APPLICATIONS OF REISSNER'S PRINCIPLE
TO
STRUCTURAL DYNAMICS

by

Ahmad. Ali. Youssefi, BSc, MSc

A thesis submitted for the degree of
Doctor of Philosophy

Mechanical Engineering Department
The University of Aston in Birmingham

July 1983

THE UNIVERSITY OF ASTON IN BIRMINGHAM

Applications of Reissner's Principle to Structural Dynamics

PhD by 1983

Ahmad. A. Youssefi

The analysis and prediction of the dynamic behaviour of structural components plays an important role in modern engineering design. In this work, the so-called "mixed" finite element models based on Reissner's variational principle are applied to the solution of free and forced vibration problems, for beam and plate structures. The mixed beam models are obtained by using elements of various shape functions ranging from simple linear to complex cubic and quadratic functions. The elements were in general capable of predicting the natural frequencies and dynamic responses with good accuracy.

An isoparametric quadrilateral element with 8-nodes was developed for application to thin plate problems. The element has 32 degrees of freedom (one deflection, two bending and one twisting moment per node) which is suitable for discretization of plates with arbitrary geometry. A linear isoparametric element and two non-conforming displacement elements (4-node and 8-node quadrilateral) were extended to the solution of dynamic problems. An auto-mesh generation program was used to facilitate the preparation of input data required by the 8-node quadrilateral elements of mixed and displacement type.

Numerical examples were solved using both the mixed beam and plate elements for predicting a structure's natural frequencies and dynamic response to a variety of forcing functions. The solutions were compared with the available analytical and displacement model solutions.

The mixed elements developed have been found to have significant advantages over the conventional displacement elements in the solution of plate type problems. A dramatic saving in computational time is possible without any loss in solution accuracy. With beam type problems, there appears to be no significant advantages in using mixed models.

Key words:

REISSNER'S PRINCIPLE

MIXED FINITE ELEMENT

BEAM AND THIN PLATE

DYNAMIC ANALYSIS

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my project supervisor Mr T.H. Richards for his valuable help, guidance and encouragement during the entire period of the project.

It is also my pleasure to acknowledge the assistance of various members of the Department of Mechanical Engineering at the University of Aston in Birmingham. My particular gratitude goes to Dr C.W. Chuen, R. Firoozian and Gerald Seet.

The patience and understanding of Mrs. S. Glendon during the typing of the thesis is also worth special mention.

The Department of Mechanical Engineering of the University of Aston in Birmingham is acknowledged for the financial assistance during the course of this work.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOTATIONS

The following is a list of the principal symbols used in this thesis. Rectangular matrices are indicated by [ ], and column vectors by { }. Overbars denote specified quantities. Dot over a symbol denotes derivative with respect to time.

a, b	dimensions of a plate in x and y directions, respectively
A	cross sectional area of a beam, plate middle plane area
B_n	jump term
[B]	operational matrix
c	damping coefficient
[C]	compliance matrix; damping matrix
D	flexural rigidity of a plate
[D]	elasticity matrix
d.o.f.	degree of freedom
E	elastic modulus
F_x, F_y, F_z	components of body forces per unit volume
$\{F\}_{nc}$, $\{F\}_c$	vectors of non-conservative and conservative body forces respectively
[g], [h]	mixed element matrices
G	shear modulus
h	plate thickness
[G], [H]	Overall mixed matrices
I	second moment of inertia
[I]	Identity matrix
[J]	Jacobian matrix
i,j,k	dummy subscripts
[K]	overall stiffness matrix
l	beam element length
L	length of a beam

$[l]$, $[L_k]$ direction cosine transformation matrices

$[L]$ strain-displacement matrix

l,m,n direction cosines

$[m]$ element mass matrix

$[M]$ overall mass matrix

M Bending moment in a beam

M_x, M_y, M_{xy} Bending and twisting moments in a plate

$\{M\}_e$ vector of nodal bending (and twisting) moments for element (e)

n,s normal and tangential directions

N_1, N_2, ..., N_8 Interpolation functions

p distributed load on a beam or plate

P concentrated load

$\{p\}_e$ distributed load intensities

$\{q\}$ principal coordinates vector

Q_x, Q_y, Q_n shearing forces in a plate

$\{Q\}_e$ specific impressed forces

$\{r\}_e$ element consistent load vector

$\{R\}$ overall load vector

S, S_u, S_σ General surface and surfaces where displacements and stresses are prescribed respectively

t Time

T kinetic energy

$\{T\}$, T_x, T_y, T_z surface traction vector and components

To kinetic energy density

$\{u\}$, u,v,w displacement vector and components

$\{u\}_e$ element nodal displacement vector

$\{U\}$, $\{u\}_o$ overall displacement vectors

$\{\hat{U}\}$ mode shape vector

U, U* potential energy and complementary potential energy

V volume

Vn effective shearing force

W	transverse deflection
$\{w\}_e$	element nodal deflection vector
$\{w^*\}$	overall deflection vector
x,y,z	cartesian coordinates
T.D.O.F.	Total number of degrees of freedom in mixed models (displacements and moments).
α,β	angles, parameters
βx, βy	rotations
δ	variational operator
$\{\varepsilon\}$	strain vector (includes both normal and shear strains)
ζ	modal damping ratio
$\lambda_1, \lambda_2, \ldots\lambda_9$	Lagrange multipliers
ν	poisson ratio
ξ η	natural coordinates
π_p	potential energy functional
π_R, π_R^D	Reissner functional for static and dynamic analysis, respectively
ρ	mass per unit volume
ω	natural frequency
ω_D	damped natural frequency
$\{\sigma\}$	stress vector (includes both normal and shear stresses)

CHAPTER 1

INTRODUCTION

1. INTRODUCTION

The increased complexity of engineering structures, and demands for increased precision in design predictions has brought about a need for obtaining accurate and efficient models to represent the behaviour of the structure under various conditions of loading. Over the past two decades, the finite element technique has played an important role as a means of obtaining adequate solutions to problems which are otherwise intractable. In this work, the so-called "mixed formulation" is employed to develop finite element models for beam and plate type structures. Free and forced vibration problems are tackled and the efficiency of these models are ascertained with reference to the conventional displacement type formulation.

The finite element technique pioneered by Turner, Clough, Martin and Topp (1) in 1956, rapidly became a very popular means for the computer solution of complex problems, particularly in the field of structural engineering. In this method, an actual continuum is imagined divided into a series of elements which are connected at a finite number of points known as nodal points. This reduces the problem from one having an infinite number of degrees of freedom to one with a finite number. The approach then involves the approximation of a variational expression (functional) in terms of nodal variables of unknown magnitudes within each element. The extremization of the functional with respect to these unknowns yields the element characteristic matrices. The procedure is repeated for each element in turn and the overall structural properties are computed by adding contributions from individual elements. Finally, standard solution algorithms for discrete parameter systems are utilized to determine the unknowns. Various schemes have been offered which use either the displacements or the stresses or a combination of both as the basic variables. Most practical elements are formulted by use of assumed displacement fields and the potential energy principle. In this method the displacements are chosen as the prime unknowns, with the stresses being determined from the calculated displacement field. The approximation involved is that the equilibrium equations are not satisfied exactly, but only in an integral sense. The continuity of displacements is required because of the method's dependence on the potential energy theorem. Alternatively it is possible to proceed with the stresses as the primary unknowns, an approach which is called "the equilibrium method". In this method, the minimum complementary energy principle is used. The assumed stress field is chosen so as to satisfy the equilibrium conditions within and across the element boundaries, and the compatibility conditions are satisfied in a 'mean'. A more general variational principle is that of Reissner (2), in which the primary field variables, are both displacements and stresses. The application of this principle results in finite element discretisations with nodal displacement and stress variables: these are classed as "mixed models". Since both compatibility and equilibrium are violated, on a strictly point by point basis, no preference is given to either displacement or stress fields. The mixed finite element models have received wide spread applications in problems dealing with bending components such as plates and shells. Mixed formulation permits the relaxation of the interelement compatibility conditions which are generally difficult to satisfy in problems dealing with flexural components. This allows the use of low order shape functions for displacements and moments which results in a decrease of computational effort.

The mixed element was first introduced by Herrmann (3), used in the solution of static plate bending problems and demonstrated the specific features of the mixed formulation in the finite element method. Numerous other mixed plate elements have been presented, (4), (5), (6). Of particular interest are the works by Mota Soares (7), and Tsay and Reddy (8) who applied the isoparametric concept to mixed element formulation for the solution of free vibration problems. Excellent results were reported for this type of formulation.

In this thesis mixed models have been applied to beam and plate type structures to take advantage of the superiority of the mixed element over the displacement type element in this class of problems. The following problems are investigated:

(i) Free vibration analysis (Mode shapes and frequencies).

(ii) Forced vibration analysis (Time history response of displacements and moments).

A generalized version of Reissner principle has been derived which incorporates damping forces as non-conservative external forces.

The beam elements were examined by employing various interpolations for deflection and moment fields. It has been shown in this work that the element can favourably predict the natural modes and frequencies of the free vibration problem. The elements have also been tested in the forced vibration problems to determine the time history response of displacements and moments. The results compare favourably with known exact and displacement type element solutions. Only the two beam elements with (parabolic-linear) and (linear-parabolic) interpolations for displacement and moment, failed to produce meaningful results: in these elements the mixed matrix rank is of lower order than required. Redundant zero energy modes are produced which cannot be removed by application of kinematic boundary conditions. The success of the linear isoparametric element in Reference (7) prompted the development of a new quadratic mixed element, to be used in the solution of thin plate problems. The element has eight nodes with four degrees of freedom at each node. (One deflection and three moments). The geometric, displacement and moment fields are assumed to vary parabolically within each element. Two computer programs were written which dealt with the free and forced vibration problems separately. The programs are capable of analyzing plates of variable thickness, and various loading and support conditions. Furthermore the plate may have orthotropic properties coinciding with the coordinate axes. The element can be applied to plates of arbitrary plan form with the proper transformation of the moments on the boundary. The transient solutions are obtained by either direct integration or modal analysis techniques.

In order to obtain a basis for the comparison of results, the following elements have also been applied and extended to the dynamic case.

(i) 4-node and 8-node non-conforming displacement type elements, Ref. (9).

(ii) Linear quadrilateral mixed element, (Ref. (7).

Using the developed quadratic element, some free vibration problems are solved for plates with various edge conditions. The results from these are then compared with the exact solution (10) and those obtained using the elements named in (i) and (ii). Despite their simplicity, mixed elements yield reasonable (good) accuracy for the modal frequencies. It is also observed that, for a particular number of degrees of freedom the quadratic mixed element yields higher accuracy in the prediction of modal frequencies.

The transient displacements and moments are obtained for a simply supported square plate under dynamic loading. The results are presented graphically with the exact and other types of elements. The advantages gained in using the mixed models over the displacement type elements may be summed up as follows:

(i) Mixed models calculate the transient displacements and moments with comparable degree of accuracy.

(ii) The eigen problem is condensed to yield a set of equations in terms of nodal displacements only. The condensation of moment degrees of freedom is an exact operation whereas in displacement formulation some accuracy is lost in the reduction of the eigen problem.

(iii) In engineering application, stress is often the quantity which is of prime interest. With a mixed model, this quantity is obtained directly through a simple matrix transformation procedure. With a displacement model, however, this quantity is obtained using a differentiation process from an approximate displacement field . This procedure is somewhat lengthy, time consuming and inherently yields reduced accuracy compared with the displacements, whereas the matrix multiplication required by mixed models offer a faster and more efficient way for the calculation of stresses. In forced vibration applications, where the stress field is to be calculated at incrementals of time, this effect is most noticable.

CHAPTER 2

VARIATIONAL METHODS IN STRUCTURAL MECHANICS

2.1 INTRODUCTION

The variational or energy methods have long been used to study the behaviour of elastic structures as an alternative to the direct "vectorial" approach. Much of the interest and of the fascination of variational principles lies in the fact that a set of equations is replaced by the stationarity of a single functional in characterizing the dynamics of a system. The variational principles of an elasticity problem do, however, provide the governing equations of the problem as the stationary conditions of a functional and, in that sense, are equivalent to the governing equations. However, the variational approach has several advantages:

(i) The functional subject to variation has usually a definite physical meaning and is invariant under coordinate transformation. Thus, the problem can be easily formulated in any coordinate system.

(ii) When a problem of elasticity cannot be solved exactly, variational method provides a convenient means for obtaining approximate solutions. The accuracy of the solution is improved by increasing the number of degrees of freedom.

(iii) A variational problem with subsidiary conditions may be transformed into an equivalent problem that can be solved more easily than the original. Transformation is achieved by the Lagrange multiplier technique. Thus a family of variational principles which are equivalent to each other are derived.

In this chapter, the basic equations which govern the distribution of stress and deformation in elastic bodies are briefly presented. Lagrange's principle will be introduced as the root of all modern variational principles, from which the principle of minimum potential energy and Hamilton's principle are derived. Other variational principles such as minimum complementary potential energy may also be derived in a similar manner (11).

Generalized principles including that of Reissner's are summarised with reference to three dimensional dynamical problems. Some approximate methods of analysis, applicable to problems involving deformations and vibrations of elastic bodies are discussed. The following notations are used:

(a) The matrix notation.

(b) The generally employed scalar notation.

(c) Cartesian coordinates (x,y,z) are used throughout.

2.2 BASIC RELATIONS

The formulation of the governing differential equations of elasticity is well established (12), (13). Thus, suppose a body deforms under the action of external and inertial forces, which are in equilibrium in accordance with d'Alembert's principle, and each point undergoes a small displacement represented by the components u, v, w parallel to the directions of the coordinate axes, Figure (2.1). The state of stress at a point of the body is defined by nine components of stress, Figure (2.2).

$$\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ & \sigma_y & \tau_{zy} \\ & & \sigma_z \end{bmatrix} \qquad (2.1)$$

The governing equations may be summarized as follows:

2.2.1 Equations of dynamic equilibrium

The equations of equilibrium of an elementary particle dxdydz subject to body forces {F}, and undergoing accelerations $\frac{\partial^2}{\partial t^2}\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$ are:

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = \rho \frac{\partial^2 u}{\partial t^2}, \quad \tau_{xy} = \tau_{yx} \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = \rho \frac{\partial^2 v}{\partial t^2}, \quad \tau_{xz} = \tau_{zx} \qquad (2.2) \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = \rho \frac{\partial^2 w}{\partial t^2}, \quad \tau_{yz} = \tau_{zy}
\end{aligned}$$

2.2.2 Strain-displacement relations

The small displacement-strain relations are derived from purely geometrical considerations and are given by:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{zy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & o & o \\ o & \frac{\partial}{\partial y} & o \\ o & o & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & o \\ \frac{\partial}{\partial z} & o & \frac{\partial}{\partial x} \\ o & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = [L]\ \{u\} \qquad (2.3)$$

where $[L]$ is a matrix of differential operators.

2.2.3 Compatibility conditions

The necessary and sufficient conditions that the six strain components can be derived from three single-valued functions (equation

2.3) are called the compatibility conditions:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad ; \quad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad ; \quad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \qquad (2.4)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \quad ; \quad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

2.2.4 Stress-strain relations

The stress-strain relations are given by the generalized Hooke's law and can be represented in matrix form as:

$$\{\sigma\} = [D]\ \{\varepsilon\}$$

and

$$\{\epsilon\} = [D]^{-1}\ \{\sigma\} = [C]\ \{\sigma\} \qquad (2.5)$$

where $[C]$ is a matrix of material compliances, and

$$\{\sigma\}^t = [\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{xz}] \qquad (2.6)$$

$\{\varepsilon\}$ is given by (2.3). In the most general case, the matrix $[C]$ can contain up to twenty-one independent constants. Such a material is said to be anisotropic. A material which has three planes of elastic symmetry may be defined by nine independent constants. Such a material is said to be orthotropic and if the three planes of elastic symmetry coincide with x-y, x-z, y-z planes then:

$$[C] = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{yx}}{E_y} & \frac{-\nu_{zx}}{E_z} & o & o & o \\ & \frac{1}{E_y} & \frac{-\nu_{zy}}{E_z} & o & o & o \\ & & \frac{1}{E_z} & o & o & o \\ & & & \frac{1}{G_{xy}} & o & o \\ & & & & \frac{1}{G_{yz}} & \\ & & & & & \frac{1}{G_{xz}} \end{bmatrix} \quad (2.7)$$

For an isotropic material (which has complete elastic symmetry) only two independent constants are required, then

$$[C] = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & o & o & o \\ -\nu & 1 & -\nu & o & o & o \\ -\nu & -\nu & 1 & o & o & o \\ o & o & o & 2(1+\nu) & o & o \\ o & o & o & o & 2(1+\nu) & o \\ o & o & o & o & o & 2(1+\nu) \end{bmatrix} \quad (2.8)$$

where E is the Young's modulus for the material and ν is its Poisson's ratio.

2.2.5 Boundary conditions

The boundary of a solid, S, may be subjected either to prescribed displacements or stresses. Equilibrium requirements must be met in the interior (Eqn.2.2) and part of the surface boundary S_σ where tractions are prescribed , that is

$$\left.\begin{aligned} T_x &= \bar{T}_x \\ T_y &= \bar{T}_y \\ T_z &= \bar{T}_z \end{aligned}\right\} \text{ on } S_\sigma \quad (2.9)$$

where $\{\bar{T}\}$ denote prescribed values of tractions. The components of surface traction T, Fig. (2.3), are given by:

$$\begin{aligned} T_x &= \sigma_x . l + \tau_{xy} . m + \tau_{xz} . n \\ T_y &= \tau_{yx} . l + \sigma_y . m + \tau_{yz} . n \\ T_z &= \tau_{zx} . l + \tau_{yz} . m + \sigma_z . n \end{aligned} \qquad (2.10)$$

l, m, n being the direction cosines of the unit vector normal to the boundary. On the other hand, displacements are prescribed on part S_u of the boundary and the geometrical conditions given by:

$$u = \bar{u} \; , \; v = \bar{v}, \;\; w = \bar{w} \quad \text{on } S_u \qquad (2.11)$$

The whole surface S is therefore the sum of S_σ and S_u that is:

$$S = S_\sigma + S_u$$

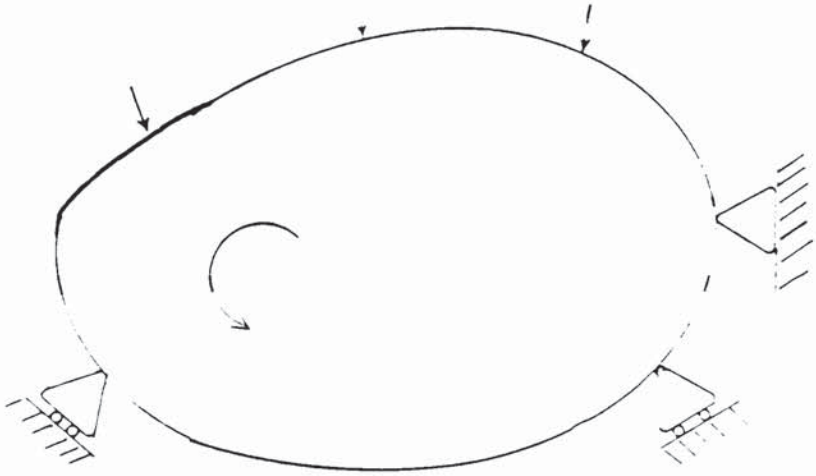


Fig 2.1 Elastic body subject to external forces.

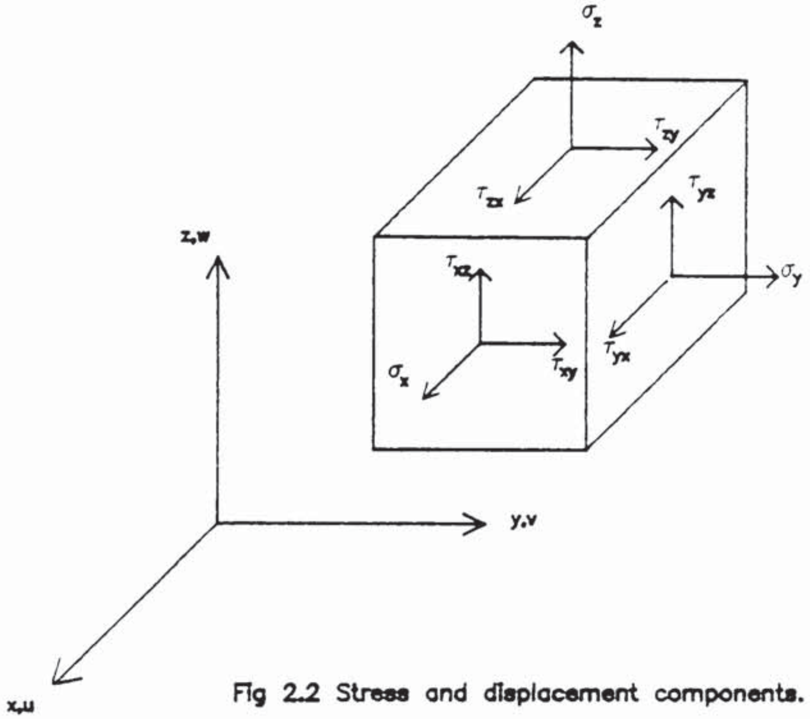


Fig 2.2 Stress and displacement components.

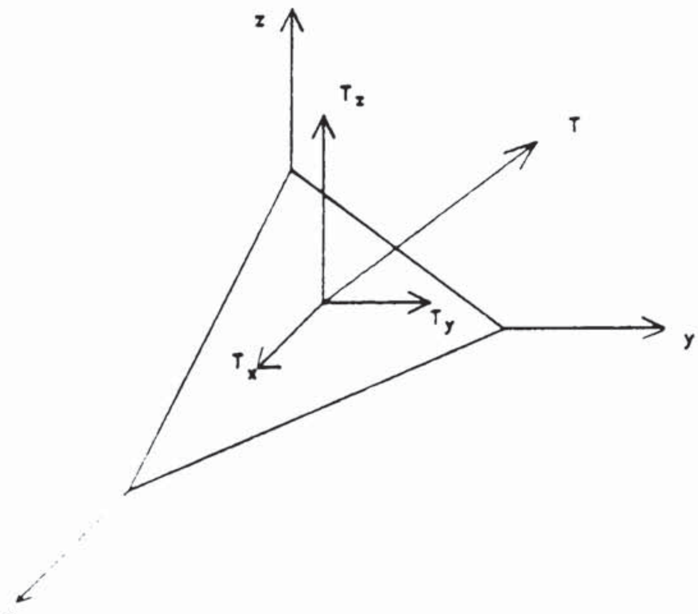


Fig 2.3 Boundary conditions.

2.3 CLASSICAL VARIATIONAL PRINCIPLES

The above summary has implied the use of Newton's laws of motion and geometry. The problem may be alternatively formulated from an integral point of view by means of d'Alembert's principle. Using the concept of variations for the interpretation of problems in mechanics. Lagrange (1736-1813) reformulated d'Alembert's principle thus yielding "Lagrange's principle". In its most general form, this principle may be stated as:

$$\int \left(\{d\ Q\}_e^t - \{\ddot{u}\}^t\ dm \right) \{\delta u\} = o, \tag{2.12}$$

where $\{Q\}_e$ are the impressed forces, m is the mass of the mechanical system and $\{u\}$ are its displacements. Lagrange's principle may be considered as the starting point for developing the more general variational principles which are freed from some or all restrictions. Thus, combining (2.12) with the Lagrange relaxation principle, one will be able to tackle any problem in mechanics in terms of a variational principle. Based on (2.12), some useful variational principles have been developed (11) which deal with problems in structural mechanics.

This section describes in brief the classical variational principles applicable to non-conservative systems. A knowledge of this is necessary to enable the understanding of the work described in the later sections.

2.3.1 Lagrange's principle for elasticity problems

For problems in structural engineering, Lagrange's principle, Equation (2.12) must be reformulated in terms of internal and external forces for an elastic, continuous body. Let V be the volume of the body. Then, set

$$\{dQ\}_e = \{Q\}_e \, dV \quad , \quad dm = \rho dV$$

where $\{Q\}_e$ are the specific impressed forces and ρ is the mass density of the body. The impressed forces may be written as:

$$\{Q\}_e = \{Q\}_{ext} + \{Q\}_{int} \tag{2.13}$$

where $\{Q\}_{ext}$ and $\{Q\}_{int}$ are the external and internal forces respectively. Therefore, (2.12), changes into:

$$\int_V \{Q\}_{ext}^t \{\delta u\} \, dV + \int_V \{Q\}_{int}^t \{\delta u\} \, dV - \int_V \rho\{\ddot{u}\}^t \{\delta u\} \, dV = o \tag{2.14}$$

In (2.14) variations $\{\delta u\}$, the so-called virtual distortions, must be small and since the reactions have not been taken into account, they must be restricted to such ones that satisfy the prescribed kinematical constraints imposed on the body at the points of application of the reactions.

