\gamma_{132} + \gamma_{030}\gamma_{231}$$

$$JACID_{0121} = \gamma_{011,2} - \gamma_{120,1} + \gamma_{121,0} - \gamma_{010}\gamma_{120} + \gamma_{011}\gamma_{020} + \gamma_{022}\gamma_{121} - \gamma_{120}\gamma_{122}$$

$$JACID_{0122} = -\gamma_{022,1} - \gamma_{120,2} + \gamma_{122,0} - \gamma_{010}\gamma_{022} + \gamma_{011}\gamma_{122} - \gamma_{020}\gamma_{120} +$$

$$+\gamma_{120}\gamma_{121}$$

$$JACID_{0123} = \gamma_{132,0} - \gamma_{231,0} + \gamma_{011}\gamma_{132} - \gamma_{011}\gamma_{231} + \gamma_{022}\gamma_{132} - \gamma_{022}\gamma_{231} -$$

$$-\gamma_{033}\gamma_{132} + \gamma_{033}\gamma_{231}$$

$$JACID_{0130} = \gamma_{010,3} - \gamma_{030,1} - \gamma_{010}\gamma_{131} - \gamma_{020}\gamma_{123} - \gamma_{020}\gamma_{231} - \gamma_{030}\gamma_{133}$$

$$JACID_{0131} = \gamma_{011,3} + \gamma_{131,0} + \gamma_{011}\gamma_{030} + \gamma_{033}\gamma_{131} - \gamma_{120}\gamma_{132} - \gamma_{120}\gamma_{231}$$

$$JACID_{0132} = -\gamma_{120,3} + \gamma_{123,0} + \gamma_{231,0} + \gamma_{011}\gamma_{123} + \gamma_{011}\gamma_{231} - \gamma_{022}\gamma_{123} -$$

$$-\gamma_{022}\gamma_{231} - \gamma_{030}\gamma_{120} + \gamma_{033}\gamma_{123} + \gamma_{033}\gamma_{231} + \gamma_{120}\gamma_{131} - \gamma_{120}\gamma_{232}$$

$$JACID_{0133} = -\gamma_{033,1} + \gamma_{133,0} - \gamma_{010}\gamma_{033} + \gamma_{011}\gamma_{133} - \gamma_{120}\gamma_{233}$$

$$JACID_{0230} = \gamma_{020,3} - \gamma_{030,2} + \gamma_{010}\gamma_{123} - \gamma_{010}\gamma_{132} - \gamma_{020}\gamma_{232} - \gamma_{030}\gamma_{233}$$

$$JACID_{0231} = \gamma_{120,3} - \gamma_{123,0} + \gamma_{132,0} + \gamma_{011}\gamma_{123} - \gamma_{011}\gamma_{132} - \gamma_{022}\gamma_{123} + \gamma_{022}\gamma_{132} +$$

$$+\gamma_{030}\gamma_{120} - \gamma_{033}\gamma_{123} + \gamma_{033}\gamma_{132} + \gamma_{120}\gamma_{131} - \gamma_{120}\gamma_{232}$$

$$JACID_{0232} = \gamma_{022,3} + \gamma_{232,0} + \gamma_{022}\gamma_{030} + \gamma_{033}\gamma_{232} + \gamma_{120}\gamma_{132} + \gamma_{120}\gamma_{231}$$

$$JACID_{0233} = -\gamma_{033,2} + \gamma_{233,0} - \gamma_{020}\gamma_{033} + \gamma_{022}\gamma_{233} + \gamma_{120}\gamma_{133}$$

$$JACID_{1230} = 0$$

$$JACID_{1231} = \gamma_{121,3} - \gamma_{123,1} - \gamma_{131,2} + \gamma_{132,1} - \gamma_{121}\gamma_{232} - \gamma_{122}\gamma_{123} + \gamma_{122}\gamma_{132} -$$

$$-\gamma_{123}\gamma_{133} - \gamma_{131}\gamma_{233} + \gamma_{132}\gamma_{133}$$

$$JACID_{1232} = \gamma_{122,3} - \gamma_{123,2} - \gamma_{231,2} + \gamma_{232,1} + \gamma_{121}\gamma_{123} + \gamma_{121}\gamma_{231} - \gamma_{122}\gamma_{131} -$$

$$-\gamma_{123}\gamma_{233} + \gamma_{133}\gamma_{232} - \gamma_{231}\gamma_{233}$$

$$JACID_{1233} = \gamma_{132,3} - \gamma_{133,2} - \gamma_{231,3} + \gamma_{233,1} + \gamma_{121}\gamma_{133} + \gamma_{122}\gamma_{233} - \gamma_{131}\gamma_{132} +$$

$$+\gamma_{131}\gamma_{231} - \gamma_{132}\gamma_{232} + \gamma_{231}\gamma_{232}$$