The external forces $\{Q\}_{ext}$ consist of $\{F\}$, (body forces per unit volume), and $\{T\}$, (distributed surface tractions per unit surface). Hence,

$$\int_V \{Q\}_{ext}^t \{\delta u\} \, dV = \int_V (F_x \delta u + F_y \delta v + F_z \delta w) \, dV$$

$$+ \int_S (T_x \delta u + T_y \delta v + T_z \delta w) \, dS = \delta U_{ext} \tag{2.15}$$

where U_{ext} is the potential of the external forces as far as these are conservative. Now, the surface S of the body consists of a part S_σ on which surface loads $\{T\} = \{\bar{T}\}$ are prescribed, and of a part S_u on which displacements $\{u\} = \{\bar{u}\}$ are prescribed. Then

$$\int_S \{T\}^t\{\delta u\}\, dS = \int_{S_\sigma} \{\bar{T}\}^t \{\delta u\}\, dS , \qquad (2.16)$$

Since $\{\delta u\} = o$ on S_u. Therefore with (2.16) in (2.15), we have:

$$\int_V \{Q\}^t_{ext}\{\delta u\}dV = \int_V \{F\}^t_{nc}\{\delta u\}dV + \int_{S_\sigma} \{\bar{T}\}^t_{nc} \{\delta u\}dS - \delta\, U_{ext} \qquad (2.17)$$

The integral involving the internal forces is given by:

$$\int_V \{Q\}^t_{int}\{\delta u\}dV = -\int_V (\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \ldots)dV = -\int_V \{\sigma\}^t\{\delta\varepsilon\}\, dV \qquad (2.18)$$

where $\{\sigma\}$ and $\{\varepsilon\}$ denote the vectors of stress and strain components respectively. Hooke's law, equation (2.5) is given by:

$$\{\sigma\} = \left[D\right] \{\varepsilon\} \qquad (2.5)$$

Thus

$$\{\sigma\}^t\{\delta\varepsilon\} = \{\varepsilon\}^t[D]\{\delta\varepsilon\} = \delta\left[\tfrac{1}{2}\{\varepsilon\}^t[D]\{\varepsilon\}\right] = \delta\, U_o \qquad (2.19)$$

The quantity

$$U_o = \tfrac{1}{2}\{\varepsilon\}^t [D] \{\varepsilon\} \qquad (2.20)$$

is the potential (strain) energy density of the internal forces.

For isotropic elastic material, Equation (2.20) may be expanded as:

$$U_o = \frac{E}{2(1+\nu)}\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \tfrac{1}{2}\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right) + \frac{E\nu}{2(1+\nu)(1-2\nu)}(\varepsilon_x+\varepsilon_y+\varepsilon_z)^2 \qquad (2.21)$$

which may also be expressed in terms of displacement components (u, v, w) by using the strain-displacement relations (2.3), thus

$$U_o\ (u,v,w) = \frac{E}{2(1+\nu)}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right) + \frac{E\nu}{2(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2$$

$$+ \frac{E}{4(1+\nu)}\left(\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right) \qquad (2.22)$$

Now using (2.19) in (2.18) yields:

$$\int_V \{Q\}^t_{int}\ \{\delta u\}dV = -\int_V \delta U_o dV = -\delta\int_V U_o dV = -\delta U_{int} \qquad (2.23)$$

where U_{int} is called the potential energy of the internal forces. Substituting (2.17) and (2.23), back into (2.14) yields:

$$\int_V \{F\}^t_{nc}\ \{\delta u\}\ dV + \int_{S_\sigma} \{\bar{T}\}^t_{nc}\{\delta u\}dS - \delta U_{ext} - \delta U_{int} - \int_V \rho\{\ddot{u}\}^t\{\delta u\}dV = o \qquad (2.24)$$

This expression represents Lagrange's principle for the elastic continuous body and is useful in applications to elasticity problems in which external forces may not be derivable from potential functions.

2.3.2 Minimum potential energy principle

In this section minimum potential energy principle will be derived from Lagrange's principle (Eqn. 2.24). The following restrictive assumptions are made:

(i) The problem is a static one. Then,

$$\{\ddot{u}\} = o \qquad (2.25)$$

(ii) The problem is a conservative one. In other words the external forces possess a potential. Then,

$$\int_V \{F\}^t_{nc}\{\delta u\}\, dV + \int_{S_\sigma} \{\bar{T}\}^t_{nc}\{\delta u\}\, dS \;=\; o \qquad (2.26)$$

Under the above assumptions, Lagrange's principle (2.24) changes into:

$$-\,\delta\,(U_{ext} + U_{int}) = o \qquad (2.27)$$

Let

$$\pi_p = U_{int} + U_{ext} \;=\; \int_V U_o dV - \int_V \{F\}^t_c \;\{u\}dV - \int_{S_\sigma} \{\bar{T}\}^t_c\, \{u\}\, dS \qquad (2.28)$$

be the total potential energy of the elastic body where now both $\{F\}_c$ and $\{\bar{T}\}_c$ are conservative forces, then

$$\delta\, \pi_p = 0, \quad \pi_p \;=\; \text{minimum} \qquad (2.29)$$

It can be proved that U_o is a positive definite quantity (12). With (2.29), one has obtained the minimum potential energy principle which may be stated as follows:

> For a kinematically admissible displacement field related to a stress field satisfying the equilibrium conditions, the total potential energy assumes a minimum value as compared to values resulting from any other admissible displacement field.

2.3.3 Hamilton's principle

Hamilton's principle may be derived from the general Lagrange's principle. For an elastic body the principle is given by Equation (2.24):

$$\int_V \{F\}^t_{nc}\{\delta u\}dV + \int_{S_\sigma} \{T\}^t_{nc}\{\delta u\}dS - \delta U_{ext} - \delta U_{int} - \int_V \rho\{\ddot{u}\}^t\{\delta u\}\, dV = 0 \qquad (2.24)$$

As before it is required that $\{\delta u\}$ be consistent with the prescribed constraint conditions. Now however, the virtual distortion is further restricted by demanding that the variations $\{\delta u\}$ be zero at all points in the body at two arbitrary instant of time t_1 and t_2, that is:

$$\{\delta u\}_{\text{at } t_1} = \{\delta u\}_{\text{at } t_2} = 0 \tag{2.30}$$

Integrating (2.24) with respect to t results in

$$\int_{t_1}^{t_2} \left[\int_V \{F\}_{nc}^t \{\delta u\} dV + \int_{S_\sigma} \{\bar{T}\}_{nc}^t \{\delta u\} dS - \delta U_{ext} - \delta U_{int} \right] dt - \int_{t_1}^{t_2} \left[\int_V \rho \{\ddot{u}\}^t \{\delta u\} dV \right] dt = 0 \tag{2.31}$$

Now since

$$\int_{t_1}^{t_2} \rho \{\ddot{u}\}^t \{\delta u\} dt = \left[\rho \{\dot{u}\}^t \{\delta u\} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho \{\dot{u}\}^t \{\delta \dot{u}\} \, dt \tag{2.32}$$

and since (2.30) shall hold true, (2.32) changes into:

$$\int_{t_1}^{t_2} \rho \{\ddot{u}\}^t \{\delta u\} dt = -\int_{t_1}^{t_2} \rho \{\dot{u}\}^t \{\delta \dot{u}\} \, dt \tag{2.33}$$

with the kinetic energy density defined as:

$$T_o = \tfrac{1}{2} \rho \, \{\dot{u}\}^t \, \{\dot{u}\} \tag{2.34}$$

then $$\delta T_o = \rho \{\dot{u}\}^t \{\delta \dot{u}\} \tag{2.35}$$

and consequently (2.31) becomes equal to

$$\int_{t_1}^{t_2}\left[\int_V \{F\}^t_{nc}\{\delta u\}dV + \int_{S_\sigma} \{\bar{T}\}^t_{nc}\{\delta u\}\, dS - \delta U_{int} - \delta U_{ext}\right]dt + \int_{t_1}^{t_2}\delta T dt = 0 \tag{2.36}$$

where T is the kinetic energy of the system.

Equation (2.36) is a general statement of Hamilton's principle for elasticity problems with non-conservative external forces. It may be re-written as follows:

$$\delta\int_{t_1}^{t_2} (T - U_{int} - U_{ext})\, dt = -\int_{t_1}^{t_2}\left[\int_V \{F\}^t_{nc}\{\delta u\}dV + \int_{S_\sigma}\{\bar{T}\}^t_{nc}\{\delta u\}dS\right]dt \neq 0 \tag{2.36a}$$

which means $(T - U_{int} - U_{ext})$ is not even stationary. For many mechanical systems, the dissipative forces can be idealised by simple viscous damping forces $\{F_d\} = -c\{\dot{u}\}$, then equation (2.36) reads as follows:

$$\delta\int_{t_1}^{t_2} (T - U_{int} - U_{ext})dt - \int_{t_1}^{t_2}\int_V c\{\dot{u}\}^t\{\delta u\}dV\, dt = 0 \tag{2.36b}$$

For conservative systems equation (2.26) holds and (2.36) changes into:

$$\delta\int_{t_1}^{t_2}(T - U_{ext} - U_{int})\, dt = 0 \tag{2.37}$$

or simply:

$$\delta\int_{t_1}^{t_2}(T - \pi_p)\, dt = 0 \tag{2.38}$$

where $\pi_p = U_{ext} + U_{int}$ is the total potential energy of the system.

Equation (2.38) represents Hamilton's classical principle and may be applied to an elastic body subjected to external conservative forces.

2.4 MULTI-FIELD VARIATIONAL PRINCIPLES - LAGRANGE'S RELAXATION PRINCIPLE

As mentioned earlier in section 2.1, variational principles may be used conveniently as a means of constructing approximate solutions to boundary value problems in linear elasticity. The crucial point in applying it is the selection of appropriate coordinate functions which should satisfy certain restrictive conditions. The principle of minimum potential energy for instance requires that the displacement field be a continuous function of position and also satisfy the geometric boundary conditions of the problem under investigation. In practice it is often desirable to relax these requirements and thus widen the function space from which coordinate functions are chosen for comparison. This may be achieved by modifying the classical variational principles so that all continuity and boundary conditions become natural ones. From the point of view of mechanics it means applying Lagrange's relaxation principle. In the next section, it will be shown how to modify the minimum potential energy and thus obtain the generalized potential energy principle. Lagrange's relaxation principle introduces new fields in the modified variational statement and thus increases the number of independent variables subject to variations.

2.4.1 The Generalization of Minimum Potential Energy Principle

In the development of the minimum potential energy principle, the assumption is made that the strains are related to displacements according to (2.3), i.e.

$$\{\varepsilon\} - \left[L\right]\{u\} = \{o\} \quad \text{in the region} \quad (2.3)$$

and that

$$\{u\} - \{\bar{u}\} = 0 \quad \text{on the boundary } S_u \qquad (2.11)$$

Restrictions on the conditions of compatibility (2.3) and the geometric boundary conditions (2.11) may be removed by means of the Lagrange multiplier technique (see references (14),(15)). Thus the functional in (2.28) is modified to yield:

$$\pi_g = \int U_o dV - \int_V \{F\}_c^t \{u\} dV - \int_{S_\sigma} \{\bar{T}\}_c^t \{u\} dS$$
$$- \int_V \left[(\varepsilon_x - \frac{\partial u}{\partial x}) \lambda_1 + \ldots + (\gamma_{xz} - \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}) \lambda_6 \right] dV \qquad (2.39)$$
$$- \int_{S_u} \left[(u - \bar{u}) \lambda_7 + (v - \bar{v}) \lambda_8 + (w - \bar{w}) \lambda_9 \right] dS$$

where λ_1 to λ_9 are the corresponding multipliers. The modified principle is therefore stated as follows:

$$\delta \pi_g = 0 \qquad (2.40)$$

with no auxiliary constraint conditions.

Now it may be shown that the above principle provides, indeed, the differential equation of the problem under consideration and in addition all the boundary conditions, as natural conditions of the variational principle (2.40).

The independent quantities subject to variations in the functional (2.39) are: six strain components, three displacements and nine Lagrange multipliers $\lambda_1, \ldots\ldots \lambda_6$ and $(\lambda_7, \lambda_8, \lambda_9)$. Performing the variation with respect to these quantities, it is observed that:

$$\delta\pi_g = \int_V \left[\left(\frac{\partial U_o}{\partial \varepsilon_x}\delta\varepsilon_x + \ldots + \frac{\partial U_o}{\partial \gamma_{xz}}\delta\gamma_{xz}\right) - \{\bar{F}\}^t\{\delta u\}\right] dV - \int_{S_\sigma} \{\bar{T}\}^t\{\delta u\} dS$$

$$- \int_V \left[\left(\varepsilon_x - \frac{\partial u}{\partial x}\right)\delta\lambda_1 + \ldots + \left(\gamma_{xz} - \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\delta\lambda_6 \right] dV \tag{2.41}$$

$$- \int_{S_u} \left[(u - \bar{u})\,\delta\lambda_7 + (v - \bar{v})\delta\lambda_8 + (w - \bar{w})\,\delta\lambda_9 \right] dS$$

$$- \int_V \left[\left(\delta\varepsilon_x - \frac{\partial(\delta u)}{\partial x}\right)\lambda_1 + \ldots + \left(\delta\gamma_{xz} - \frac{\partial(\delta w)}{\partial x} - \frac{\partial(\delta u)}{\partial z}\right)\lambda_6 \right] dV$$

$$- \int_{S_u} \left[\delta u\lambda_7 + \delta v\,\lambda_8 + \delta w\,\lambda_9 \right] dS = 0$$

Integrating by parts, where appropriate using Green's formula, and rearranging the terms yields:

$$\delta\pi_g = \int_V \left[\left(\frac{\partial U_o}{\partial \varepsilon_x} - \lambda_1\right)\delta\varepsilon_x + \ldots + \left(\frac{\partial U_o}{\partial \gamma_{xz}} - \lambda_6\right)\delta\gamma_{xz} \right] dV$$

$$- \int_V \left[\left(\frac{\partial\lambda_1}{\partial x} + \frac{\partial\lambda_4}{\partial y} + \frac{\partial\lambda_6}{\partial z} + F_x\right)\delta u + (\ldots)\delta v + (\ldots)\delta w \right] dV \tag{2.42}$$

$$+ \int_{S_\sigma} \left[(T_x - \bar{T}_x)\delta u + (T_y - \bar{T}_y)\delta v + (T_z - \bar{T}_z)\delta w \right] dS +$$

$$+ \int_{S_u} \left[(T_x - \lambda_7)\delta u + (\ldots)\delta v + (\ldots)\delta w \right] dS$$

$$- \int_V \left[\left(\varepsilon_x - \frac{\partial u}{\partial x}\right)\delta\lambda_1 + \ldots + \left(\gamma_{xz} - \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\delta\lambda_6 \right] dV$$

$$- \int_{S_u} \left[(u - \bar{u})\delta\lambda_7 + (\ldots)\delta\lambda_8 + (w - \bar{w})\delta\lambda_9 \right] dS = 0$$

The conditions for π_g to be stationary are, then

$$\frac{\partial U_o}{\partial \varepsilon_x} = \lambda_1 \ldots ; \quad \frac{\partial U_o}{\partial \gamma_{xz}} = \lambda_6 \quad \text{in } V \tag{2.43}$$

$$T_x = \lambda_7 ; \quad T_y = \lambda_8 ; \quad T_z = \lambda_9 \quad \text{on } S_u \tag{2.44}$$

$$\frac{\partial \lambda_1}{\partial x} + \frac{\partial \lambda_4}{\partial y} + \frac{\partial \lambda_6}{\partial z} + F_x = 0; \text{ etc.} \quad \text{in } V \tag{2.45}$$

$$T_x = \bar{T}_x ; \quad T_y = \bar{T}_y ; \quad T_z = \bar{T}_z \quad \text{on } S_\sigma \tag{2.46}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} ; \quad \ldots ; \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad \text{in } V \tag{2.47}$$

$$u = \bar{u}; \quad v = \bar{v}; \quad w = \bar{w} \quad \text{on } S_u \tag{2.48}$$

These are the so-called Euler-Lagrange equations of the Principle (2.40). The last two equations give the constraints satisfaction and from the others it is seen that the Lagrange multipliers may be identified as follows:

$$\lambda_1 = \sigma_x , \quad \lambda_2 = \sigma_y , \ldots , \quad \lambda_6 = \tau_{xz} \tag{2.49}$$

and

$$\lambda_7 = T_x, \quad \lambda_8 = T_y, \quad \lambda_9 = T_z \tag{2.50}$$

with this identification the variational principle is known as the Hu-Washizu principle and can be stated as a stationary requirement for the function (2.51).

$$\pi_{HW} = \int_V U_o dV - \int_V \{F\}_c^t \{u\} \, dV - \int_{S_\sigma} \{\bar{T}\}_c^t \{u\} \, dS - \int_V \{\sigma\}^t (\{\varepsilon\} - [L]\{u\}) \, dV$$

$$- \int_{S_u} \{T\}^t (\{u\} - \{\bar{u}\}) \, dS \tag{2.51}$$

The independent quantities subject to variation in the functional (2.51) consist of the stresses $\{\sigma\}$, strains $\{\varepsilon\}$ and displacements {u} with no subsidiary conditions. On taking variations with respect to these quantities, it is found that the stationary conditions are given by Equation (2.43) through (2.48), with λ's replaced by stresses {σ} and {T} as in Equations (2.49), (2.50) (see reference (14)).

2.4.2 E. Reissner's principle

For a linear elastic solid, the so-called complementary strain energy density (U_o^*) is defined as

$$U_o^* = \tfrac{1}{2}\{\sigma\}^t[C]\{\sigma\} \tag{2.52}$$

thus it can be easily shown that

$$U_o = \{\sigma\}^t\{\varepsilon\} - U_o^* \tag{2.53}$$

holds true.

Substituting from (2.53) into (2.51), the strain components can be eliminated from the functional (2.51) to yield another principle known as Reissner's Principle (2).

Then,

$$\pi_R = -\int_V U_o^*\, dV + \int_V \{\sigma\}^t [L]\{u\}\, dV - \int_V \{F\}_c^t \{u\}\, dV - \int_{S_\sigma} \{\bar{T}\}_c^t \{u\}\, dS$$

$$- \int_{S_u} \{T\}^t \left(\{u\} - \{\bar{u}\}\right) dS \tag{2.54}$$

where now only {σ} and {u} are independent variables, with no subsidiary conditions. Carrying out the variations we get:

$$\delta\pi_R = \int_V \left[\{\delta\sigma\}^t [L] \{u\} + \{\sigma\}^t [L]\{\delta u\} - \frac{\partial U_o^*}{\partial\{\sigma\}} \{\delta\sigma\} \right] dV$$

$$- \int_V \{F\}^t \{\delta u\} \, dV - \int_{S_\sigma} \{\bar{T}\}^t \{\delta u\} \, dS - \int_{S_u} \{\delta T\}^t (\{u\} - \{\bar{u}\}) \, dS$$

$$- \int_{Su} \{T\}^t \{\delta u\} \, dS \tag{2.55}$$

The second term on the right of the above equation may be recast as follows (making use of the integral theorem of Gauss).

$$\int_V \{\sigma\}^t [L] \{\delta u\} \, dV = \int_{S = S_\sigma + S_u} \left[(\sigma_x . l + \tau_{xy} . m + \tau_{xz} . n) \delta u + (\ldots) \delta V \right.$$

$$\left. + (\tau_{xz} . l + \tau_{yz} . m + \sigma_z . n) \delta w \right] dS \tag{2.56}$$

$$- \int_V \left[\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} \right) \delta u + (\ldots) \delta v + \left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} \right) \delta W \right] dV$$

substituting equation (2.56) into (2.55) yields:

$$\delta\pi_R = - \int_V \left[\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + F_x \right) \delta u + (\ldots) \delta v + (\ldots) \delta w \right] dV$$

$$- \int_V \left[\left(\frac{\partial U_o^*}{\partial\sigma_x} - \frac{\partial u}{\partial x} \right) \delta\sigma_x + \ldots + \left(\frac{\partial U_o^*}{\partial\tau_{xz}} - \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \delta\tau_{xz} \right] dV \tag{2.57}$$

$$- \int_{S_\sigma} \{\delta u\}^t (\{T\} - \{\bar{T}\}) \, dS - \int_{S_u} \{\delta T\}^t (\{u\} - \{\bar{u}\}) \, dS$$

with $\{\delta u\}$ and $\{\delta\sigma\}$ as aribtrary independent quantities, the following relations are obtained as Euler equations and natural boundary conditions of the functional (2.54).

(a) - The equations of equilibrium in V

(b) - The strain-displacement relations in V

(c) - The requirements that $T_x = \bar{T}_x$, etc. on S_σ

(d) - The requirements that $u = \bar{u}$, etc. on S_u

2.4.3 Reissner's principle - extension to dynamical problems

Reissner's principle, equation (2.54), is applicable to static problems in which all forces, internal and external ones are derivable from a potential. In this section the principle is modified to the case of dynamic problems with non-conservative damping forces. To the author's knowledge this has not been attempted before.

Hamilton's principle, equation (2.36) may be modified in a manner similar to section (2.4.1) for static problems. Thus the dynamical version of Reissner's principle is obtained. Assuming that all the external forces are conservative and derivable from potential functions, the new Principle may be written as follows:

$$\delta \int_{t_1}^{t_2} \pi_R^D \, dt = \delta \int_{t_1}^{t_2} \Big[\int_V (-T_o - U_o^* + \{\sigma\}^t [L]\{u\}) \, dV - \int_V \{F\}_c^t \{u\} dV$$

$$- \int_{S_\sigma} \{\bar{T}\}_c^t \{u\} dS - \int_{S_u} \{T\}^t (\{u\} - \{\bar{u}\}) \, dS \Big] dt = 0 \qquad (2.58)$$

where

$$\pi_R^D = \pi_R - T$$

may be referred to as Reissner's dynamical functional. Carrying out the variations indicated in (2.58), we find that, as in the static case, the stationary conditions are the differential equations of dynamic equilibrium, strain-displacement relations and all mechanical and geometrical boundary conditions. Reissner's principle is thus seen to give equal emphasis to the conditions of equilibrium and compatibility since both appear as Euler-Lagrange equations of the functional π_R.

When damping forces are also included, Hamilton's principle equation (2.36b) may be generalized by means of Lagrange multipliers, thus generalized Reissner's principle is obtained which may be stated as follows:

$$\delta \int_{t_1}^{t_2} \Big[\int_V (-T_o - U_o^* + \{\sigma\}^t [L]\{u\}) dV - \int_V \{F\}_c^t \{u\} dV - \int_{S_\sigma} \{\bar{T}\}_c^t \{u\} \, dS - \int_{S_u} \{T\}^t (\{u\} - \{\bar{u}\}) dS \Big] dt + \int_{t_1}^{t_2} \int_V c\{\dot{u}\}^t \{\delta u\} dV \, dt = 0 \qquad (2.59)$$

2.5 APPROXIMATE METHODS

Problems of any complexity are governed by a set of simultaneous differential equations stemming from Newton's laws of motion. These equations can be regarded as the Euler-Lagrange equations of a functional with one or several dependent variables. The principal approach in approximate methods is to work with the functional for the purpose of finding approximate solution to the corresponding differential equations. In this connection, the variational methods of Ritz (16), Galerkin (16) and Kantorovich (17) have been extensively used, with the displacement finite element method becoming increasingly applied in recent years. In the next section, Rayleigh-Ritz method is outlined. The finite element method which may be interpreted as a piece-wise Rayleigh-Ritz method will be described in detail in the following chapters.

2.5.1 Rayleigh-Ritz method

The most notable approximate procedure is the Rayleigh-Ritz method (16), which was originally developed for use with the potential energy functional. In this method the structure's displacement field is approximated by functions which contain a finite number of independent coefficients. The assumed functions are chosen to satisfy the kinematic boundary conditions, but they need not satisfy the mechanical boundary conditions (ones involving forces and moments). The following displacement field components are thus employed for expressing the total potential energy.

$$
\begin{aligned}
u &= \phi_o(x,y,z) + \sum_{i=1}^{n} a_i \, \phi_i(x,y,z) \\
v &= \psi_o(x,y,z) + \sum_{i=1}^{n} b_i \, \psi_i(x,y,z)
\end{aligned}
\qquad (2.60)
$$

continued:

$$w = \gamma_o(x,y,z) + \sum_{i=1}^{n} c_i \gamma_i(x,y,z) \qquad (2.60)$$

where ϕ_o, ψ_o and γ_o satisfy the kinematic boundary conditions on S_u while the remaining functions are zero there. The scheme for the Ritz method is to choose the values of the unknown coefficients so as to minimize the total potential energy. Thus substituting equation (2.60) into the potential energy functional equation (2.28), and performance of the integration results in $\pi_p = \pi_p (a_i, b_i, c_i)$ $i=1,2,..n$ then for a stationary π_p, $\delta\pi_p = 0$ which is equivalent to

$$\frac{\partial \pi_p}{\partial a_i} = 0$$

$$\frac{\partial \pi_p}{\partial b_i} = 0 \qquad i = 1,2,\ldots,n \qquad (2.61)$$

$$\frac{\partial \pi_p}{\partial c_i} = 0$$

This process yields 3n simultaneous algebraic equations in the undetermined coefficients a_i, b_i, c_i. For dynamical problems the Rayleigh-Ritz procedure can be used in conjunction with Hamilton's principle (16). Thus the equations of motion are obtained which may be expressed in matrix notation as

$$\begin{bmatrix} K \end{bmatrix} \{a\} + \begin{bmatrix} M \end{bmatrix} \{\ddot{a}\} = \{R\} \qquad (2.62)$$

where {a} and {R} are the generalized coordinates and generalized forces respectively. The response is obtained by solving equation (2.62), using an appropriate direct integration or mode superposition method (see section 4.6).

For free vibration {R} = 0 in (2.62) and, with harmonic motion:

$$([K] - \omega^2[M]) \{a\} = \{o\} \qquad (2.63)$$

which may be solved by standard eigen value solution routines (18).

The Rayleigh-Ritz method may also be applied with the Reissner's functional. Now forces and displacements are independently represented by shape functions satisfying the "forced" boundary conditions (16), and the constants are found as before by rendering the functional stationary.

2.5.2 The finite element method

The finite element method pioneered by Turner et al (1) and Clough (19), is the most significant development in structural analysis in recent years. With the development of powerful digital computers the f.e.m. has gained considerable popularity and become a very important tool in the analysis of structural problems and in the broad field of continuum mechanics (20), (21). The basic concept of the method, when applied to problems of structural analysis, is that a continuum (structure) can be modelled analytically by its subdivision into regions (the finite elements) in each of which the behaviour is described by a separate set of assumed functions representing the stresses or displacements in that region. Then it is possible by the use of the appropriate energy functional and a procedure similar to the Rayleigh-Ritz technique to derive an element matrix equation which may have generalized displacements, stresses or both, at the nodal points, as unknowns to be evaluated.

The Rayleigh-Ritz technique is applied to each element in turn and the overall problem is examined by assembling all the individual element properties in a suitable manner.

Although the two procedures of Ritz and finite element are theoretically identical, in practice, the finite element method has most important advantages over a conventional Ritz analysis. A particular difficulty associated with a conventional Ritz analysis is the selection of appropriate Ritz functions. In order to solve accurately for large displacement or stress gradients, many functions may be needed. However, these functions also unnecessarily cover the regions in which the displacements and stresses vary slowly and where not many functions are required. Another difficulty arises when the total region of interest is made up of subregions with different kinds of strain. In such a case, the Ritz functions used for one region are not appropriate for the other regions and special displacement continuity conditions must be enforced. No such difficulties arise in the finite element procedure and it may be applied to represent highly irregular and complex structures and loading conditions.

The so-called displacement finite element method, based on the principle of minimum potential energy, is the most well known of all and has been applied to static, dynamic, buckling and a whole range of other problems (20), (21), (22), (23). The compatibility conditions imposed on the assumed displacement field can be satisfied without major difficulties in CØ continuity problems; for example, in plane stress and plane strain problems or the analysis of three-dimensional solids. However, in the analysis of bending problems, such as plate and shell analysis (C1 problems), continuity of displacement first derivatives along inter-element boundaries is difficult to maintain. Furthermore, considering complex analyses in which completely different finite elements must be used to idealize different regions of the structure, compatibility may be almost impossible to maintain. Difficulties with inter-element compatibility requirements render attractive alternative formulations based on a mixed variational principle.

2.5.3 The mixed finite element method

The most general variational principle is that of Reissner, in which the primary field variables are both displacements and stresses. The application of this principle results in finite element discretizations with nodal displacements and stress variables, referred to as mixed models. By using the Reissner functional in one of the several possible alternatives (22), (24), inter-element continuity conditions may be conveniently relaxed allowing the use of stress/displacement shape functions of lower order which ease the computational effort. Another advantage of finite element mixed models is that stresses and displacements are obtained with similar degrees of accuracy, thus avoiding the decrease in accuracy characteristic of the displacement method due to the process of differentiating approximate displacements to obtain the strains (and hence the stresses) once the displacement are evaluated. The mixed method was first investigated by Herrmann (3) in the static plate bending analysis.

CHAPTER 3

DYNAMIC ANALYSIS OF ELASTIC BEAMS AND PLATES

3.1 INTRODUCTION

The prediction of dynamic behaviour of structural elements in the form of beams and plates due to transient forces is a problem of practical importance, with applications in the design of vehicles, aircraft, missiles, etc.

In the absence of continuously applied external forces, the structure undergoes a motion due to inertia and elastic forces only. Natural frequencies and modal shapes can be determined from a free vibration analysis of the structure. Knowledge of the natural frequencies helps the designer avoid the peak resonances which occur in the vicinity of the natural frequencies. More detailed knowledge of the mode shapes may be used to estimate bending stresses excited in a vibratory mode. The free vibration results only give information for each mode independent of the rest. The more important class of problems is when the structure experiences external dynamic loads. Displacements and stresses developed under such circumstances are of great importance to the structural analyst.

A recent survey by Leissa (25, 26) uncovers more than 200 references which deal with problems involving the free, undamped vibration of plates. Forced vibration problems, however, have not received as much attention, largely due to the increased complexity of such problems. For convenience for subsequent references the basic equations governing the motions of elastic beams and plates are reviewed. Hamilton's principle (2.36) and the Reissner functional (2.54) will be specialized for plate problems. Finally, the available classical methods of solving dynamic plate problems will be outlined.

3.2 RESPONSE OF A BEAM TO AN APPLIED FORCE

3.2.1. Equation of motion

In this section, the equations of motion for a straight, non-uniform beam, Fig. (3.1) are formulated. It is assumed that vibration occurs in one of the principal planes of the beam and the effects of rotatory inertia and of transverse shear deformation are negligible.

The significant physical properties of the beam include the flexural stiffness EI(x), and the mass per unit length ρA(x). In addition, the resistance to transverse velocity, c(x) is included to represent the damping mechanisms in the beam. A distributed force p(x) is applied on the beam, which is a function of time f(t) and acts in the z direction. The equations of motion can readily be derived by considering the equilibrium of forces acting on the differential segment of the beam. Fig. (3.1b). Thus summing all the forces acting vertically leads to the first dynamic-equilibrium relationship:

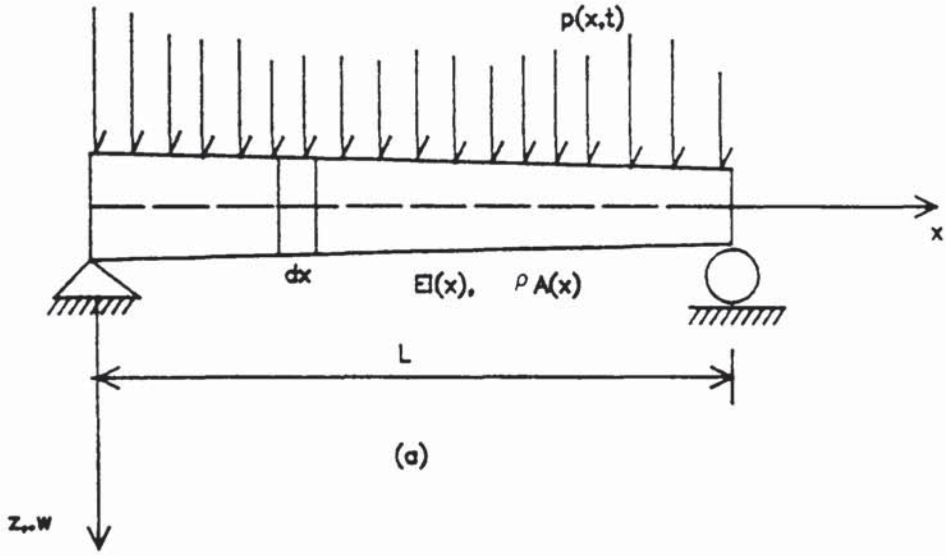
$$\frac{\partial Q}{\partial x} = - p(x)\ f(t) + \rho A \frac{\partial^2 w}{\partial t^2} + c(x) \frac{\partial w}{\partial t} \qquad (3.1)$$

where w(x,t) is the deflection at any section x at time t and Q (x,t) is the shearing force.

The second equilibrium relationship is obtained by summing moments about the centre line of the element (neglecting products of small quantities), gives:

$$\frac{\partial M(x,t)}{\partial x} = Q(x,t) \qquad (3.2)$$

Differentiating Eqn. (3.2) with respect to x and substituting into



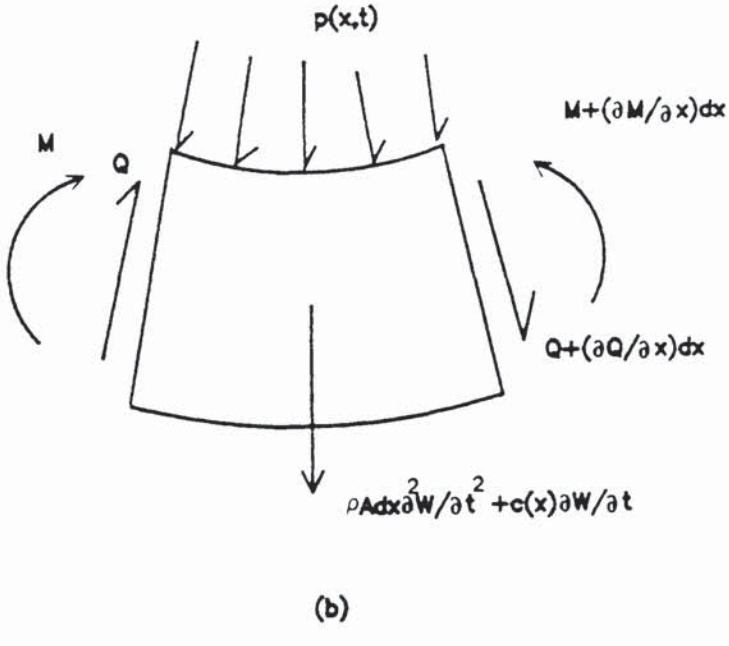


Fig 3.1 Beam subjected to dynamic loading.(a) beam properties and coordinates.(b) forces acting on a differential element.

Equation (3.1) yields after rearrangements:

$$\frac{\partial^2 M(x,t)}{\partial x^2} - \rho A \frac{\partial^2 w(x,t)}{\partial t^2} - c(x) \frac{\partial w}{\partial t} = - p(x) f(t) \qquad (3.3)$$

Finally, introducing the basic moment-curvature relationship of elementary beam theory ($M = -EI\frac{\partial^2 w}{\partial x^2}$) leads to the partial differential equation of motion in terms of w and its derivatives only.

$$\frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 w}{\partial x^2}) + \rho A_{(x)} \frac{\partial^2 w}{\partial t^2} + c(x) \frac{\partial w}{\partial t} = + p(x)f(t) \qquad (3.4)$$

Equation (3.4) is valid for both uniform and non-uniform beams.

3.2.2 Solution of the equations of motion

Free vibration - the general equation for transverse undamped free vibration of a beam may be obtained from equation (3.4) with $p(x,t)=c(x)\frac{\partial w}{\partial t} = 0$, thus:

$$\frac{\partial^2}{\partial x^2}(EI \frac{\partial^2 w}{\partial x^2}) = - \rho A \frac{\partial^2 w}{\partial t^2} \qquad (3.5)$$

For the free vibration, w(x,t) must be a harmonic function of time, i.e.

$$w(x,t) = \hat{w}(x) \quad \text{Sin } (\omega t + \alpha) \qquad (3.6)$$

substituting (3.6) in (3.5) and assuming that EI(x) is constant we have:

$$\frac{d^4\hat{w}}{dx^4} - \frac{\rho A \omega^2}{EI} \hat{w} = 0 \qquad (3.7)$$

The general form of the solution for equation (3.7) becomes:

$$\hat{w} = c_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh x\lambda$$

where $$\lambda = \left(\frac{\rho A \omega^2}{EI}\right)^{\frac{1}{4}} \qquad (3.8)$$

Two conditions expressing the displacement, slope, moment, or shear force will be defined at each end of the beam. These may be used to determine the four constants c_1 to c_4 (to within an arbitrary constant) and will also provide an expression (called the frequency equation) from which the frequency parameter λ can be evaluated. The total response is thus obtained by superimposing the individual mode shapes. That is:

$$w(x,t) = \sum_{i=1}^{\infty} \hat{w}_i \sin(\omega_i t + \alpha) \qquad (3.9)$$

The natural frequencies and the mode shapes for the first few modes of beams with different end conditions have been tabulated in Ref. (27).

The orthogonality conditions for uniform and non-uniform beams with simple and general end conditions are derived in Ref. (28). The following orthogonality relationships exist for a beam with the standard (simply supported, clamped, free) end conditions:

$$\int_0^L \rho A \, \hat{w}_i(x) \, \hat{w}_j(x) = 0 \qquad (a)$$

$$\int_0^L \hat{w}_i(x) \frac{d^2}{dx^2}\left[EI(x) \frac{d^2\hat{w}_j(x)}{dx^2} \right] dx = 0 \qquad (b) \qquad (3.10)$$

$$\int_0^L \hat{w}_i''(x) \, \hat{w}_j''(x) \, EI(x) \, dx = 0 \qquad (c)$$

Also it can be shown that:

$$\int_0^L \hat{w}_i \frac{d^2}{dx^2}\left[EI\frac{d^2\hat{w}_i}{dx^2}\right]dx = \omega_i^2 \int_0^L \rho A\ \hat{w}_i^2\ dx \qquad (3.11)$$

Response: A solution of equation (3.4) will be sought in the form of an infinite series of the normal modes multiplied by the time-dependent generalized coordinates. That is:

$$w(x,t) = \sum_{i=1}^{\infty} \hat{w}_i(x)\ q_i\ (t) \qquad (3.12)$$

substituting for w from (3.12) in (3.4), multiplying by w_j and integrating with respect to x over the length of the beam,

$$\int_0^L \rho A\ \hat{w}_j \sum_i (\hat{w}_i \ddot{q}_i)\ dx + \int_0^L \hat{w}_j \frac{d^2}{dx^2}\left[EI\sum_i \frac{d^2\hat{w}_i}{dx^2}\ q_i\right]\ dx$$

$$+ \int_0^L c(x)\ \ \hat{w}_j \sum_i (\hat{w}_i\ \dot{q}_i)\ dx = \int_0^L p(x)\ \hat{w}_j\ f(t)\ dx \qquad (3.13)$$

Applying orthogonality relations, equations (3.10) and (3.11) together with the assumption of proportional damping leads to:

$$\ddot{q}_i\ (t) + 2\ \xi_i\ \omega_i\ \ \dot{q}_i\ (t) + \omega_i^2\ q_i\ (t) = p_i(x)\ f(t) \qquad (3.14)$$

where

$$p_i(x) = \frac{\int_0^L p(x)\ \hat{w}_i(x)\ dx}{\int_0^L \rho A\,[\,\hat{w}_i(x)\,]^2\ dx} \qquad (3.15)$$

If the variation of the applied force with time is given, the principal coordinates q_i, may be determined from equation (3.14), using Duhammel integral or other direct numerical integration methods. The complete dynamic response is found by substituting in equation (3.12).

3.3 REISSNER PRINCIPLE APPLIED TO FLEXURAL MOTION OF BEAMS

To discuss the Reissner principle for the one dimensional technical theory of beams, consider a beam of length L, subject to a uniform transverse load p(x,t) per unit length, Fig. (3.1a). For this beam, the stress field $\{\sigma\}$ is the moment M, the displacement field $\{u\}$ is the transverse displacement w, and the strain field is the curvature w" ($\varepsilon_x = -zw''$). Hence, the Reissner principle, Equation (2.59), can be written as:

$$\delta \int_{t_1}^{t_2} \left[-\int_0^L \left(\rho A \left(\frac{dw}{dt}\right)^2 + \frac{M^2}{2EI} + M \frac{d^2w}{dx^2}\right) dx - \int_0^L w^t\, p(x,t)\, dx + \frac{dw}{dx} \bar{M} \Big|_0^L \right] dt$$

$$+ \int_{t_1}^{t_2} \left[\int_0^L c(x)\, \dot{w}\, \delta\dot{w}\, dx \right] dt = 0 \qquad (3.16)$$

where $\bar{M}$ represents the prescribed end moments. Taking variations with respect to w and M, and equating to zero yields the stationary conditions for (3.16) as:

(i) The equilibrium equations (3.3)

(ii) The moment-curvature relation ($M = -EI \frac{d^2w}{dx^2}$)

(iii) The appropriate Boundary Conditions on x = o and x = L.

In finite element applications, the variables M and w can be approximated independently, but the latter would have to show continuous slope according to standard 'Integrating' rules (20). It is possible to relax this condition by integrating by parts of the term $M \frac{d^2w}{dx^2}$ in Eqn. (3.16). Thus Reissner's principle may be re-written as:

$$\delta \int_{t_1}^{t_2} \left[\int_0^L (-\rho A\, \dot{w}^2 - \frac{M^2}{2EI} + M'w')\, dx - \int_0^L w^t\, p(x,t)\, dx + \frac{\partial w}{\partial x} (M - \bar{M}) \Big|_0^L \right] dt$$

$$+ \int_{t_1}^{t_2} \int_0^L c(x)\, \dot{w}\, \delta\dot{w}\, dxdt = 0 \qquad (3.17)$$

In the present work, equations (3.16) and (3.17), have been used to develop several beam finite element models with different interpolation functions (see section 6.1). The behaviour of these elements in free and forced vibration problems is studied and numerical examples are presented in Chapter 8.

3.4 BASIC EQUATIONS - THIN PLATE THEORY

A plate of uniform thickness (h) is considered (Figure 3.2), such that its middle surface coincides with the x-y plane and the free surfaces of the plate are the planes $z = \pm \frac{1}{2} h$. If h is small compared to other in plane dimensions, the following assumptions may be made with regard to small deflections of the plate.

(i) The direct stress in the transverse direction σ_z is considered negligible.

(ii) Membrane stresses in the middle plane of the plate are neglected.

(iii) Plane sections that are initially normal to the middle plane remain plane and normal to it. This is equivalent to neglecting the transverse shear effects ($\gamma_{xz} = \gamma_{yz} = 0$).

(iv) Transverse displacement w of any point of the plate is identical to that of the point (below or above it) in the middle surface.

3.4.1 Plate displacement components

From the third and fourth assumptions, the plate displacement field is given as

$$w\,(x,y,z,t) = w\,(x,y,o,t) = W\,(x,y,t)$$

$$u = -z\,\frac{\partial W}{\partial x} \qquad (3.18)$$

$$v = -z\,\frac{\partial W}{\partial y}$$

Therefore the strain in a plane at a distance z from the middle surface is given by the expression

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = -z \begin{bmatrix} \frac{\partial^2 W}{\partial x^2} \\ \frac{\partial^2 W}{\partial y^2} \\ \frac{2\partial^2 W}{\partial x \partial y} \end{bmatrix} \qquad \text{(a)} \qquad (3.19)$$

and $\sigma_z = \gamma_{xz} = \gamma_{yz} = 0$ (b)

3.4.2 Stress-strain relations

With $\sigma_z = 0$, the stress-strain relations for an orthotropic plate with principal directions of orthotropy coinciding with the x and y axes can be written in matrix notations as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 1/E_x & \frac{-\nu_{yx}}{E_y} & 0 \\ \frac{-\nu_{xy}}{E_x} & 1/E_y & 0 \\ 0 & 0 & 1/G_{xy} \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \qquad (3.20)$$

Assuming that the material is isotropic the equations become:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \qquad (3.21)$$

Stresses τ_{xz} and τ_{yz} can only be evaluated from the equilibrium conditions (2.2).

3.4.3 Relations between internal moments, stresses and displacements

Integration of the direct stresses across the thickness of the plate yields stress resultants in the form of direct (M_x, M_y) and twisting (M_{xy}) moments per unit length (Fig. 3.3).

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z\,dz \qquad (3.22)$$

and the shear force intensities (Q_x,Q_y) are given by:

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} dz \qquad (3.23)$$

Using equations (3.19) - (3.22), the following expression may be derived for stress resultants (M_x, M_y, M_{xy}) in terms of curvatures.

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 W}{\partial x^2} \\ -\frac{\partial^2 W}{\partial y^2} \\ -2\frac{\partial^2 W}{\partial x \partial y} \end{bmatrix} \qquad (3.24)$$

where D is the plate bending rigidity; $D = \frac{Eh^3}{12(1-\nu^2)}$

Comparing equations (3.19) and (3.24), the following relation is obtained

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{12z}{h^3} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \qquad (3.25)$$

If a transformation of coordinates (n,s,z) is required, simple equilibrium considerations yields (see Fig. 3.4)

$$\begin{bmatrix} M_s \\ M_n \\ M_{ns} \end{bmatrix} = \begin{bmatrix} \sin^2\alpha & \cos^2\alpha & -\sin 2\alpha \\ \cos^2\alpha & \sin^2\alpha & \sin 2\alpha \\ -\frac{1}{2}\sin 2\alpha & \frac{1}{2}\sin 2\alpha & \cos 2\alpha \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \qquad (3.26)$$

and

$$Q_n = \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} \qquad (3.27)$$

where α is the angle between outward normal n and the x-axis.

3.4.4 Derivation of the governing differential equations

The governing differential equation of plate flexural motion can be derived by examining, on a differential element, the equilibrium of forces with respect to the vertical direction z and of moments about the x and y axes, respectively. In addition to the applied transverse force per unit area, $p(x,y)\ f(t)$ there is an inertia force ($\rho h \frac{\partial^2 W}{\partial t^2}$) and a damping force ($c \frac{\partial W}{\partial t}$) per unit area acting in the z direction. (Fig. 3.5).

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \qquad (3.28)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p(x,y,t) = \rho h \frac{\partial^2 W}{\partial t^2} + c(x,y) \frac{\partial W}{\partial t}$$

Eliminating Q_x and Q_y from above equations yields

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x,y) f(t) = \rho h \frac{\partial^2 W}{\partial t^2} + c \frac{\partial W}{\partial t} \quad (3.29)$$

Substituting from equation (3.24) into (3.29) gives the equilibrium equation for an element of the plate in terms of W and its derivatives.

$$D\left[\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}\right] + \rho h \frac{\partial^2 W}{\partial t^2} + c \frac{\partial W}{\partial t} = p(x,y) f(t) \qquad (3.30)$$

which may also be written as:

$$D\left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \nabla^2 W\right) + \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \nabla^2 W\right)\right] - p(x,y,t) = -\rho h \ddot{W} - c\dot{W}$$

Comparing this with the last relation in (3.28) results in

$$Q_x = -D \frac{\partial}{\partial x} (\nabla^2 W)$$

$$Q_y = -D \frac{\partial}{\partial y} (\nabla^2 W) \qquad (3.31)$$

For a dynamic problem W(x,y,t) must satisfy equation (3.30) together with the boundary conditions.

3.4.5 Boundary Conditions

To solve the plate equation (3.30) one needs to satisfy the boundary conditions for the given plate problem. Since equation (3.30) is a 4th order differential equation no more than two, either geometrical or mechanical boundary conditions can be imposed at a boundary. The mechanical boundary conditions may consist of the normal moment M_n, the twisting moment M_{ns} and the normal shear force intensity Q_n. Since 3 conditions are too many for the thin plate theory, the twisting moment M_{ns} and the normal shear force intensity Q_n must be reduced into one quantity, the so-called normal effective shear force intensity given by (29) as

$$V_n = Q_n + \frac{\partial M_{ns}}{\partial s} \qquad (3.32)$$

the boundary conditions can thus be imposed as:

either $M_n = \bar{M}_n$ or $\frac{\partial W}{\partial n}$ is prescribed

either $V_n = \bar{V}_n$ or W is prescribed

For a simply-supported boundary

$$M_n = -D\left[\frac{\partial^2 W}{\partial n^2} + \nu \frac{\partial^2 W}{\partial s^2}\right] = 0$$

$$W = 0$$

For a built in boundary

$$\frac{\partial W}{\partial n} = 0$$

$$W = 0$$

and for a free boundary

$$M_n = 0, \qquad V_n = Q_n + \frac{\partial M_{ns}}{\partial s} = 0$$

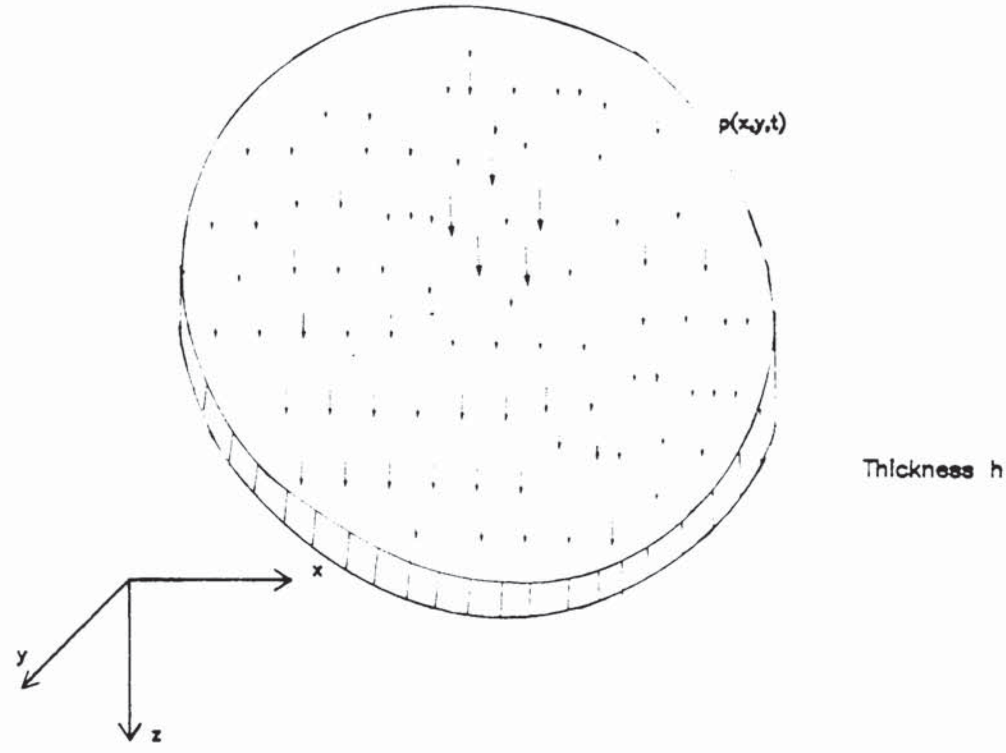


Fig 3.2 Thin plate subjected to distributed loading.

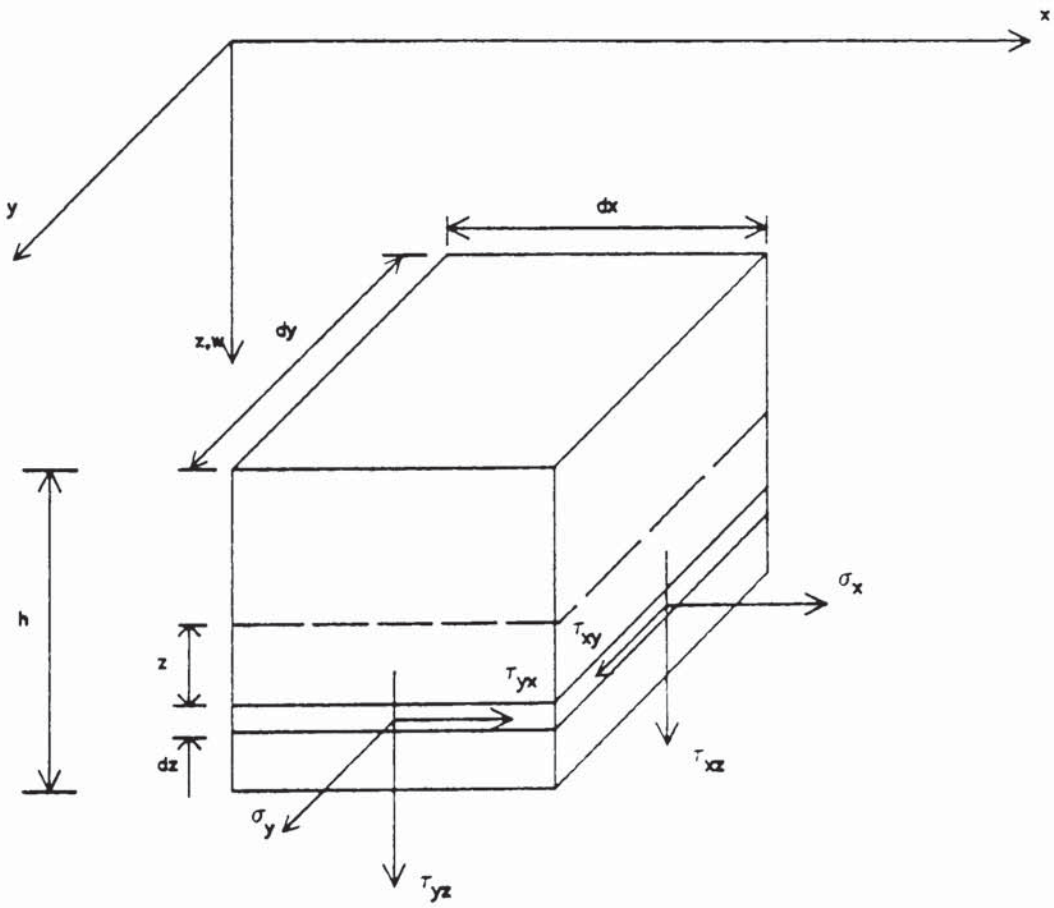
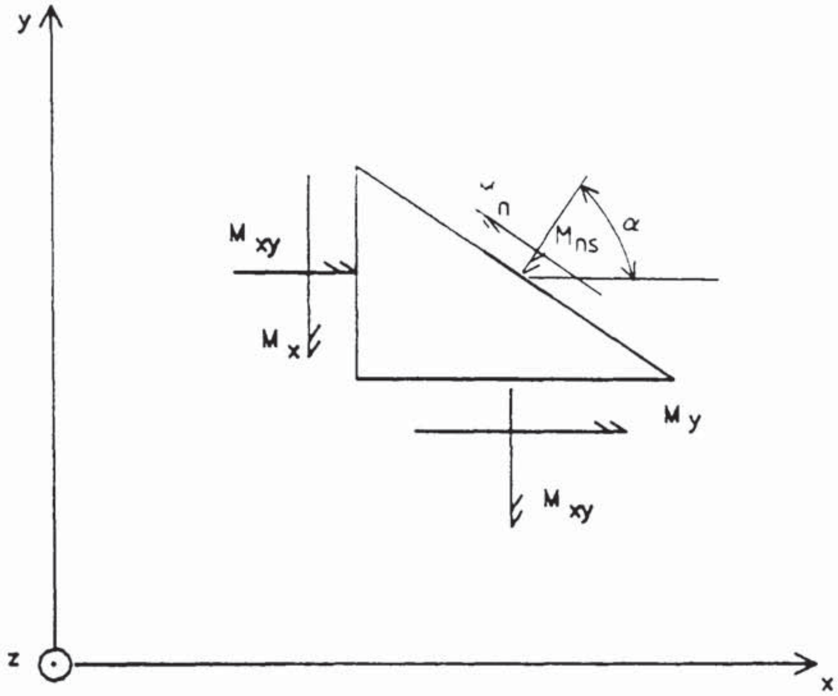


Fig 3.3 Stress components on a plate element.



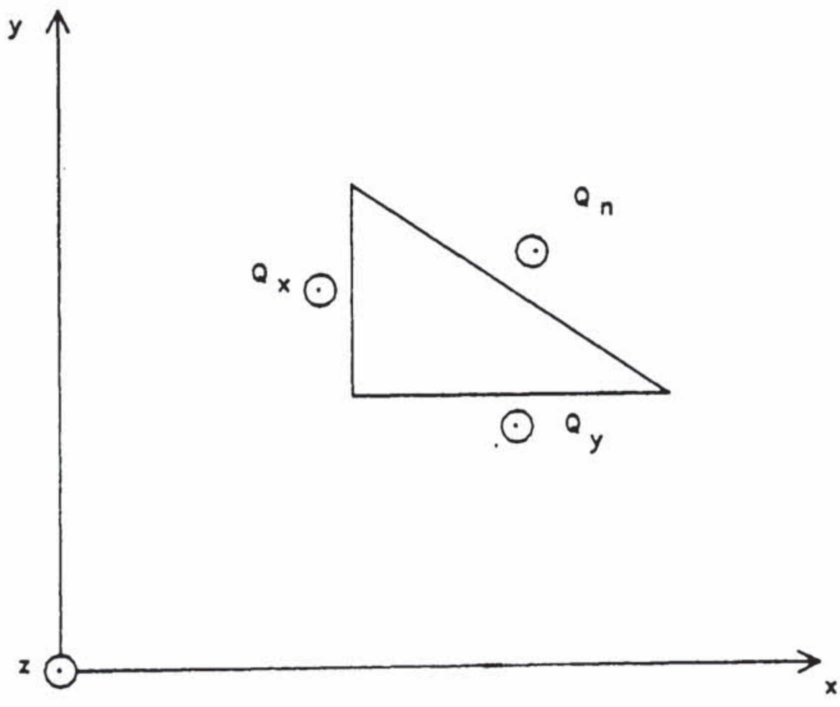


Fig 3.4 Moments and shear notations.

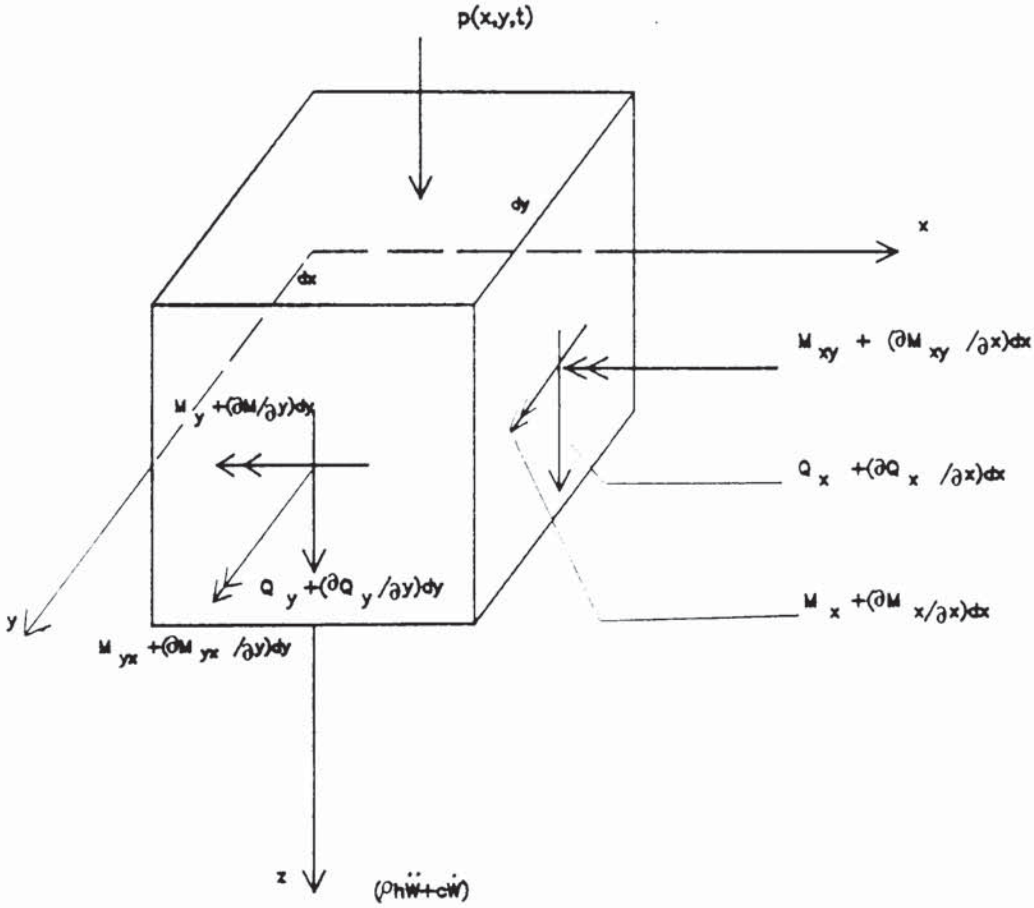


Fig 3.5 Forces and moments on an element of a plate

3.5 HAMILTON'S PRINCIPLE - THIN PLATE THEORY

Hamilton's principle, equation (2.36) may be specialized for the plate bending theory. Let the plate be subject to a distributed lateral load p(x,y,t) per unit area of the middle surface in the direction of the z-axis (Fig. 3.2). On the part of the side boundary S_σ, external forces are prescribed, defined per unit area of the side boundary ($\bar{T}_x$, $\bar{T}_y$, $\bar{T}_z$). On the remaining part of the boundary S_u, geometrical boundary conditions are prescribed. Then, Hamilton's principle for the present problem can be written as follows:

$$\int_{t_1}^{t_2}\left[\int_{S_\sigma}\{\bar{T}\}^t_{nc}\,\{\delta u\}\,dS + \int_A p(x,y,t)\,\delta W dA\right]dt + \int_{t_1}^{t_2}(\delta T - \delta U)\,dt = 0 \qquad (3.33)$$

The kinetic energy (T) is given by

$$T = \tfrac{1}{2}\int_A\int_{-h/2}^{h/2}(\rho\,\dot{W}^2)\,dz\,dx\,dy = \tfrac{1}{2}\int_A \rho h\left(\frac{\partial W}{\partial t}\right)^2 dx\,dy \qquad (3.34)$$

where the effects of rotary inertia are neglected and the strain energy U_{int} becomes

$$U = \frac{E}{2(1-\nu^2)}\int_A\int_{-h/2}^{h/2}\left(\varepsilon_x^2 + 2\nu\,\varepsilon_x\varepsilon_y + \varepsilon_y^2 + \frac{(1-\nu)}{2}\,\gamma_{xy}^2\right)dz\,dx\,dy \qquad (3.35)$$

substituting from equation (3.19) and integrating with respect to z over the plate thickness yields:

$$U = \tfrac{1}{2}\int_A D\left\{\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 W}{\partial x^2}\cdot\frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right]\right\}dx\,dy \qquad (3.36)$$

Using equations (2.10) and (3.18), the integral involving the boundary tractions may be written as:

$$\int_{S_\sigma} (\bar{T}_x \, \delta u + \bar{T}_y \, \delta v + \bar{T}_z \, \delta w) \, dS = - \int\int_{s_\sigma}\int_{-h/2}^{h/2} \Big[(\bar{\sigma}_x \, l + \bar{\tau}_{xy} m) \delta \, (W,x)$$

$$+ \, (\bar{\tau}_{xy} \, l + \bar{\sigma}_y m) \, \delta \, (W,y) \Big] \, zdzds + \int\int_{s_\sigma}\int_{-h/2}^{h/2} (\bar{\tau}_{zx} l + \bar{\tau}_{yz} m) \, \delta W \, dzds \tag{3.37}$$

Integrating over the thickness yields

$$\int_{S_\sigma} \{\bar{T}\}^t \, \{\delta u\} \quad dS = - \int_{s_\sigma} \Big[(\bar{M}_x \, l + \bar{M}_{xy} \, m) \delta \, (W,x) + (\bar{M}_{xy} \, l + \bar{M}_y m) \delta \, (W,y)$$

$$-(Q_x \, l + Q_y m) \delta \, W \Big] \, ds \tag{3.38}$$

The quantities $\delta \, (\frac{\partial W}{\partial x})$ and $\delta \, (\frac{\partial W}{\partial y})$ can be expressed in terms of

$\delta \, (\frac{\partial W}{\partial n})$ and $\delta (\frac{\partial W}{\partial s})$. Thus

$$\delta \, (\frac{\partial W}{\partial x}) = \delta \, (\frac{\partial W}{\partial n}) \, l - \delta \, (\frac{\partial W}{\partial s}) \, m \tag{3.39}$$

$$\delta \, (\frac{\partial W}{\partial y}) = \delta \, (\frac{\partial W}{\partial n}) m + \delta \, (\frac{\partial W}{\partial s}) \, l$$

Substituting from (3.39) into (3.38) yields:

$$\int_{S_\sigma} \{\bar{T}\}^t \, \{\delta u\} \, dS = - \int_{s_\sigma} \Big[\bar{M}_n \, \delta(W,n) - \bar{M}_{ns} \, (\delta W,s) - \bar{Q}_n \, \delta W \Big] ds \tag{3.40}$$

Substituting from (3.34), (3.36) and (3.40) into Equation (3.33), Hamilton's principle is finally reduced to:

$$\delta \int_{t_1}^{t_2} \tfrac{1}{2} \int_A \{\rho h \dot{W}^2 - D\Big((\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2})^2 - 2(1-\nu)\big[\frac{\partial^2 W}{\partial x^2} \cdot \frac{\partial^2 W}{\partial y^2} - (\frac{\partial^2 W}{\partial x \partial y})^2\big]\Big)\} dx\, dy\, dt$$

$$+ \int_{t_1}^{t_2} \int_A P_{(x,y,t)} \; \delta W \, dx\, dy\, dt + \int_{t_1}^{t_2} \int_{S_\sigma} \Big[-\bar{M}_n \delta\,(W,n) + \bar{M}_{ns} \delta\,(W,s)$$

$$+ Q_n \, \delta W \Big] ds\, dt = 0 \qquad (3.41)$$

With the geometrical boundary conditions satisfied a priori, the above principle yields the equation of motion (3.30) and mechanical boundary conditions on S_σ.

3.6 REISSNER'S PRINCIPLE APPLIED TO PLATE BENDING

3.6.1 Introduction

Reissner's principle for static problems, equation (2.54) has been used to develop a system of two-dimensional equations for transverse bending of plates (2). This system of equations is of such a nature that three boundary conditions can and must be prescribed along the edge of the plate. In this section, the dynamic Reissner principle, Eqn. (2.58) will be specialised for an elastic plate where the effects of transverse shear stresses τ_{xz}, τ_{yz} as well as rotary inertia are included. This derivation is similar to the one used in (2). The Principle will then be simplified to correspond to the classical plate theory. The first derivation is referred to as "moderately thick plate" theory.

3.6.2 Reissner's functional for plate bending

As before, a plate of thickness h is considered. The faces of the plate are the planes $z = \pm h/2$ which are taken to be free from tangential traction but under normal pressure p(x,y,t). Thus

$$\tau_{xz} = \tau_{yz} = 0 \quad \text{at } z = \pm h/2, \; (\sigma z)_{z=-h/2} = p(x,y,t) \qquad (3.42)$$
$$(\sigma z)_{z=h/2} = 0$$

For an isotropic material which obeys Hooke's law, the variational principle (2.58) may be written as follows:

$$\delta \int_{t_1}^{t_2} \left\{ \int_A \int_{-h/2}^{h/2} (-T_o - U_o^* + \{\sigma\}^t [L]\{u\}) \, dz \, dx \, dy - \int_A P_{(x,y,t)} W \, dx \, dy \right.$$

$$\left. - \int_{s_\sigma} \int_{-h/2}^{h/2} \{\bar{T}\}_c^t \{u\} \, dz \, ds - \int_{s_u} \int_{-h/2}^{h/2} \{T\}^t (\{u\} - \{\bar{u}\}) \, dz \, ds \right\} dt = 0$$

(3.43)

As in the classical theory of thin plates, it is assumed that the bending stresses are distributed linearly over the plate thickness, i.e.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{12z}{h^3} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \qquad (3.25)$$

Expressions for the transverse shear stresses may be obtained by means of the differential equations of equilibrium which satisfy the conditions that the faces of the plate are free from shear stress, then

$$\begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \frac{3}{2h} \left(1 - \frac{4z^2}{h^2}\right) \begin{bmatrix} Q_x \\ Q_y \end{bmatrix} \qquad (3.44)$$

and $\sigma_z = 0$

For the displacement field, it is assumed that

$$\begin{aligned} u &= z\,\beta_x \\ v &= z\,\beta_y \\ w &= (x,y,z,t) = w\,(x,y,t) = W \end{aligned} \qquad (3.45)$$

where β_x and β_y are "average rotations" of the normal to the middle plane of the plate such that

$$\int_{-h/2}^{h/2} \sigma_x \, u \, dz = M_x \, \beta_x, \quad \int_{-h/2}^{h/2} \sigma_y \, v \, dz = M_y \beta_y \qquad (3.46)$$

and W is a mean transverse deflection with respect to the plate thickness such that

$$\int_{-h/2}^{h/2} \tau_{xz} \, W_{(x,y,z)} \, dz = Q_x \, W \qquad (3.47)$$

For the boundary terms in (3.43) we have

$$\int_{-h/2}^{h/2} \bar{T}_x \, z \, dz = \bar{M}_x \, l + \bar{M}_{xy} . m, \quad \int_{-h/2}^{h/2} \bar{T}_y \, z \, dz = M_{xy} . l + M_y \, m$$

and

$$\int_{-h/2}^{h/2} \bar{T}_z \, dz = Q_x \, l + Q_y \, m \qquad (3.48)$$

Introducting the above assumptions into the functional (3.43), and integrating with respect to z we obtain:

$$\delta \int_{t_1}^{t_2} \left[\int_A \frac{1}{2} \left\{ \frac{\rho h^3}{12} \left[\left(\frac{\partial \beta_x}{\partial t}\right)^2 + \left(\frac{\partial \beta_y}{\partial t}\right)^2 \right] - \rho h \dot{w}^2 - \frac{12}{Eh^3} (M_x^2 + M_y^2 - 2\nu M_x M_y) \right. \right.$$

$$+ 2(1+\nu) M_{xy}^2) - \frac{12}{5hE} (1+\nu) (Q_x^2 + Q_y^2) + 2\left[M_x \frac{\partial \beta_x}{\partial x} + M_y \frac{\partial \beta_y}{\partial y} + M_{xy}\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right)\right.$$

$$\left.\left. + Q_y \left(\beta_y + \frac{\partial W}{\partial y}\right) + Q_x \left(\beta_x + \frac{\partial W}{\partial x}\right)\right] \right\} dx\, dy - \int_A P_{(x,y)} W \, dx\, dy$$

$$- \int_{s_\sigma} (\bar{M}_n \beta_n + \bar{M}_{ns} \beta_s + \bar{Q}_n W) \, ds - \int_{s_u} \left[M_n (\beta_n - \bar{\beta}_n) + M_{ns} (\beta_s - \bar{\beta}_s) \right.$$

$$\left. \left. + Q_n (W - \bar{W}) \right] ds \right] dt = 0 \qquad (3.49)$$

The stationary conditions for the above functional are:

(i) The equations of equilibrium:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \frac{\rho h^3}{12} \ddot{\beta}_x$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = \frac{\rho h^3}{12} \ddot{\beta}_y \qquad (3.50)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p(x,y,t) = \rho h \ddot{W}$$

(ii) Stress-displacement relations:

$$M_x = D \left(\frac{\partial \beta_x}{\partial x} + \nu \frac{\partial \beta_y}{\partial y}\right)$$

$$M_y = D \left(\nu \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}\right)$$

$$M_{xy} = \frac{Gh^3}{12} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) \qquad (3.51)$$

$$Q_x = \frac{5Gh}{6} \left(\beta_x + \frac{\partial W}{\partial x}\right)$$

$$Q_y = \frac{5Gh}{6} \left(\beta_y + \frac{\partial W}{\partial y}\right)$$

(iii) Boundary conditions

Geometrical boundary conditions are

$$\beta_n = \bar{\beta}_n, \quad \beta_s = \bar{\beta}_s, \quad W = \bar{W}, \text{ on } s_u \tag{3.52}$$

and mechanical boundary conditions

$$M_n = \bar{M}_n, \quad M_{ns} = \bar{M}_{ns}, \quad Q_n = \bar{Q}_n \quad \text{on } s_\sigma \tag{3.53}$$

These are the Euler equations corresponding to $\delta\, \pi_R^D = 0$ which govern the behaviour of plates, including the effect of transverse shear deformation and rotatory inertia.

For thin plates, the complementary strain energy due to the stresses σ_z, τ_{xz} and τ_{yz} are assumed negligible, i.e.

$$\frac{12}{5hE} \quad (Q_x^2 + Q_y^2) = 0 \tag{3.54}$$

and the rotations are:

$$\beta_x = -\frac{\partial W}{\partial x} \qquad \beta_y = -\frac{\partial W}{\partial y}$$

$$\beta_n = -\frac{\partial W}{\partial n} \qquad \beta_s = -\frac{\partial W}{\partial s} \tag{3.55}$$

in accordance with the classical assumptions as presented in section (3.4). Using equations (3.54) and (3.55) in the expression for Reissner's principle equation (3.49), we obtain:

$$\delta\int_{t_1}^{t_2}\Big\{\int_A -\Big[\frac{\rho h}{2}\left(\frac{\partial W}{\partial t}\right)^2 + \frac{6}{Eh^3}\left(M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_{xy}^2\right) + M_x\frac{\partial^2 W}{\partial x^2}$$

$$+ M_y\frac{\partial^2 W}{\partial y^2} + M_{xy}\frac{\partial^2 W}{\partial x\partial y}\Big]dx\,dy - \int_A p(x,y,t)\,W\,dx\,dy$$

$$+ \int_{s_\sigma}\left(\bar{M}_n\frac{\partial W}{\partial n} + \bar{M}_{ns}\frac{\partial W}{\partial s} - \bar{Q}_n w\right)ds - \int_{s_u}\Big(M_n\left(\frac{\partial W}{\partial n} - \frac{\partial \bar{W}}{\partial n}\right) + M_{ns}\left(\frac{\partial W}{\partial s} - \frac{\partial \bar{W}}{\partial s}\right)$$

$$+ Q_n(W - \bar{W})\Big)ds\Big\}dt + \int_{t_1}^{t_2}\int_A cW\,\delta\hat{W}\,dx\,dy\,dt = 0 \qquad (3.56)$$

In which the term due to damping is included according to section (2.4.3). The quantities subject to variations in (3.56) are M_x, M_y, M_{xy}, W. The Euler-Lagrange equations can be shown to be the equations of equilibrium (3.29), and the curvature-moment relations (3.24). As boundary conditions we will obtain:

(i) geometrical boundary conditions

$$W = \bar{W}, \qquad \frac{\partial W}{\partial n} = \frac{\partial \bar{W}}{\partial n} \qquad \text{on } s_u \qquad (3.57)$$

(ii) mechanical boundary conditions

$$V_n = \bar{V}_n, \qquad M_n = \bar{M}_n \qquad \text{on } s_\sigma \qquad (3.58)$$

where V_n is the effective shear force intensity. Reissner's principle, (3.56) may be transformed into simpler forms for use with the finite element method (see section 4.3.2).

3.7 METHODS FOR THE SOLUTION OF DYNAMIC PLATE PROBLEMS

3.7.1 Free vibration of thin rectangular plates

The subsequent study of the forced motion of elastic plates will require certain basic relations which are obtained from the study of free vibrations with homogeneous boundary conditions. The familiar equation of motion for free vibration of thin plates is obtained by setting of p = 0 in equation (3.30), then

$$D \left(\nabla^4 W(x,y,t)\right) + \rho \frac{\partial^2 W}{\partial t^2} = 0 \qquad (3.59)$$

where $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic differential operator and the effect of damping is neglected.

Assuming a harmonic motion, we may write

$$W(x,y,t) = \hat{W}(x,y) \sin(\omega t) \qquad (3.60)$$

Here $\hat{W}(x,y)$ is the shape function describing the modes of vibration of the middle plane of the plate and ω is the natural frequency of the vibrations. Substitution of equation (3.60) into equation (3.59) gives:

$$\nabla^4 \hat{w} = \lambda^* \hat{w} \qquad (3.61)$$

where

$$\lambda^* = \frac{\rho h}{D} \cdot \omega^2 \qquad (3.62)$$

Equation (3.61) is an eigenvalue equation whose exact solution will consist of infinite series of frequencies and associated normal modes (eigenvalues and eigenvectors). We shall now briefly illustrate exact and approximate methods to a few situations of the type that we will subsequently treat by finite elements.

a) Exact solution method

Exact solutions to the eigenvalue equation (3.61) exist for very few cases where the shape and boundary conditions of the plate are suitable. In the case of a rectangular plate with simply supported edges (Fig. 3.6), Navier's method (29) is the classical method of analysis. The shape functions $\hat{W}_{(x,y)}$ can be given by double trigonometric series in the form of equation (3.63).

$$\hat{W}\,(x,y) \quad = \quad \mathrm{Sin}\,\frac{m\pi x}{a} \quad \mathrm{Sin}\,\frac{n\pi y}{b} \tag{3.63}$$

This function completely satisfies the conditions at the edges which require that

$$\hat{W} = \frac{\partial^2\hat{W}}{\partial x^2} = 0 \qquad \text{at } x = 0 \quad \text{and } x = a$$

$$\hat{W} = \frac{\partial^2\hat{W}}{\partial y^2} = 0 \qquad \text{at } y = 0 \quad \text{and } y = b \tag{3.64}$$

Substituting equation (3.63) into equation (3.61) yields:

$$D\left[\left(\frac{\pi m}{a}\right)^4 + \frac{2\pi^4 m^2 n^2}{a^2 b^2} + \left(\frac{\pi n}{b}\right)^4\right] = \rho h\,\omega^2 \tag{3.65}$$

Associating ω, with the corresponding integers m and n, equation (3.65) can be represented as:

$$\frac{\rho h}{D}\,\omega_{mn}^2 \quad = \quad \pi^4\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n^2}{b}\right)\right]^2 \tag{3.66}$$

solving for ω_{mn} gives

$$\omega_{mn} \quad = \quad \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}} \tag{3.67}$$

for m, n = 1, 2, 3, ...

ω_{mn} are the natural frequencies (eigenvalues) and the corresponding natural modes (eigenfunctions) are:

$$\hat{W}_{mn} (x,y) = \text{Sin} \frac{m\pi x}{a} \quad \text{Sin} \frac{n\pi y}{b} \tag{3.63}$$

The free vibration of the plate is a superposition of all the modes with proper amplitudes.

$$W(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\text{Sin} \frac{m\pi x}{a} \ \text{Sin} \frac{n\pi y}{b}) (A_{mn} \text{Sin} \ \omega_{mn}t + B_{mn} \text{Cos} \ \omega_{mn}t) \tag{3.68}$$

where the double infinity of constants A_{mn} and B_{mn} are determined to satisfy the initial conditions:

$$\begin{aligned} W(x,y,o) &= \phi \ (x,y) \\ \frac{\partial W(x,y,o)}{\partial t} &= \psi \ (x,y) \end{aligned} \tag{3.69}$$

with ϕ and ψ as known functions . Now making use of the orthogonality property of the eigenfunctions that is

$$\int_A (\hat{W}_{rs}) \ (\hat{W}_{pq}) \ dx \ dy = \frac{ab}{4} \left(\delta_{rs} \cdot \delta_{pq} \right) \tag{3.70}$$

it may be shown that the unknowns A_{mn} and B_{mn} are determined from the following relations

$$A_{mn} = \frac{4}{ab\omega_{mn}} \int_A \psi(x,y) \operatorname{Sin} \frac{m\pi x}{a} \operatorname{Sin} \frac{n\pi y}{b} \, dx\, dy \tag{3.71}$$

and

$$B_{mn} = \frac{4}{ab} \int_A \phi(x,y) \operatorname{Sin} \frac{m\pi x}{a} \operatorname{Sin} \frac{n\pi y}{b} \, dx\, dy$$

Thus the free vibration problem for a rectangular simply supported plate is solved. Levy's type of solution can be applied to rectangular plates which are simply supported along a pair of opposite edges (say at $x = 0$ and at $x = a$) while the other edges ($y = 0$ and $y = b$) are supported in an arbitrary manner (Fig. 3.7). The shape function $\hat{W}(x,y)$ can take the form of equation (3.72) which satisfies the boundary conditions on $x = 0$ and $x = a$

$$\hat{W}(x,y) = Y_{mn}(y) \operatorname{Sin} \left(\frac{m\pi x}{a}\right) \tag{3.72}$$

$Y_{mn}(y)$ is yet to be determined and must satisfy appropriate boundary conditions at $y = 0$ and $y = b$. Substituting equation (3.72) into equation (3.61), a fourth order ordinary differential equation in $Y(y)$ is obtained

$$Y_{mn}^{IV} - 2\frac{m^2\pi^2}{a^2} Y_{mn}'' + \left(\frac{m^4\pi^4}{a^4} - \frac{\rho h\omega_{mn}^2}{D}\right) Y_{mn} = 0 \tag{3.73}$$

The general solution of equation (3.73) is given in reference (30) as:

$$Y_{mn}(y) = c_1 e^{-\alpha y} + c_2 e^{\alpha y} + c_3 \operatorname{Cos} \beta y + c_4 \operatorname{Sin} \beta y \tag{3.74}$$

where

$$\alpha = \sqrt{\omega_{mn} \sqrt{\frac{\rho h}{D}} + \frac{m^2\pi^2}{a^2}} \quad (a)$$

$$(3.75)$$

$$\beta = \sqrt{\omega_{mn} \sqrt{\frac{\rho h}{D}} - \frac{m^2\pi^2}{a^2}} \quad (b)$$

The constants c_1, c_2, c_3 and c_4 can be eliminated by application of boundary conditions at $y = 0$ and $y = b$ to give the frequency equations from which ω is determined. Details will not be given here. The equation has been solved for different plate ratios $^a/_b$ and the results for all possible combinations of clamped, free and simply-supported conditions on $y = 0$ and $y = b$ are given by Leissa (31).

(b) Approximate solution - Rayleigh-Ritz method

When the plate does not have two parallel edges simply supported, no single expression of the form

$$\hat{W}(x,y) = X(x)\ Y(y)$$

satisfies the plate equation and all the boundary conditions. It is therefore necessary to resort to various approximate methods for this purpose, the method of Rayleigh and Ritz and the finite element method have gained increased popularity in the solution of plate problems with complex geometry, loading and boundary conditions. The general method of Rayleigh and Ritz was described in section 2.5.1. In the application to plates, the series approximation for $\hat{W}$ is taken in the form

$$\hat{W}_{(x,y)} = \sum_{i=1}^{I} \sum_{j=1}^{J} X_i(x)\ Y_j(y) \qquad (3.76)$$

The functions $X_i(x)$ and $Y_j(y)$ must be chosen to satisfy any geometric boundary conditions. Leissa (10) has used appropriate beam modal functions for X(x) and Y(y) to determine natural frequencies for several modes for all combinations of clamped, simply supported and free edges. Application of the finite element method to plate problems will be described in the following chapter.

3.7.2 Forced vibration analysis of thin plates

The equation of motion for the forced, damped vibration of a plate is given by equation (3.30).

$$D\nabla^4 W + \rho h \frac{\partial^2 W}{\partial t^2} + c \frac{\partial W}{\partial t} = p(x,y)\ f(t) \tag{3.30}$$

where p(x,y). is the applied force per unit surface area.

The solution of the above equation can be obtained by normal mode superposition method using the normal modes $\hat{W}(x,y)$, of the undamped system (32). Thus the exact and approximate methods of determining natural frequencies and modes can be extended to the determination of response. For a rectangular plate which is simply supported on two parallel edges (x = 0 and x = a) the normal modes are given by equation (3.72),

$$\hat{W}(x,y) = Y_{mn}(y)\ \text{Sin}\ \frac{m\pi x}{a} \tag{3.72}$$

It can be shown (28) that for any combination of homogeneous boundary conditions on y = 0 and y = b,

$$\int_0^b Y_{mn}(y)\ Y_{im}(y)\ dy = 0 \qquad i \neq n \tag{3.77}$$

Also the modal functions $Y_{mn}(y)$ must satisfy the differential equation (3.73). We seek a solution for equation (3.30) in the form

$$W(x,y,t) = \sum_{m}\sum_{n} Y_{mn}(y) \; \mathrm{Sin}\, \frac{m\pi x}{a} \; q_{mn}(t) \tag{3.78}$$

where $q_{mn}(t)$, a principal coordinate, is a function of time. Substituting for W from equation (3.78) in equation (3.30) and using equation (3.73) yields:

$$\sum_{m}\sum_{n} \rho h \, Y_{mn} \, \mathrm{Sin}\, \frac{m\pi x}{a} \left[\ddot{q}_{mn} + \frac{c}{\rho h} \dot{q}_{mn} + \omega_{mn}^2 \, q_{mn} \right] = p(x,y)\, f(t) \tag{3.79}$$

multiplying equation (3.79) by $Y_{ns}(y)$ Sin $\frac{s\pi x}{a}$, integrating over the area of the plate and using equation (3.77) and the orthogonal property of functions Sin $\frac{m\pi x}{a}$, a set of uncoupled equations is obtained: (damping is assumed proportional)

$$\ddot{q}_{mn} + 2\,\xi_{mn}\,\omega_{mn}\,\dot{q}_{mn} + \omega_{mn}^2\, q_{mn} = P_{mn}\; f(t) \tag{3.80}$$

$$m,n = 1,2,\ldots.$$

$$P_{mn} = \int_0^a \int_0^b p(x,y) \;\; Y_{mn}(y) \;\; \mathrm{Sin}\, \frac{m\pi x}{a}\, dx\, dy \tag{3.81}$$

The solution of equation (3.80) may be obtained using the Duhammel integral (section 4.6) and then the complete dynamic response is found by substituting in equation (3.78).

For the response analysis of a plate with general boundary conditions, either The Rayleigh-Ritz (section 2.5.1) or finite element methods (section 2.5.2) may be used.

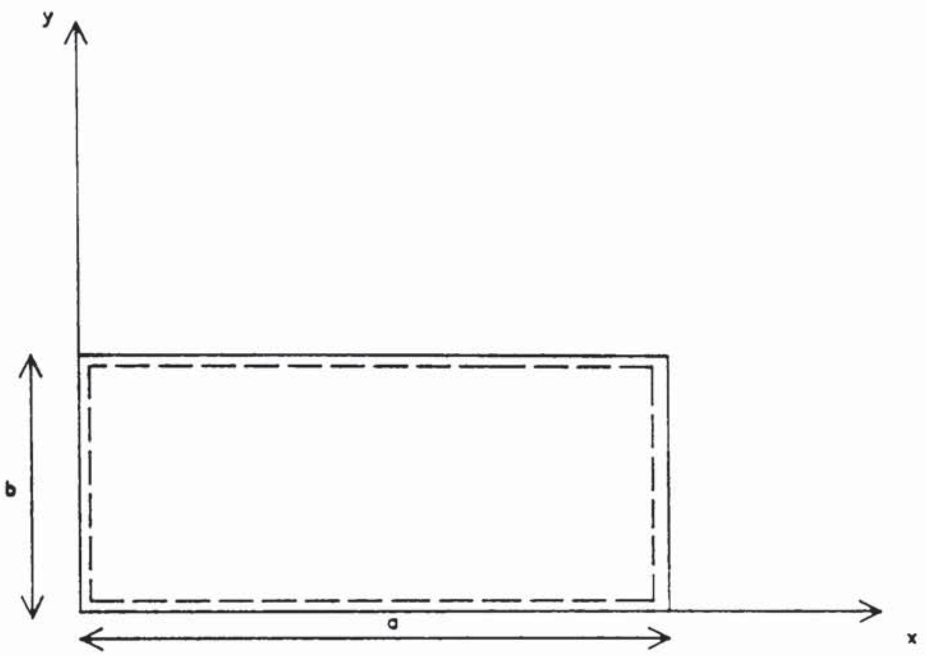


Fig 3.6 Simply supported rectangular plate ,Navier's method

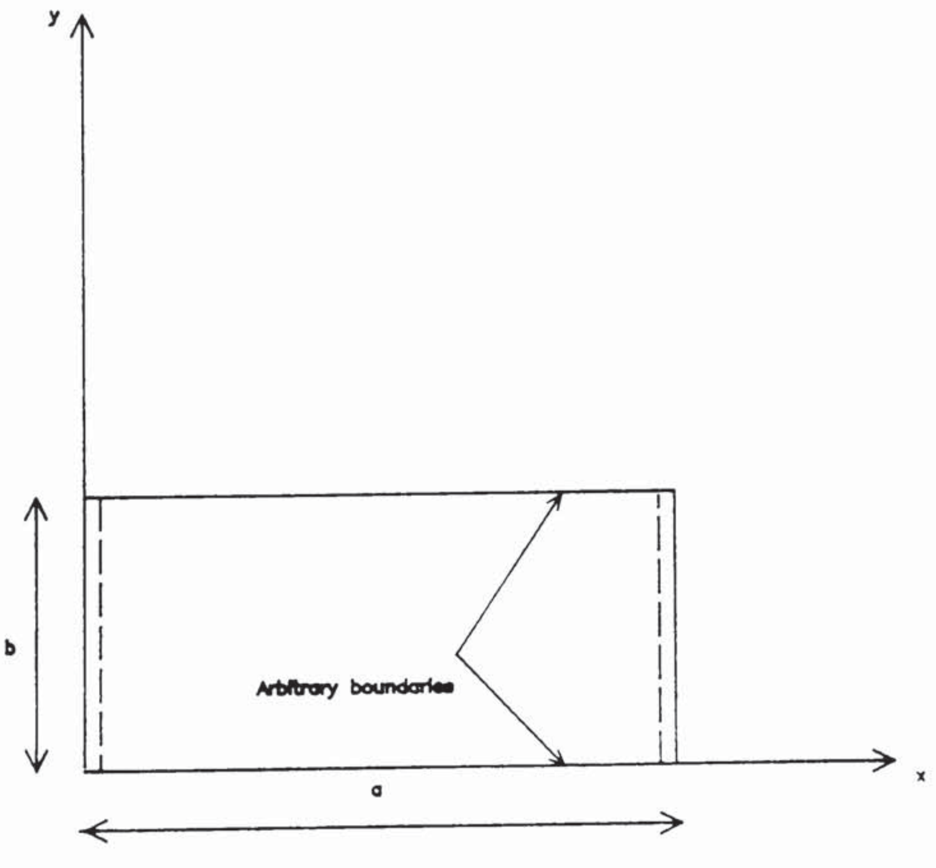


Fig 3.7 Rectangular plate with two opposite edges simply supported , Levy's method

CHAPTER 4

APPLICATIONS OF REISSNER'S PRINCIPLE IN FINITE ELEMENT FORMULATION

4.1 INTRODUCTION

Prior to the work of Reissner "on a variational theorem in elasticity" published in 1950 (2), approximate solutions to elasticity problems were obtained by means of the principle of stationary potential energy and the principle of stationary complementary energy. The principle of minimum potential energy is well adapted to elasticity problems that are formulated in terms of displacements. In this theorem, the stress-displacement relations (2.3) are used as equations of constraint which define the components of stress in terms of appropriate displacement derivatives. The corresponding variational equations (Euler-Lagrange equations) are the equilibrium equations in the interior and on the boundary S_σ of the solid. The complementary energy principle is, on the other hand, suited to problems that are formulated in terms of stresses. In this theorem, the differential equations of equilibrium serve to restrict the class of admissible stress variations and the variational equations are equivalent to the system of stress-displacement relations. As a result of the above constraint conditions introduced in the variational principles, the approximate solutions are such that part of the complete system of differential equations is satisfied exactly while the remaining equations are satisfied only approximately. Reissner's principle may be derived from either the potential energy or the complementary energy principle by introducing the appropriate constraint conditions into the variational statement through the Lagrange multiplier technique. The resulting variational theorem simultaneously provides the differential equations of equilibrium, the stress-displacement relations and the boundary conditions. Thus approximate solutions can be obtained in such a manner that there is no preferential treatment of either one of the two kinds of differential equations which occur in practice. In this section, the current finite element models are briefly reviewed. A new version of Reissner principle is derived which is suitable for finite element analyses of plate and shell type structures. Finally, the essential steps in formulating the mixed element equations are described.

4.2 FINITE ELEMENT MODELS

Variational principles in structural mechanics have acquired a significant practical importance as the basis for numerical methods of analysis. When used in conjunction with finite element techniques, variational principles exhibit comparative advantages pertaining to algebraic simplicity of forming and solving the equations, number of unknowns per nodal point, accuracy of the various types of unknowns, convergence properties, etc. According to a study carried out by Pian and Tong (24) the current finite element models may be divided into four basic types:

- (i) Compatible models (compatibility satisfied, equilibrium violated).
- (ii) Equilibrium models (equilibrium satisfied, compatibility violated).
- (iii) Hybrid models.
- (iv) Mixed models (equilibrium violated, compatibility violated).

The first class contains the compatible displacement model which is derived from minimum potential energy principle. Based on an assumed displacement field continuous over the entire solid, the principle yields a system of equations with the nodal displacements as unknowns. Although the potential energy principle is the predominant approach to the formulation of finite element equations, it is not always the most convenient approach. In many practical situations, it becomes extremely difficult to choose an element displacement field that will satisfy all the conditions of inter-element displacement continuity. Plate and shell elements, for example, require the continuity of both displacement and its derivatives across the element boundaries. No simple displacement functions are capable of satisfying these requirements. A modified potential energy functional may be derived for application to finite element analysis. In the new formulation, the displacement functions are chosen independently for each individual element while inter-element compatibility conditions are accommodated by including Lagrange multiplier terms in the functional. In application to finite element analysis, equilibrating tractions are assumed along the interelement boundaries in addition to the assumed continuous displacement fields in each element. This method is thus called a hybrid-displacement method (33). Both compatible and hybrid displacement models produce better results for displacements than stresses.

The second class contains the equilibrium model (34) which is derived from the principle of minumum complementary energy and is based on an assumed equilibrium stress field within and across the element boundaries. It is customary to use stress functions as primary field variables and the nodal values of such variables are the unknowns of the final system of equations. A dual hybrid method can be formulated, for which compatible displacement functions are assumed along the interelement boundaries in addition to the assumed equilibrating stress field in each element (35). According to the above classification hybrid models fall into the third category. The results from the equilibrium and hybrid-stress models are, as one would expect, more in favour of stresses. The fourth method, derived from the Reissner's principle, presented in section (4.4.2) is called the mixed method (3), (36) with nodal values of both displacements and stresses as unknowns. In mixed models, the field variables should only maintain a degree of continuity such that the functional of the variational problem is defined, i.e. it must be finite. Mixed formulation, in general yields a solution with balanced accuracy in displacements and stresses. It will be seen in the following section that there exists a wide degree of freedom in the application of Reissner's principle to the finite element method.

In the solution of boundary value problems by approximate methods, the continuity requirements placed on the approximating functions depend on the order of the governing differential equations and its variational formulation. Reissner's principle leads directly to mixed formats of the element force-displacement equations. Because the Euler equations of this functional are the more basic equations of elasticity, with lower order derivatives, the continuity requirements on the assumed fields are of lower order than for the conventional variational principles. In the finite element formulation, the functional for the complete system is comprised of the sum of functionals of (n) individual regions (elements) π^j, such that

$$\pi = \sum_{j=1}^{n} \pi^j \qquad j = 1, 2 \ldots, n \qquad (4.1)$$

Thus approximating functions must be such that their derivatives up to the highest order occurring in the corresponding Euler equations are continuous within each discrete element. The admissibility on the inter element boundary conditions may be broadened to the degree that the assumed functions shall only possess continuous derivatives in such a manner that the functional of the variational problem is defined (24). The interelement boundary conditions may be further relaxed by considering the displacement continuity or traction reciprocity conditions* as conditions of constraint that can be included in the variational statement by means of Lagrange multiplier terms as additional variables along the element boundary. General

* That is $\{T\}_b$ $-\{T\}_a$ on S_g where $\{T\}$ are the boundary tractions and a, b denote the elements at the two sides of the boundary.

discussions of this topic have been made by Prager (37), Pian (24) and by Nemat-Nasser (38).

In accordance with equation (4.1), Reissner's Principle (2.54) can be written in a discretized form as

$$\pi R = \sum \Bigg\{ \int_{V_n} \Big[-U_o^* (\sigma) + \{\sigma\}^t[L]\{u\} - \{\bar{F}\}^t \{u\} \Big] dV - \int_{S_{\sigma n}} \{\bar{T}\}^t\{u\} \, dS$$

$$- \int_{S_{u_n}} \{T\}^t(\{u\} - \{\bar{u}\}) \; dS - B_n \Bigg\} \qquad (4.2)$$

where $\{T\} = [1]\{\sigma\}$ represents boundary tractions. V_n indicates the volume of the nth element. For the boundary of the nth element, $S_{\sigma n}$ is the portion over which the surface tractions $\{\bar{T}\}$ are prescribed while over S_{u_n} the displacements $\{u\}$ are prescribed. The term B_n, arises from possible jump functions of the derivatives of $\{u\}$ across the interelement boundaries. For example if the displacements $\{u\}$ are continuous, $B_n = 0$ and if $\{u\}$ are not continuous along S_n of the nth element while the surface tractions are in equilibrium with the tractions of the adjacent element, then

$$B_n = \int_{S_n} \{T\}^t \; \{u\} \; dS \qquad (4.3)$$

The independent variables subject to variations are still $\{\sigma\}$ and $\{u\}$ with subsidiary conditions that $\{T\}$ are in equilibrium along the inter-element boundary, i.e.

$$\{T\}_a = -\{T\}_b \quad \text{on } S_n \qquad (4.4)$$

where (a) and (b) are the elements at the two sides of the boundary. When tractions are not in equilibrium at the two sides of the boundary, equation (4.4) must be introduced as a condition of constraint. The corresponding Lagrange multipliers are the boundary displacements $\{\tilde{u}\}$ which are independent of the displacements $\{u\}$. Thus if along S_n, $\{u\}$ are discontinuous and $\{T\}$ are non-reciprocal, then

$$B_n = \int_{S_n} \{T\}^t \{u\}\, dS - \int_{S_n} \{T\}^t \{\tilde{u}\}\, dS \qquad (4.5)$$

The independent variables subject to variations are $\{\sigma\}$ and $\{u\}$ in each element, and $\{\tilde{u}\}$ along the interelement boundaries. There are still many more versions of the π_R based on the additional variables introduced along the interelement boundaries. These functionals have been studied by Pian and Tong (39).

The functional in Reissner's principle may be transformed to a different form by integrating by parts the second term in the volume integral of equation (4.2). Then

$$\pi_R' = \sum \Bigg\{ \int_{V_n} \left[-U_o^*(\sigma) - ([L]'\{\sigma\})^t \{u\} - \{\bar{F}\}^t \{u\} \right] dV$$
$$- B_n' - \int_{S_{\sigma_n}} \{\bar{T}\}^t \{u\}\, dS - \int_{S_{u_n}} \{T\}^t (\{u\} - \{\bar{u}\})\, dS \Bigg\} \qquad (4.6)$$

where now

$$B'_n = B_n - \int_{S_n + S_{\sigma_n} + S_{u_n}} \{T\}^t \{u\} \, dS \qquad (4.7)$$

$[L]'$ is the differential operator obtained in the process of integration by parts. It is seen that the new version of Reissner's principle imposes some new continuity on the stresses, but relaxes those on the displacements. This version of the Reissner functional has practical importance in application to plate and shell type structures.

4.3.1 Discretized Reissner's Principle - Beam Bending Problems

For application to beam bending problems, Reissner's Principle in the form of equations (3.16) or (3.17) may be directly employed to formulate the element relationships. The approximate shape functions for the displacement w, and the bending moment M_x, must satisfy the necessary interelement continuity conditions. This follows from the requirement that the functional be defined (20). Therefore, when using the variational principle (3.16), it is necessary to ensure the continuity of w and its slope between elements (C1 continuity). On the other hand, the variational principle (3.17) requires the shape functions to satisfy the displacement (w) continuity only (C0 continuity). Thus the latter formulation permits the use of simpler shape functions. The beam element formulation is described in section (6.2).

4.3.2 Discretized Reissner's Principle - Plate Bending Problems

For thin plates, the expression of π_R, that is equivalent to equation (4.2), is

$$\pi_R = \sum_n \Bigg\{ \int_{A_n} -\left[\frac{6}{Eh^3} \left(M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu) M_{xy}^2\right) + M_x \frac{\partial^2 W}{\partial x^2} + M_y \frac{\partial^2 W}{\partial y^2} + M_{xy} \frac{\partial^2 W}{\partial x \partial y}\right] dx\, dy - \int_{A_n} P(x,y)\, W\, dx\, dy - B_n$$

$$- \int_{S_{\sigma_n}} (\bar{Q}_n W - \bar{M}_{ns} \bar{W},_s - \bar{M}_n W,_n)\, ds - \int_{S_{u_n}} \left[Q_n (W-\bar{W}) - M_{ns}(W,_s - \bar{W},_s) - M_n (W,_n - \bar{W},_n)\right] ds \Bigg\} \quad (4.8)$$

where B_n depends on the different continuity conditions along the interelement boundaries. The following expression is used if all displacement continuity requirements are to be relaxed along the boundaries:

$$B_n = \int_{S_n} \left[Q_n (W - \tilde{W}) - M_{n_s} (W,_s - \tilde{W},_s) - M_n (W,_n - \tilde{W},_n) \right] ds \quad (4.9)$$

But when W is continuous and M_n are in equilibrium across the inter-element boundary then

$$B_n = - \int_{S_n} M_n W,_n\, ds \quad (4.10)$$

which accounts for the discontinuity of $W,_n$.

A convenient version for finite element implementation of plate bending problems is one which corresponds to equation (4.6) and (4.10), then

$$\pi'_R = \sum \Bigg\{ \int_{A_n} - \frac{6}{Eh^3} \left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu) M_{xy}^2 \right] dx\, dy$$

$$+ \int_{A_n} \left[\left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}\right) \frac{\partial W}{\partial y} + \left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) \frac{\partial W}{\partial x} \right] dx\, dy$$

$$- \int_{A_n} P(x,y)\, W dx\, dy - \int_{s_n} M_{ns} W,_s\, ds - \int_{s_{\sigma_n}} \Big[\bar{Q}_n W - (\bar{M}_{ns} - M_{ns}) W,_s$$

$$- (\bar{M}_n - M_n) W,_n \Big] ds - \int_{s_{u_n}} \left[Q_n (W - \bar{W}) + M_{ns} \bar{W},_s + M_n \bar{W},_n \right] ds \Bigg\}$$

(4.11)

Which only requires the continuity of W and bending moment components across the element boundaries. Herrmann (3) was the first to use the above principle in the development of a finite element mixed model for static plate bending analysis. The dynamic version of this principle may be simply obtained by including the inertia and time varying forces in the functional. This will be the starting point for the development of mixed dynamic plate elements in this thesis.

4.4 FINITE ELEMENT FORMULATION

4.4.1 General Approach

The finite element method is formulated by approximating the variables in the variational functional in terms of a finite number of unknown parameters. The application of the variational principle then leads to the final matrix equation to be solved. The procedure consists of the following steps:

1) Definition of the finite element mesh. Depending on the problem at hand, the complete region (continuum) is subdivided into one, two or three dimensional sub-regions (finite-elements). The elements are separated by imaginary lines or surfaces inter-connected at certain nodal points. For the two dimensional continuum, the elements may be of triangular, rectangular or general quadrilateral shapes. An improvement over the straight-sided triangular and rectangular elements are those with curved sides which are more easily adaptable to any given geometry.

2. Modelling of unknown variables. The field variables in the variational functional are represented by interpolating functions and generalized displacements and/or stresses at a finite number of nodal points of each element. In most cases, the interpolating functions must be such that the continuity requirements inside and across the element boundaries are satisfied.

3. Formulating the element equations. On the basis of the assumed functions of (2) above, the energy functional is expressed in terms of element generalized coordinates (displacements

and/or stresses). The application of the variational principle then leads to a set of matrix equations for individual elements. The final matrix equations representing the structure as a whole is then synthesized from element matrices.

4. Solution of the resulting system of equations. The overall matrix equation of the structure is solved for the unknown displacements and/or stresses, after imposing the appropriate geometric and/or mechanical boundary conditions. The solution of equations is a standard procedure in matrix algebra. This as well as the generation of element characteristics and synthesis of system characteristics are performed on a digital computer.

4.4.2 Derivation of the mixed element equations

If we choose to satisfy the displacement boundary conditions with our field variables models the Reissner generalized principle (equation 2.59) in matrix notation becomes:

$$\delta \int_{t_1}^{t_2} \left[\int_V \left(-\tfrac{1}{2}\rho \{\dot{u}\}^t \{\dot{u}\} - \tfrac{1}{2}\{\sigma\}^t [D]^{-1}\{\sigma\} + \{\sigma\}^t [L] \{u\} \right) dV - \int_V \{F\}_c^t \{u\} \, dV - \int_{S_\sigma} \{\bar{T}\}_c^t \{u\} \, dS \right] dt + \int_{t_1}^{t_2} \int_V c\{\dot{u}\}^t \{\delta u\} \, dV dt = 0 \tag{4.12}$$

Let the displacement and stress fields within an element be represented independently by:

$$\{u\} = [\Phi]\{\gamma\} \quad (a)$$
$$\{\sigma\} = [\Psi]\{\alpha\} \quad (b) \qquad (4.13)$$

where $\{u\}$ and $\{\sigma\}$ are vectors that contain all possible displacement and stress components, within the element, in the direction of the coordinate axes. $[\Phi]$ and $[\Psi]$ are matrices of position which in general are of different order, and $\{\gamma\}$ and $\{\alpha\}$ are the generalized parameters. The nodal values of the displacements and stresses will be

$$\{u\}_e = \left[A \right]\{\gamma\} , \qquad \{\sigma\}_e = \left[P \right]\{\alpha\} \qquad (4.14)$$

For a two dimensional element such as the one in figure (4.1), the nodal displacements and stresses are:

$$\{u\}_e^t = \left[u_1, v_1, \ldots\ldots, u_4, v_4 \right]$$

$$\{\sigma\}_e^t = \left[\sigma_{x_1}, \sigma_{y_1}, \tau_{xy_1}, \ldots\ldots, \sigma_{x_4}, \sigma_{y_4}, \tau_{xy_4} \right]$$

From equations (4.13) and (4.14), the element displacement and stresses will be:

$$\{u\} = [\Phi][A]^{-1}\{u\}_e = \left[N_u \right]\{u\}_e \quad (a)$$
$$\{\sigma\} = [\Psi][P]^{-1}\{\sigma\}_e = \left[N_\sigma \right]\{\sigma\}_e \quad (b) \qquad (4.15)$$

If the interpolating functions $\left[N_u \right]$ and $\left[N_\sigma \right]$ satisfy the interelement continuity requirements, then equation (4.12) may be utilized to derive the element matrices. Thus substituting the mixed variable model equations (4.15) into equation (4.12), we get for an element:

$$\delta \int_{t_1}^{t_2} \left\{ \int_{V_n} \left(-\tfrac{1}{2}\rho \{\dot{u}\}_e^t [N_u]^t [N_u]\{\dot{u}\}_e - \tfrac{1}{2}\{\sigma\}_e^t [N_\sigma]^t [D]^{-1} [N_\sigma]\{\sigma\}_e \right.\right.$$

$$\left. + \{\sigma\}_e^t [N_\sigma]^t [L][N_u]\{u\}_e \right) dV_n - \int_{V_n} \{F\}^t [N_u]\{u\}_e$$

$$\left. + \int_{S_\sigma} \{\bar{T}\}^t [N_u]\{u\}_e \; dS_n \right\} dt + \int_{t_1}^{t_2} \int_{V_n} c\{\dot{u}\}_e^t [N_u]^t [N_u]\{\delta u\}_e$$

$$dV_n \; dt = 0 \qquad (4.16)$$

Now taking variations with respect to the generalized parameters $\{u\}_e$ and $\{\sigma\}_e$ yields:

$$\int_{t_1}^{t_2} \Big[\{\delta u\}_e^t [m]\{\ddot{u}\}_e - \{\delta\sigma\}_e^t [k_{\sigma\sigma}]\{\sigma\}_e + \{\delta\sigma\}_e^t [k_{\sigma u}]\{u\}_e$$

$$+ \{\delta u\}_e^t [k_{\sigma u}]^t \{\sigma\}_e - \{\delta u\}_e^t \{r\}_e \Big] dt + \int_{t_1}^{t_2} \{\delta u\}_e^t [c]\{\dot{u}\}_e \, dt = 0 \qquad (4.17)$$

collecting terms in $\{\delta u\}_e$ and $\{\delta\sigma\}_e$ and equating to zero yields:

$$\int_{t_1}^{t_2} \{\delta u\}_e^t \left([m]\{\ddot{u}\}_e + [k_{\sigma u}]^t \{\sigma\}_e + [c]\{\dot{u}\}_e - \{r\}_e \right) dt = 0 \quad (a)$$

and (4.18)

$$\int_{t_1}^{t_2} \Big[\{\delta\sigma\}_e^t \left(-[k_{\sigma\sigma}]\{\sigma\}_e + [k_{\sigma u}]\{u\}_e \right) \Big] dt = 0 \qquad (b)$$

therefore:

$$[m]\{\ddot{u}\}_e + [k_{\sigma u}]^t \{\sigma\}_e + [c]\{\dot{u}\}_e = \{r\}_e \qquad (a)$$

and (4.19)

$$[k_{\sigma u}]\{u\}_e - [k_{\sigma\sigma}] \{\sigma\}_e = \{0\}$$

where

$$[k_{\sigma\sigma}] = \int_{V_n} [N_\sigma]^t [D]^{-1} [N_\sigma]\, dV \qquad (a)$$

$$[k_{\sigma u}] = \int_{V_n} [N_\sigma]^t [L][N_u]\, dV \qquad (b)$$

$$[m] = \int_{V_n} \rho [N_u]^t [N_u]\, dV \qquad (c) \qquad (4.20)$$

$$[c] = \int_{V_n} c [N_u]^t [N_u]\, dV \qquad (d)$$

$$\{r\}_e = \int_{V_n} [N_u]^t \{F\}_c\, dV + \int_{S_{\sigma_n}} [N_u]^t \{\bar{T}\}_c\, dS \qquad (e)$$

The mixed element matrices and load vector in equation (4.20) can be assembled for the overall structure, in accordance with the rules of assembly. Thus after introducing the boundary conditions the mixed equations for the assembled structure are:

$$[M]\{\ddot{u}\}_o + [K_{\sigma u}]^t \{\sigma\}_o + [C]\{\dot{u}\}_o = \{R\} \qquad (a)$$

(4.21)

$$[K_{\sigma u}] \{u\}_o - [K_{\sigma\sigma}]\{\sigma\}_o = \{o\} \qquad (b)$$

where $\{u\}_o$ and $\{\sigma\}_o$ are the unknown stress and displacement vectors. For the dynamic case we solve (4.21 b) for $\{\sigma\}_o$ and substitute into (4.21a), thus,

$$\{\sigma\}_o = [K_{\sigma\sigma}]^{-1} [K_{\sigma u}] \{u\}_o \qquad (4.22)$$

and

$$[M]\{\ddot{u}\}_o + [K_{\sigma u}]^t [K_{\sigma\sigma}]^{-1} [K_{\sigma u}]\{u\}_o + [C]\{\dot{u}\}_o = \{R\} \qquad (4.23)$$

or simply

$$[M]\{\ddot{u}\}_o + [K]\{u\}_o + [C]\{\dot{u}\}_o = \{R\}$$

where

$$[K] = [K_{\sigma u}]^t [K_{\sigma\sigma}]^{-1} [K_{\sigma u}] \qquad (4.24)$$

is a full symmetric matrix. The solution of equation (4.23) yields the time history of displacements. The stresses may be obtained by substituting for the displacements into equation (4.22).

In the case of undamped free vibrations, $[C]$ and $\{R\}$ are zero, therefore, equations (4.21) becomes:

$$-\omega^2[M]\{\hat{u}\}_o + [K_{\sigma u}]^t\{\hat{\sigma}\}_o = \{o\} \qquad (a)$$

$$[K_{\sigma u}]\{\hat{u}\}_o - [K_{\sigma\sigma}]\{\hat{\sigma}\}_o = \{o\} \qquad (b) \qquad (4.25)$$

where it is assumed that $\{u\}_o$ and $\{\sigma\}_o$ vary harmonically with time, i.e.

$$\{u\}_o = \{\hat{u}\}_o \sin \omega t \qquad (a)$$

$$\{\sigma\}_o = \{\hat{\sigma}\}_o \sin \omega t \qquad (b) \qquad (4.26)$$

The eigenvalue equation may be obtained by solving (4.25 b) for $\{\hat{\sigma}\}$ and substituting into (4.25 a), thus:

$$[K]\{\hat{u}\}_o - \omega^2 [M] \{\hat{u}\}_o = \{o\} \qquad (4.27)$$

where

$$[K] = [K_{\sigma u}]^t [K_{\sigma\sigma}^{-1}] [K_{\sigma u}] \qquad (4.28)$$

conversely, it is possible to write the eigenvalue equation in terms of stress vector $\{\sigma\}_o$. Solving (4.25 a) for $\{\hat{u}\}_o$ and substituting in (4.25 b) yields:

$$[K^*] \{\hat{\sigma}\}_o - \omega^2 [K_{\sigma\sigma}] \{\hat{\sigma}\}_o = \{o\} \qquad (4.29)$$

where

$$[K^*] = [K_{\sigma u}][M]^{-1}[K_{\sigma u}]^t \qquad (4.30)$$

This represents the eigenvalue equation in terms of stress $\{\hat{\sigma}\}_o$. Either equations (4.27) or (4.29) may be solved to yield the system eigenvalues (ω) and eigenvectors ($\{\hat{u}\}_o$ or $\{\hat{\sigma}\}_o$).

It should be noted that the number of parameters in $\{\hat{u}\}_o$ is in general different from that in $\{\hat{\sigma}\}_o$, (i.e. the number of displacement degrees of freedom is different from the number of stress degrees of freedom). This affects the rank of the matrices involved in equations (4.28) and (4.30). If the number of parameters in $\{\hat{\sigma}\}_o$ exceeds that of $\{\hat{u}\}_o$, matrix $[K^*]$ (Eqn. 4.30) will be deficient in rank and the eigenvalue equation (4.29) yields extra very low or zero eigenvalues which have no physical significance. The converse is also possible (i.e. when the number of parameters in $\{\hat{u}\}_0$ exceeds that of $\{\hat{\sigma}\}_0$, matrix $[K]$ in (4.28) becomes deficient in rank). It is therefore advisable to attempt the solution of the eigenvalue equation with smaller matrices. This ensures that only the true system eigenvalues are obtained, thus overcoming the need to compute unwanted zero eigenvalues.

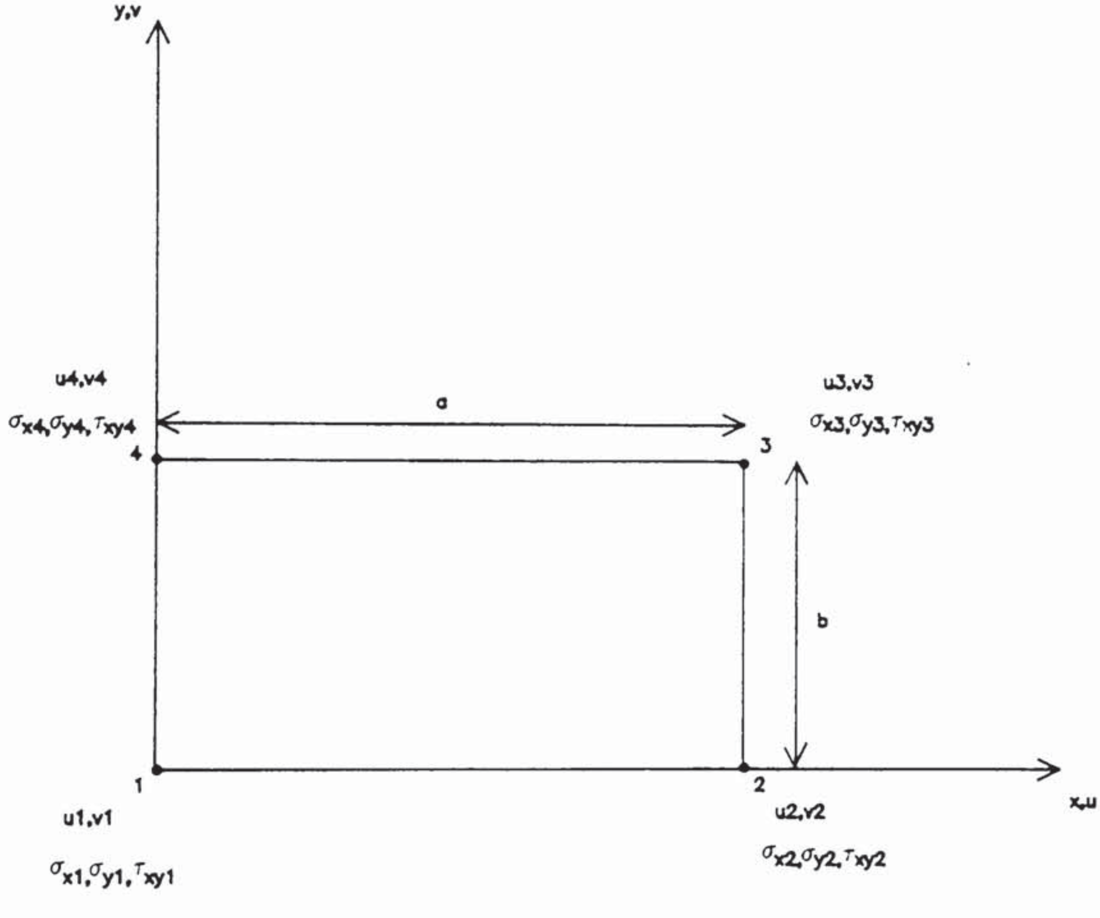


Fig 4.1 A rectangular plane stress/strain finite element

4.5 EVALUATION OF THE DAMPING MATRIX

4.5.1 Importance of Damping

Before discussing methods of solving the equations of motion, it is worth considering the importance of the damping matrix. Damping is the removal of energy from a vibratory system. The energy lost is either transmitted away from the system by some mechanism of radiation or dissipated within the system. Damping is responsible for the eventual decay of free vibrations and is of primary importance in controlling response amplitudes under conditions of steady state resonance excitation. Most structures are lightly damped (1% of the critical damping) and if they are subject to periodic force containing at least one frequency component coinciding with a structural resonance then the damping will be important. However, if the excitation is slightly off-resonance, then the response will be controlled almost entirely by the distribution of mass and stiffness properties.

The energy loss mechanisms in practical structures may be basically divided into external and internal ones. The acoustic radiation, fluid flow resistance and coloumb friction are some examples of models of external energy dissipation sources. Internal friction (damping) in mterials is caused by different physical micromechanisms (40). In metals, for instance, these mechanisms include thermoelasticity, grain boundary viscosity, eddy current effects and to some extent electronic effects. For most non-metallic materials, little is known about such physical mechanisms. However, for one important class of these, namely, polymers and elastomers considerable information has been obtained as the rheological behaviour of such materials may be adequately represented by simple mathematical models (41).

4.5.2 The element damping matrix

The finite element method can be used to generate a damping matrix for a structure where definite damping mechanisms can be recognized. If damping is viscous then equation (4.20 d) yields the so-called consistent damping matrix.

$$[c] = \int_{V_n} c_n [N_u]^t [N_u] \, dV_n \qquad (4.20\ d)$$

Viscous damping coefficients (c) equivalent to a number of different damping mechanisms can be determined by measuring the energy dissipated per cycle (E) in a dashpot undergoing sinusoidal motion $u_o \sin\omega t$. The expression for E is (41)

$$E = \pi c \, \omega \, u_o^2 \qquad (4.31)$$

The overall damping matrix C is constructed from contributions of all the elements. That is

$$\left[C \right] = \sum_n \int_{V_n} c_n [N_u]^t [N_u] \, dV_n \qquad (4.32)$$

In practice, it is very difficult, if not impossible, to determine for general finite element assemblage the element damping parameters, in particular because the damping properties are frequency dependent. For this reason, matrix $\left[C \right]$ is in general not assembled from element damping matrices. Instead, direct methods are availalbe (42, 43) which incorporate the mass and stiffness matrices of the complete assemblage together with experimental results on the amount of damping in order to derive an orthogonal damping matrix for the overall structure. A knowledge of modal damping ratios is thus a prerequisite. Some experimental techniques for identification of modal parameters in lightly damped structures with uncoupled modes are described in Reference (41). Wilson and Penzien (43) have presented a direct method for the numerical evaluation of an orthogonal damping matrix. This method is applicable to lightly damped structures where the effect of modal coupling can be ignored. The final matrix is expressed as the sum of a series of matrices, each of which produces damping in a particular mode. The procedure is described in Appendix A.

4.6 SOLUTION OF DYNAMIC EQUILIBRIUM EQUATIONS

Having established the system characteristics matrices and load vector, we can proceed with the solution of dynamic equations,

$$\left[\ M\ \right]\{\ddot{U}\} + \left[\ K\ \right]\{U\} + \left[\ C\ \right]\{\dot{U}\} = \{R(t)\} \qquad (4.33)$$

{U} is the overal displacement vector and {R(t)} is the overall load vector. The various forms of force inputs are shown in figure (4.2). The analysis of the response of any specified structural system to a prescribed dynamic loading is defined as a deterministic analysis. The non-deterministic analysis, on the other hand, corresponds to the analysis of response to a random dynamic loading. Only the deterministic analysis is considered here.

There are basically two methods of solving these equations: direct step-by-step integration or the mode superposition method (44). In the first method, the response is obtained at a series of sequential time intervals whereas the mode superposition method requires the application of a coordinate transformation prior to the numerical integration. This causes the equations to become uncoupled in the new coordinates.

The choice of which method depends on both the type of force input and the required form of response. It has been found (45) that direct step-by-step integration is most useful when only the initial transient response is required for a small number of loading cases. The normal mode superposition is preferred when there are many loading cases or when the steady state response is required.

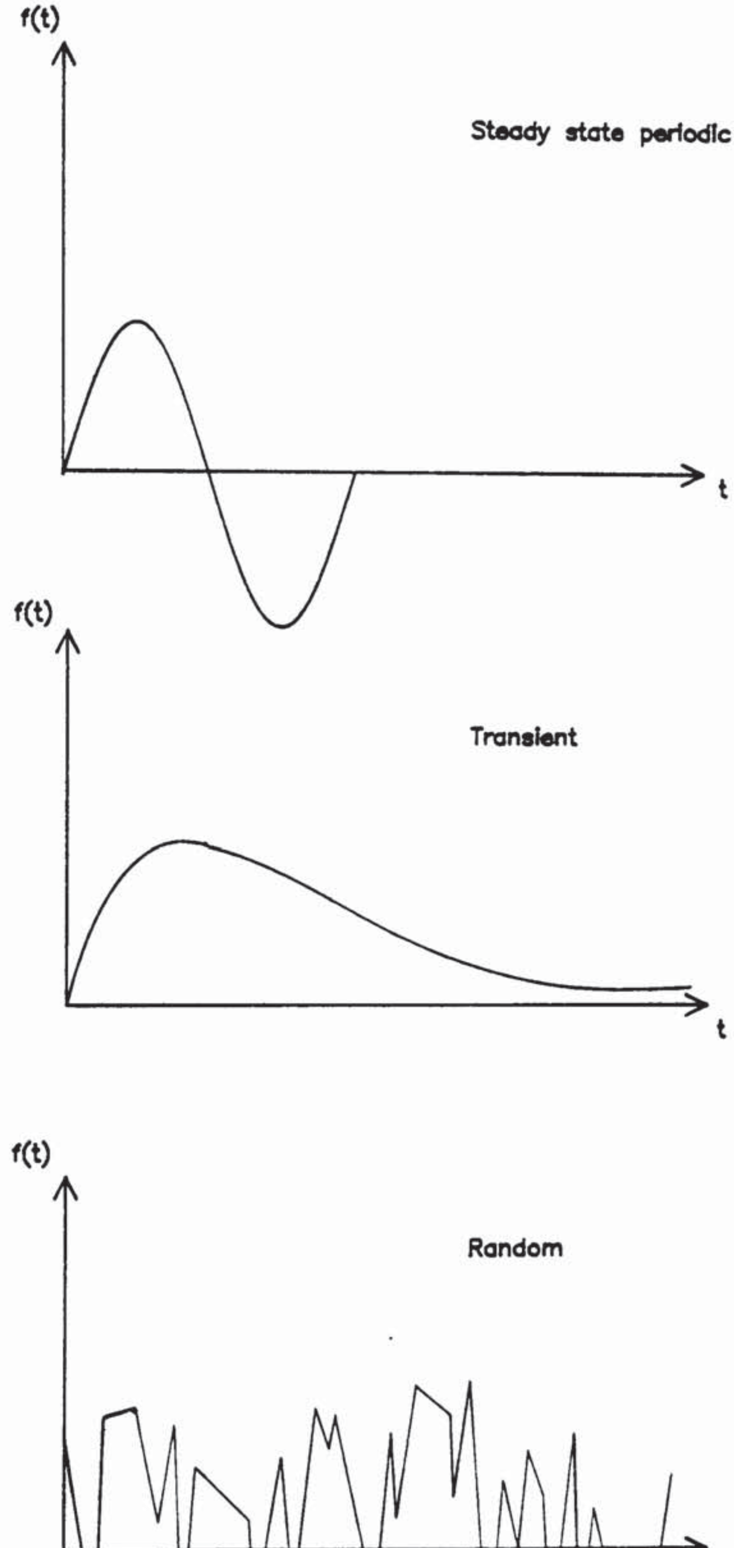


Fig 4.2 Characteristics of typical dynamic loading.

4.6.1 Direct integration method - Wilson θ method

In this method, the equations in (4.33) are integrated using a numerical step-by-step procedure. A polynomial is assumed to represent the variation of displacements, velocities and acceleration within each time interval Δt. In the solution process, it is assumed that the displacement, velocity and acceleration vectors at time 0, denoted by $\{U\}_0$, $\{\dot{U}\}_0$ and $\{\ddot{U}\}_0$, respectively are known. The time span under consideration, T, is subdivided into n equal time intervals Δt (i.e. $\Delta t = \frac{T}{n}$) and the integration scheme is employed to establish an approximate solution at times 0, Δt, $2\Delta t$, ...t, $t + \Delta t$, ...T. Algorithms are derived by assuming that the solutions at time 0, Δt, $2\Delta t$, ..., t are known and that the solution at time $t + \Delta t$ is required next. The calculations performed to obtain the solution at time $t + \Delta t$ are typical for calculating the solution at time Δt later than considered so far, and thus establish the general algorithm which can be used to calculate the solution at all discrete time points. Some commonly used effective step-by-step solution methods are presented in reference (44). An important consideration is the choice of time interval, Δt, which is somewhat arbitrary, but should be less than the time period of the highest natural frequency to enable the complete transient response to be determined. Thus

$$\Delta t \leqslant \Delta t_{cr} = \frac{T_N}{\pi}$$

where T_N is the smallest period of the finite element assemblage and N is the order of the finite element system.

The Wilson θ method

The step-by-step technique of Wilson θ (44) is one of the most popular techniques for the integration in time of the equation of motion of linear structural systems.

One of the main features of the Wilson θ method is that it can be made unconditionally stable, i.e. however large the time step length Δt used in the time integration, the predicted response remains bounded. The cost of a direct integration analysis is directly proportional to the number of time steps required for solution. In the present work, the computing time and computer storage are of primary concern. Thus an unconditional stable scheme provides an attractive method of solution. The effect of time step size, Δt on numerical stability and solution accuracy is demonstrated in Chapter 8.

In Wilson θ method, a linear variation of acceleration from time t to time $t + \theta\,\Delta t$ is assumed, where $\theta \geqslant 1.0$. For unconditional stability, it is required that $\theta \geqslant 1.37$ and usually $\theta = 1.40$ is employed. The acceleration at any time in the interval $(t, t + \theta\,\Delta t)$ is

$$\{\ddot{U}\}_{t+\tau} = \{\ddot{U}\}_t + \frac{\tau}{\theta\Delta t}\left(\{\ddot{U}\}_{t+\theta\Delta t} - \{\ddot{U}\}_t\right) \qquad (4.35)$$

where $0 \leqslant \tau \leqslant \theta\,\Delta t$

Integrating (4.35) yields:

$$\{\dot{U}\}_{t+\tau} = \{\dot{U}\}_t + \{\ddot{U}\}_t\,\tau + \frac{\tau^2}{2\theta\Delta t}\left(\{\ddot{U}\}_{t+\theta\Delta t} - \{\ddot{U}\}_t\right) \qquad (4.36)$$

and

$$\{U\}_{t+\tau} = \{U\}_t + \{\dot{U}\}_t \tau + \tfrac{1}{2} \{\ddot{U}\}_t \tau^2 + \frac{1}{6\theta\Delta t} \tau^3 (\{\ddot{U}\}_{t+\theta\Delta t} - \{\ddot{U}\}_t) \quad (4.37)$$

Using (4.36) and (4.37), we have, at time $t + \theta\Delta t$

$$\{\dot{U}\}_{t+\theta\Delta t} = \{\dot{U}\}_t + \frac{\theta\Delta t}{2} (\{\ddot{U}\}_{t+\theta\Delta t} + \{\ddot{U}\}_t) \quad (4.38)$$

$$\{U\}_{t+\theta\Delta t} = \{U\}_t + \theta\Delta t \{\dot{U}\}_t + \frac{\theta^2\Delta t^2}{6} (\{\ddot{U}\}_{t+\theta\Delta t} + 2 \{\ddot{U}\}_t) \quad (4.39)$$

from which we can solve for $\{\ddot{U}\}_{t+\theta\Delta t}$ and $\{\dot{U}\}_{t+\theta\Delta t}$ in terms of $\{U\}_{t+\theta\Delta t}$

$$\{\ddot{U}\}_{t+\theta\Delta t} = \frac{6}{\theta^2\Delta t^2} (\{U\}_{t+\theta\Delta t} - \{U\}_t) - \frac{6}{\theta\Delta t} \{\dot{U}\}_t - 2\{\ddot{U}\}_t \quad (4.40)$$

and

$$\{\dot{U}\}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t} (\{U\}_{t+\theta\Delta t} - \{U\}_t) - 2 \{\dot{U}\}_t - \frac{\theta\Delta t}{2} \{\ddot{U}\}_t \quad (4.41)$$

Now the equilibrium equations (4.33) are considered at time $t + \theta\Delta t$, i.e.

$$[M]\{\ddot{U}\}_{t+\theta\Delta t} + [C]\{\dot{U}\}_{t+\theta\Delta t} + [K]\{U\}_{t+\theta\Delta t} = \{R\}_{t+\theta\Delta t} \quad (4.42)$$

where

$$\{R\}_{t+\theta\Delta t} = \{R\}_t + \theta (\{R\}_{t+\Delta t} - \{R\}_t) \quad (4.43)$$

substituting (4.40) and (4.41) into (4.42), an equation is obtained from which $\{U\}_{t+\theta\Delta t}$ can be solved. Then substituting $\{U\}_{t+\theta\Delta t}$, into (4.40) we obtain $\{\ddot{U}\}_{t+\theta\Delta t}$, which is used in (4.35), (4.36) and (4.37), all evaluated at $\tau = \Delta t$ to calculate $\{\ddot{U}\}_{t+\theta\Delta t}$, $\{\dot{U}\}_{t+\Delta t}$, and $\{U\}_{t+\Delta t}$.

The complete algorithm used in the integration is given in reference (44).

4.6.2 Mode-superposition method - Duhammel integral

In this method, the response of the general system to prescribed time-dependent forces is obtained as a sum of contributions from individual modes. The system coordinates are transformed to a new set of coordinates in order to obtain new system stiffness, mass and damping matrices which have a smaller bandwidth than the original system matrices. In systems with proportional damping an effective transformation matrix is the modal matrix which contains the eigenvectors of the free vibration equation, i.e.

$$\{U\} = \left[\hat{U}\right]\{q\} \qquad (4.44)$$

If (4.44) is used to transform the variables (displacement, etc.) in equation (4.33) from the original set {U} to a new set {q}, it can be shown that the equations in terms of the transformed variables are uncoupled. (Equation 4.45)

$$\ddot{q}_r + 2\,\zeta_r\omega_r\,\dot{q}_r + \omega_r^2\,q_r = \frac{R_{(r)}}{m_r} \qquad (4.45)$$

$$R_{(r)} = \{\hat{U}\}_r^t\,\{R\} \qquad r = 1,2,\ldots,N$$

Each equation can then be solved as a single degree of freedom problem. The solution to equations (4.45) is obtained by evaluating the Duhammel integral which is given by

$$q_r(t) = \frac{1}{m_r \omega_{D_r}} \int_0^t R_r(\tau)\, e^{-\zeta_r \omega_r (t-\tau)} \sin \omega_{D_r} (t-\tau)\, d\tau$$

$$+ e^{-\zeta_r \omega_r t} \left[\frac{\dot{q}_r(o) + q_r(o)\, \zeta_r \omega_r}{\omega_{D_r}} \sin \omega_{D_r} t + q_r(o) \cos \omega_{D_r} t \right] \qquad (4.46)$$

where $\omega_{D_r} = \omega_r \sqrt{1 - \zeta_r^2}$ (4.47)

and $q_r(o)$, $\dot{q}_r(o)$ represent the initial modal displacement and velocity. These can be obtained from the specified initial displacements $\{U\}_o$ and velocity $\{\dot{U}\}_o$ expressed in the original geometric coordinates as follows for each modal component

$$q_r(o) = \frac{\{\hat{U}\}_r^t [M]\{U\}_o}{m_r} \qquad (a)$$

$$\dot{q}_r(o) = \frac{\{\hat{U}\}_r^t [M]\{\dot{U}\}_o}{m_r} \qquad (b) \qquad (4.48)$$

When the response for each mode $q_r(t)$ has been determined from equation (4.46), the displacements expressed in original coordinates are given by the normal coordinate transformation, equation (4.44).

In summary, the response analysis by mode superposition requires

(i) The solution of the eigenvalue and eigenvectors of the problem in (4.27).

(ii) The solution of the decoupled equilibrium equations in (4.45).

(iii) The superposition of the response in each eigenvector as given by (4.44).

4.6.3 Comparison between mode superposition and direct integration methods

In the last two sections, the methods of direct integration and mode superposition were presented which can be used in the solution of dynamic equilibrium equations of (4.33). The solutions obtained using either procedures are identical, within the numerical errors of the time integration scheme. It can therefore be said, that the choice between mode superposition analysis and direct integration is only one of numerical effectiveness. The effectiveness of a mode superposition procedure depends on the number of modes that must be included in the analysis. It has been shown by experience that for many types of practical loading (e.g. earthquake), only a fraction of the total number of decoupled equations need be considered, in order to obtain a good approximation to the actual response of the system. This means that only the first p equilibrium equations in (4.45) need be used, and that only the lowest p eigenvalues and the corresponding eigenvectors need be solved. The summation in (4.44) is carried out in the first p modes ($p<<N$).

In general the finite element analysis approximates the lowest exact frequency accurately, little or no accuracy can, however, be expected in approximating the higher frequencies and mode shapes. Thus, there is usually little justification for including the response corresponding to higher modes in the analysis. If the lower modes of a finite element system are predicted accurately, little response is calculated in the higher modes and the inclusion of the system high-frequency response will not seriously affect the accuracy of the solution.

From the above discussion it can be concluded that the mode superposition procedure may be more advantageous to direct integration. Significant saving in computational time can be achieved by calculating only the response for the lower modes. As the response corresponding to higher modes is in most instances inaccurate, there is no advantage in computing the higher modes of the system. A direct integration method can also be used to integrate only the first p equations in (4.45) and neglect the high frequency response of the system. This may be achieved by using an unconditionally stable scheme (Wilson θ for example) and selecting an integration time step Δt, which is much larger than the integration step used with a conditionally stable scheme.

In the present work, computer subroutines for the direct integration and mode superposition methods are provided. The subroutines can be called in the main program routine to solve the equilibrium equations.

CHAPTER 5

SURVEY OF LITERATURE ON PLATE ELEMENTS

5. SURVEY OF LITERATURE ON PLATE ELEMENTS

Much effort has been devoted to the development of finite elements for the bending of plates. Most of this effort has been oriented towards the classical poisson-Kirchhoff theory of bending, which neglects the effect of the transverse shear deformation. The Kirchhoff assumption reduces the number of independent variables in the variational statement but introduces higher order derivatives in the formulation of plate elements. The continuity requirements imposed by this theory on "displacement" finite element models has prevented the development of simple and natural elements. Because of this an exceptionally wide variety of alternative formulation has been proposed. A survey by Gallagher (46) shows the extensive amount of literature on the subject. Some of the finite element models which have been developed in the past for the analysis of thin plates are quite briefly summarized, pointing out their advantages and shortcomings.

In the application of the finite element method to thin plate flexure, reliable and accurate formulations are available for assumed displacement (compatible) models obtained by means of potential energy principle. However, the construction of a fully compatible element is rather complicated and involves nodal derivative degrees of freedom of order greater than one. Thus the interelement compatibility inevitably leads to extensive algebraic operations in the formation of the basic element stiffness coefficients and consequently to large storage requirements and computational time. A number of investigators have developed displacement compatible (conforming) models for plate analysis. Bogner, Fox and Schmit (47) developed rectangular elements with 16 degrees of freedom, i.e. with W, $W,_x$, $W,_y$, and $W,_{xy}$ at each corner as generalized coordinates.

In this case, the displacement W and the normal slope $W,_n$ all vary as cubic functions along each edge, hence the interelement compatibility is satisfied.

Butlin and Leckie (48), and Mason (49) also proposed some other conforming rectangular elements. Later Cowper et al. (50) presented a general triangular element suitable for plates with arbitrary boundary shapes. The element has 18 degrees of freedom with the transverse deflection and its first and second derivatives appearing as generalized coordinates at each vertex. The element is reported to be more accurate than the conforming triangular ones previously developed by Bazeley et al. (51) and by Clough and Toucher (52). But a higher order polynomial is used to represent the displacement variation within the element.

There is a very serious drawback in using the conforming elements for practical engineering purposes such as plates with varying thickness, plates with stiffners and plates meeting at angles. The difficulty arises since $W,_{xy}$ and other higher derivatives (strains) appear as nodal degrees of freedom. At a node where there is a change in section or a stiffener then it is wrong to require strain continuity.

The difficulties associated with compatible displacement functions have led to several attempts at ignoring the complete slope continuity while still preserving the other necessary criteria for solution convergence. Therefore non-conforming plate bending elements may be formulated which require simpler displacement fields. Since the "lower bound" solution characteristics of a rigorous minimum potential energy principle is lost, the convergence of such elements is not obvious and should be proved either by the application of the patch test (20) or by comparison with the finite difference algorithms. Successful application of several non-conforming elements have been reported by Bazelay et al. (51). Henshell et al. (9), in particular, developed a family of curvilinear plate bending elements with non-conformable shape functions, for plate vibration and stability tests. Their basic element is a quadrilateral with four nodes but the extensions to this element provide for mid-side nodes making eight and twelve nodes in all and enabling the element to have curved sides. The elements may be used in very general folded plate structures. They concluded that the 8-node element performance was superior to the other two.

By abandoning the Kirchhoff assumption, the interelement compatibility requirement is no longer a serious problem. In the principle of minimum potential energy for plate bending, the rotation angles are used as independent variables in addition to normal delfection (53). But it is known that the so-called thick plate theory does not give reliable solutions for thin plate problems. The difficulty lies on the existence of severe constraints because of the condition of zero transverse shear strain. To capture the behaviour of thin plate theory, Wempner et al. (54) introduced the concept of "discrete Kirchhoff hypothesis" in which the constraint of zero shear strains is imposed at a discrete number of points. The method is effective, but the implementation tends to be somewhat complicated. Some improvements over this have been proposed by Fried (55). On the other hand, reduced integration by Zienkiewicz et al. (56) and by Pawsey and Clough (57) utilizes a lower order of integration and has proved to be very successful in relaxation of constraints on the transverse shear strains.

These elements are based on the assumed-displacement method, but have been shown to be equivalent to elements derived from a mixed formulation (58). An accurate quadrilateral element for thick and thin plates has been developed by Zienkiewicz et al. (56). This element possesses eight node-four corner and four mid-side with the basic three degrees of freedom per node. The transverse displacement and rotation shape functions are selected from 'serendipity' family (20). Two by two Gaussian quadrature is an essential requirement for good performance of the element.

Difficulties in the establishment of admissible displacement fields may be avoided by resorting to complementary or mixed variational principles. Equilibrium elements are based on assumed stress fields and the complementary energy principle. Forces, not displacements, are the primary unknowns of the assembled structure. Displacements are obtained by means of the stress-strain relations and integration of the strain-displacement relations.

The solution for displacements depends on the chosen integration path and in general is not a unique solution. Morely (59) has developed triangular equilibrium elements using the unknown stress resultants (values of stress function) as generalized coordinates. Fraeijs de Veubeke and Sander (50) formulated an equilibrium model which has generalized displacements as unknowns in the final matrix equations.

The specific feature of the mixed model in finite element method was first demonstrated by Herrmann (61), (3). He used the Reissner principle to develop two triangular plate elements. The first element is based on linear variation in W and in the three stress couples, while the second is based on linear variation in W and constant moment distributions. The former has the corner values of W, Mx, My, and Mxy as unknowns, hence it has twelve degrees of freedom, the latter has the corner values of W and edge values of $W,_n$ as unknown, hence it has only six degrees of freedom.

Herrmann's second plate bending element was particularly remarkable for its algebraic simplicity and gave fairly reasonable results in the distribution of moments. The transverse displacement was, however, predicted with less accuracy, leaving room for some improvements. Based on a similar formulation, Visser (4) developed a triangular plate element with six nodes based on a parabolically varying lateral displacement distribution combined with a linearly varying moment distribution, within each element. The element has twelve degrees of freedom and is suitable for thin plate problems only.

Tahiani (62) presented two mixed elements, by considering linear distribution for the transverse displacement and moments, and parabolic variations for the transverse displacement and moments respectively. The concept of area-natural coordinates was used for the first time in a mixed formulation and the shape functions were formed in terms of these natural coordinates.

Mixed formulations for flat plates of rectangular shape were made by Kikuchi and Ando (6). The transverse displacement is assumed to vary linearly. Mx and My are assumed constant within the element and they are expressed in terms of normal moments along the sides of the rectangle. The element has eight degrees of freedom and is compatible with Herrmann triangular element (3).

Bron and Dhatt (63) made a detailed study of the influence of various types of mesh subdivision on the convergence properties of the mixed elements in references (61), (62). They showed that certain types of subdivision for the mixed triangular elements lead to wrong solutions. In an attempt to overcome these shortcomings, Bron and Dhatt (63) proposed general quadrilateral shape elements. The elements were reported to give excellent precision for moments and displacements.

Only a few investigations have been published on mixed models in plate dynamics. Cook (5) developed a triangular thin plate element which was tested in the solution of dynamic and buckling problems. The results, although converging to the correct answers were disappointing due to the slow rate of convergence. Mota Soares (7) developed an isoparametric linear element for moderately thick plates and the results compared favourably with other mixed and displacement models. Reddy and Tsay (8) formulated linear and quadratic isoparametric elements for vibration of thin plates. Each element has three degrees of freedom (the transverse displacement and two normal moments) at each node. Despite the simplicity, the elements yield good accuracy for frequencies.

This literature survey highlights the ability of mixed formulation in generating simple and efficient plate finite elements. The works by Kikuchi and Ando (6) and by Reddy and Tsay (8) show that in general, quadrilateral type elements are more accurate and reliable than triangles. In particular the simple formulation of isoparametric elements prompted us to develop an eight node quadrilateral element for the solution of free and forced plate vibration problems.

CHAPTER 6

TREATMENT OF BEAM & PLATE VIBRATION PROBLEMS BY MIXED FINITE ELEMENT METHOD

6.1 INTRODUCTION

The mixed beam elements properties are briefly described and the types of elements which have been developed for the solution of free and forced vibration of beams are illustrated. Then free and forced vibration problems of thin plates are treated by means of mixed finite element technique. Reissner principle (4.11) is used which does not require the continuity of slope across element boundaries. Based on this theorem, an isoparametric quadrilateral element with 8-nodes is developed which is applicable to thin plate theory only. The geometric, deflection and moment fields are expressed as quadratic functions of position. Mixed element matrices are evaluated by means of numerical integration in which the Gauss quadrature rule is employed. Models based on this element were used to calculate the natural frequencies and modes of vibration, and the transient displacements and moments in plate type structures.

Two computer programs are developed as described in Chapter 7, which incorporate the 8-node quadrilateral element presented in this section. Examples of results will be given in Chapter 8 to show the order to accuracy which can be achieved compared with other types of elements.

6.2 DERIVATION OF THE MIXED BEAM ELEMENT PROPERTIES

Reissner's principle, equation (3.16) for application to dynamic beam problems was developed in section (3.3). We also presented the modified version of this principle, equation (3.17). This version imposes CO continuity requirements on the fields of bending moment and deflection. These principles can be directly incorporated in finite element formulation of beam bending problems. Several such elements have been developed in this work which are shown in Table (6.1), with the corresponding shape functions. The mixed element matrices for one of these elements will be derived in here. Other elements may be formulated in a similar manner.

6.2.1 Mixed finite element properties

Let us divide the beam into finite elements, for the eth element: (from equation 3.17)

$$(\delta\pi_R^D)_e = \delta \int_{t_1}^{t_2} \left[\int_0^1 \left(-\rho A\dot{w}^2 - \frac{M_x^2}{2EI} + M'w' \right) dx - \int_0^1 w^t \, p(x,t) \, dx \right] dt$$

$$+ \int_{t_1}^{t_2} \left[\int_0^1 c(x) \, \dot{w} \, \delta w \, dx \right] dt = 0 \qquad (6.1)$$

A natural coordinate, ξ, is assumed within the element (Fig. 6.1) such that

$$x = \left[\tfrac{1}{2}(1-\xi) \quad \tfrac{1}{2}(1+\xi) \right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad (6.2)$$

where x_1, x_2 are the nodal coordinates at node 1 and 2, and $\xi = -1$, $\xi = +1$ respectively at node 1 and node 2.

Solving equation (6.2) for ξ and differentiating with respect to x yields:

$$\frac{d}{dx} = \frac{d\xi}{dx} \frac{d}{d\xi} = \frac{2}{T} \frac{d}{d\xi} \qquad (6.3)$$

hence $$dx = \frac{1}{2}\, d\xi \qquad (6.4)$$

As an example, assume a parabolic variations for M_x and w within the element, then,

$$W_x = \left[\tfrac{1}{2}\xi(\xi-1) \quad \tfrac{1}{2}\xi(\xi+1) \quad (1-\xi^2) \right] \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} \qquad (6.5)$$

$$M_x = \left[\tfrac{1}{2}\xi(\xi-1) \quad \tfrac{1}{2}\xi(\xi+1) \quad (1-\xi^2) \right] \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} \qquad (6.6)$$

or $(M_x, w) = [N](\{w\}^e, \{M\}^e)$ where $\{w\}^e$ and $\{M\}^e$ are nodal values of deflection and bending moments respectively. Hence using equation (6.3)

$$\frac{dM_x}{dx} = \frac{2}{T} \cdot \frac{dM_x}{d\xi} = \frac{1}{T} \left[(2\xi-1) \quad (2\xi+1) \quad -2\xi \right] \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} \qquad (6.7)$$

i.e. $$\frac{dM}{dx} = \left[B \right] \{M\}_e$$

and

$$\frac{dw}{dx} = \frac{2}{T} \frac{dw}{d\xi} = \left[B \right] \{w\}_e \qquad (6.8)$$

Substituting relations (6.5) to (6.8) into the Reissner equation (6.1) yields

$$\delta \int_{t_1}^{t_2} \Big[-\tfrac{1}{2}\{\dot{w}\}_e^t [m]\{\dot{w}\}_e - \tfrac{1}{2}\{M\}_e^t [g]\{M\}_e + \{M\}_e^t [h]\{w\}_e$$

$$- \{w\}_e^t \{r\} \Big] dt + \int_{t_1}^{t_2} \{\dot{w}\}_e^t [c]\{\delta w\}_e \, dt = 0 \qquad (6.9)$$

The corresponding matrices are then evaluated from:

$$[g] = \int_{-1}^{1} [N]^t \frac{1}{EI} [N] \frac{1}{2} d\xi \qquad (a)$$

$$[h] = \int_{-1}^{1} [B]^t [B] \frac{1}{2} d\xi \qquad (b)$$

$$[m] = \int_{-1}^{1} \rho A [N]^t [N] \frac{1}{2} d\xi \qquad (c) \qquad (6.10)$$

$$[c] = \int_{-1}^{1} c [N]^t [N] \frac{1}{2} d\xi \qquad (d)$$

$$\{r\} = \int_{-1}^{1} [N]^t p(x,t) \frac{1}{2} d\xi \qquad (e)$$

Variations of Reissner's principle then yields the mixed governing equations (4.19).

The behaviour of the beam elements in connection with free and forced vibration problems is investigated in Chapter 8.

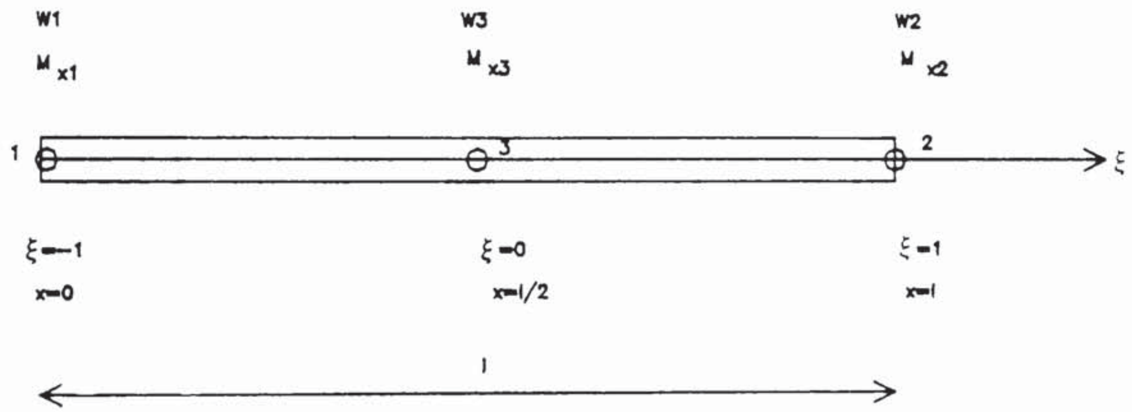


Fig 6.1 Beam finite element(MB5)

Table 6.1 Mixed beam elements,(C1 and C0 continuous elements)

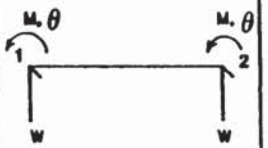
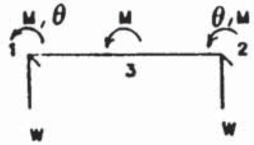
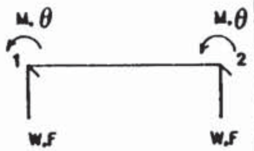
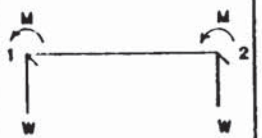
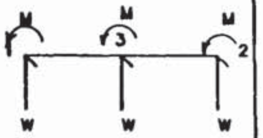
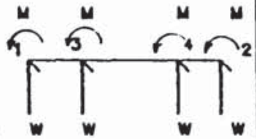
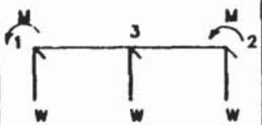
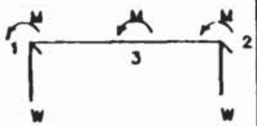
Type of element	Symbol	Displacement/Stress Interpolations $0 \le x \le l$, $-1 \le \xi \le +1$	Displacement freedoms	Stress freedoms
	MB1 (C1)	$[N_w] = [1-3(x^2/l^2)+2(x^3/l^3),\ x-2(x^2/l)+x^3/l^2,\ 3(x^2/l^2)-2(x^3/l^3),\ -x^2/l+x^3/l^2]$ $[N_m] = [1-x/l, x/l]$	W_x , $\dot{\theta}_x$	M_x
	MB2 (C1)	$[N_w] = [1-3(x^2/l^2)+2(x^3/l^3),\ x-2(x^2/l)+x^3/l^2,\ 3(x^2/l^2)-2(x^3/l^3),\ -x^2/l+x^3/l^2]$ $[N_m] = [(2x-l)(x-l)/l^2,\ 2(x^2/l^2)-x/l,\ (4xl-4x^2)/l^2]$	W_x , θ_x	M_x
	MB3 (C1)	$[N_w] = [N_m] =$ $[1-3(x^2/l^2)+2(x^3/l^3),\ x-2(x^2/l)+x^3/l^2,\ 3(x^2/l^2)-2(x^3/l^3),\ -x^2/l+x^3/l^2]$	W_x , θ_x	M_x , F_x
	MB4 (C0)	$[N_w] = [N_m] =$ $[1/2(1-\xi), 1/2(1+\xi)]$	W_x	M_x

Table 6.1 Continued..

Type of element	Symbol	Displacement/Stress Interpolations	Displacement freedoms	Stress freedoms
	MB5 (C0)	$[N_w] = [N_m] =$ $[1/2(\xi^2 - \xi), 1/2(\xi^2 + \xi), (1 - \xi^2)]$	W_x	M_x
	MB6 (C0)	$[N_w] = [N_m] = [N1, N2, N3, N4]$ $N1 = 1/2(1-\xi) + 9/16(-\xi^3 + \xi^2 + \xi - 1)$ $N2 = 1/2(1+\xi) + 9/16(\xi^3 + \xi^2 - \xi - 1)$ $N3 = 9/16(3\xi^3 - \xi^2 - 3\xi + 1)$ $N4 = 1/16(-27\xi^3 - 9\xi^2 + 27\xi + 9)$	W_x	M_x
	MB7 (C0)	$[N_w] = [1/2(\xi^2 - \xi), 1/2(\xi^2 + \xi), (1 - \xi^2)]$ $[N_m] = [1/2(1-\xi), 1/2(1+\xi)]$	W_x	M_x
	MB8 (C0)	$[N_w] = [1/2(1-\xi), 1/2(1+\xi)]$ $[N_m] = [1/2(\xi^2 - \xi), 1/2(\xi^2 + \xi), (1 - \xi^2)]$	W_x	M_x

6.3 COMPUTER IMPLEMENTATION OF THE MIXED BEAM ELEMENTS

The series of programs written for the study of beam elements can be divided into two groups, on the basis of their functions, these being free vibration programs and response analysis programs. The former produces the eigenvalues and eigenvectors of the undamped free vibration problem. The second group of programs performs the response analysis and outputs the time history plots of the displacements and bending moments.

An important consideration in using the one-dimensional beam elements is the similarity between the calculation of different elements. For this reason and because of the familiarity and ease of formulation of one-dimensional elements, the related programs are not described in detail. However, in Appendix C, the computer listing for the forced vibration of element MB5 (defined in Table 6.1) is provided. It is believed that by showing the actual computer implementation of this element, the implementing of other beam elements is self explanatory. The input and output (I/O) variables and the flow of the program are documented within this listing. The package of mixed beam elements includes the programs VREIS1 to VREIS8 for elements MB1 to MB8 which perform the free vibration tests and the programs MBRSP1 to MBRSP5 which perform the forced vibration tests for elements MB1 to MB5.

6.4 MIXED FINITE ELEMENT FORMULATION - THIN PLATES

Reissner's principle applied to thin plate vibration theory may be derived from equation (2.59). Assuming that the prescribed normal moments, twisting moment and transverse deflection are satisfied, that is:

$$M_n = \bar{M}_n, \quad M_{ns} = \bar{M}_{ns} \quad \text{on } (s_\sigma)_e \tag{6.11}$$

and

$$W = \bar{W} \quad \text{on } (s_u)_e$$

We will obtain the following expression for Reissner's principle:

$$(\delta\pi_R^D)_e = \delta \int_{t_1}^{t_2} \Big[-\tfrac{1}{2} \int_{A_e} \rho h \ \dot{W}^2 \, dA - \tfrac{1}{2} \int_{A_e} \{M\}^t [D]^{-1} \{M\} \ dA$$

$$+ \int_{A_e} \{Q\}^t \{W'\} \, dA - \int_{A} p(x,y,t) \ W dA - \int_{s_e} M_{ns} \frac{\partial W}{\partial s} \, ds \Big] dt$$

$$+ \int_{t_1}^{t_2} \Big(\int_{A} c\dot{W}^t \, \delta W dA \Big) \, dt = 0 \tag{6.12}$$

in which

$$\{M\} = \begin{bmatrix} M_x & M_y & M_{xy} \end{bmatrix}^t$$

$$\{Q\} = \begin{bmatrix} Q_x & Q_y \end{bmatrix}^t = \left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) \quad \left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}\right) \right]^t$$

$$\{W'\} = \left[\frac{\partial W}{\partial x} \quad \frac{\partial W}{\partial y} \right]^t$$

and the homogeneous natural boundary conditions are:

$$\frac{\partial W}{\partial n} = 0 \quad \text{on } (s_u)_e$$
$$\bar{V}_n = 0 \quad \text{on } (s_\sigma)_e \qquad (6.13)$$

The surface integrals are evaluated over the entire area of the element and the line integral is evaluated (in an anti-clockwise direction) around each element boundary s_e.

Now consider a general thin plate divided into an arbitrary grid of finite elements (fig. 6.2). The transverse displacement W and the moments $\{M\} = \begin{bmatrix} M_x & M_y & M_{xy} \end{bmatrix}^t$ may be independently assumed within each element by:

$$W = \begin{bmatrix} N_W \end{bmatrix} \{W\}_e \quad (a)$$
$$\{M\} = \begin{bmatrix} N_M \end{bmatrix} \{M\}_e \quad (b) \qquad (6.14)$$

$[N_W]$ and $[N_M]$ are the element displacement and bending moment shape functions respectively. For the present formulation, the trial functions should be at least linear in x and y. The element nodal parameters are given by:

$$\{W\}_e = \left[W_1, W_2, \ldots, W_n \right]^t \qquad \text{(a)}$$

(6.15)

$$\{M\}_e = \left[M_{x_1} \; M_{y_1} \; M_{xy_1} \; , \; M_{x_2} \; M_{y_2} \; M_{xy_2}, \; \ldots, \; M_{x_m} \; M_{y_m} \; M_{xy_m} \right]^t \qquad \text{(b)}$$

where n and m depend on the order of shape functions (trial functions). Note that independent approximations for the displacement and moments are used. From equation (6.14), the slopes within the element are given by:

$$\{W'\} = \left[N'_W \right] \{W\}_e \tag{6.16}$$

and on the boundary s_n,

$$\frac{\partial W}{\partial s} = \left[L_W \right] \{W'\} = \left[L_W \right] \left[N'_W \right] \{W\}_e = \left[Y \right] \{W\}_e \tag{6.17}$$

Shear force intensities are derived by differentiating (6.14b), in the interior.

$$\{Q\} = \left[N'_W \right] \{M\}_e \tag{6.18}$$

and on the boundary

$$M_{ns} = \left[L \right] \left[N_M \right] \{M\}_e = \left[L_{ns} \right] \{M\}_e \tag{6.19}$$

$\left[L \right]$ and $\left[L_W \right]$ are the direction cosine matrices.

Upon substitution of the above derived equations into equation (6.12), we obtain

$$(\delta\pi_R^D) = \delta \int_{t_1}^{t_2} \Big(-\tfrac{1}{2} \{\dot{W}\}_e^t [m] \{\dot{W}\}_e - \tfrac{1}{2} \{M\}_e^t [g]\{M\}_e$$

$$+ \{M\}_e^t [h] \{W\}_e + \{W\}_e^t \{r\}\Big) \, dt + \int_{t_1}^{t_2} \{\dot{W}\}_e^t [c] \{\delta W\}_e \, dt = 0 \qquad (6.20)$$

where:

$$[g] = \int_{A_e} [N_M]^t [D]^{-1} [N_M] \, dA \qquad (a)$$

$$[h] = \int_{A_e} [N'_M]^t [N'_W] \, dA + \int_{s_n} [L_{ns}]^t [Y] \, ds \qquad (b) \qquad (6.21)$$

$$[m] = \int_{A_e} \rho h \, [N_W]^t [N_W] \, dA \qquad (c)$$

$$[c] = \int_{A_e} c [N_W]^t [N_W] \, dA \qquad (d)$$

$$\{r\} = \int_{A_e} [N_W]^t \, p(x,y,t) \, dA_e \qquad (e)$$

$[m]$ and $[c]$ represent the consistent mass and damping matrices and $\{r\}$ is the vector of equivalent nodal forces for element (e). In order to obtain a consistent set of nodal forces corresponding to a general distributed load, the following assumptions may be made.

$$p(x,y) = \begin{bmatrix} N_p \end{bmatrix} \{p\}_e \tag{6.22}$$

in which $\begin{bmatrix} N_p \end{bmatrix}$ contains the assumed functions and $\{p\}_e$ are the nodal load intensities. Thus equation (6.21e) may be re-written as:

$$\{r\} = \left(\int_{A_e} \begin{bmatrix} N_W \end{bmatrix}^t \begin{bmatrix} N_p \end{bmatrix} dA \right) \{p\}_e \tag{6.23}$$

Variation of $(\pi_R^D)_e$ with respect to $\{M\}_e$ and $\{W\}_e$, in succession, yields

$$\begin{aligned} -\begin{bmatrix} g \end{bmatrix}\{M\}_e + \begin{bmatrix} h \end{bmatrix}\{W\}_e &= 0 \\ \begin{bmatrix} h \end{bmatrix}^t \{M\}_e + \begin{bmatrix} c \end{bmatrix}\{\dot{W}\}_e + \begin{bmatrix} m \end{bmatrix}\{\ddot{W}\}_e &= \{r\}_e \end{aligned} \tag{6.24}$$

The above set of equations represent the mixed element matrices for an arbitrary plate finite element. We now confine our attention to the isoparametric quadrilateral element.

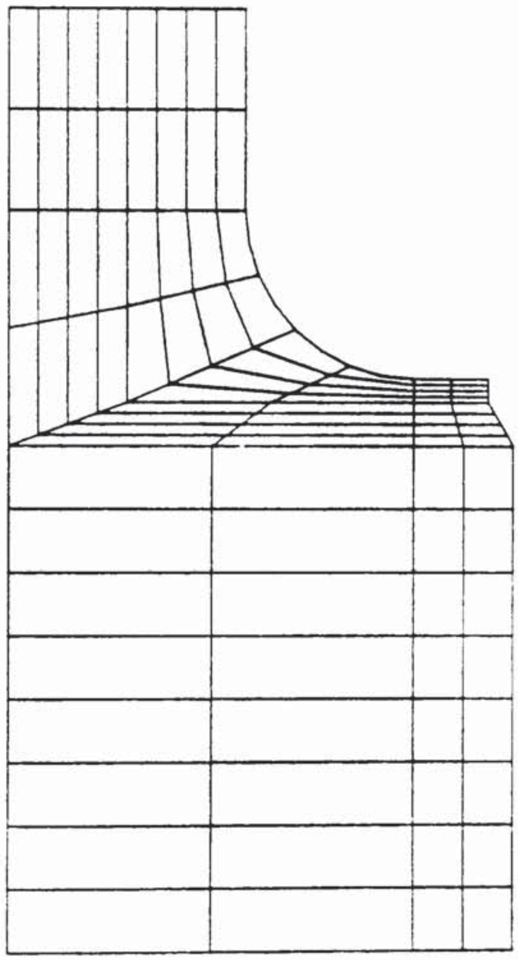


Fig 6.2 Finite element idealisation of a plate.

6.5 DERIVATION OF MIXED ELEMENT PROPERTIES - THIN PLATES

6.5.1 Element shape functions

The element under consideration is an isoparametric quadrilateral element of quadratic type (see Fig. 6.3). The element has 8 nodes with 32 degrees of freedom (one transverse deflection and two bending and one twisting moments per node). Isoparametric elements have identical geometric transformation and displacement assumptions which may be represented as:

$$(x,y) = \left[N_1, N_2, \ldots, N_8 \right] \left(\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_8 \end{Bmatrix}, \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_8 \end{Bmatrix} \right) \qquad (6.25)$$

$$W = \left[N_1, N_2, \ldots, N_8 \right] \begin{Bmatrix} W_1 \\ W_2 \\ \vdots \\ W_8 \end{Bmatrix} \qquad (6.26)$$

i.e. $W = \left[N_W \right] \{W\}_e$

and the bending moments are given by:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & & N_8 & 0 & 0 \\ 0 & N_1 & 0 & \cdots\cdots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & & 0 & 0 & N_8 \end{bmatrix} \begin{bmatrix} M_{x_1} \\ M_{y_1} \\ M_{xy_1} \\ \vdots \\ M_{x8} \\ M_{y8} \\ M_{xy8} \end{bmatrix} \qquad (6.27)$$

i.e. $\{M\} = \left[N_M \right] \{M\}_e$

A curvilinear coordinate system (ξ,η) is defined within the element in such a way that the corners of the element have coordinates of +1 or -1. The location of local node points for each element are initially defined in terms of the cartesian coordinates (x,y). The shape functions are:

$$N_i = \tfrac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)(\xi\xi_i + \eta\eta_i - 1) \quad (i = 1,2,3,4)$$

$$N_i = \tfrac{1}{2}(1 - \xi^2)(1 + \eta\eta_i) \quad (i = 5,7) \qquad (6.28)$$

$$N_i = \tfrac{1}{2}(1 - \eta^2)(1 + \xi\xi_i) \quad (i = 6,8)$$

which are the same for all three types of parameters, (geometric, displacement and moments).

6.5.2 Transformation

The evaluation of the element coefficients involves the derivatives of the shape functions (which are defined in terms of ξ and η) with respect to x and y and integration over the area of the element. Integration is performed in the transformed coordinate system and therefore various terms, such as the transformation jacobian are included in the integration to give the correct results for the original coordinate system (20).

From the chain rule of differentiation it can be shown that

$$\begin{Bmatrix} \frac{\partial Ni}{\partial \xi} \\ \frac{\partial Ni}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial Ni}{\partial x} \\ \frac{\partial Ni}{\partial y} \end{Bmatrix} = \left[j \right] \begin{Bmatrix} \frac{\partial Ni}{\partial x} \\ \frac{\partial Ni}{\partial y} \end{Bmatrix} \qquad (6.29)$$

where $\left[\, J \,\right]$ is the Jacobian operator relating the curvilinear coordinate derivatives to the local x,y coordinate derivatives. The Jacobian operator can easily be found using (6.25). Inverting (6.29) gives:

$$\begin{Bmatrix} \dfrac{\partial Ni}{\partial x} \\ \\ \dfrac{\partial Ni}{\partial y} \end{Bmatrix} = \left[\, J \,\right]^{-1} \begin{Bmatrix} \dfrac{\partial Ni}{\partial \xi} \\ \\ \dfrac{\partial Ni}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} J^*_{11} & J^*_{12} \\ \\ J^*_{21} & J^*_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial Ni}{\partial \xi} \\ \\ \dfrac{\partial Ni}{\partial \eta} \end{Bmatrix} \qquad (6.30)$$

The inverse of $\left[\, J \,\right]$ exists provided that there is a one-to-one correspondence between the natural, (ξ,η) and local (x,y) coordinates. An operating matrix B which includes all the shape function derivatives may then be represented as:

$$\left[\, B \,\right] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \cdots\cdots & \dfrac{\partial N_7}{\partial x} & \dfrac{\partial N_8}{\partial x} \\ \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \cdots\cdots & \dfrac{\partial N_7}{\partial y} & \dfrac{\partial N_8}{\partial y} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \cdots\cdots & B_{17} & B_{18} \\ \\ B_{21} & B_{22} & \cdots\cdots & B_{27} & B_{28} \end{bmatrix} \qquad (6.31)$$

in which

$$\begin{aligned} B_{11} &= J^*_{11} \; (1-\xi)\;(2\eta+\xi)\;/4 + J^*_{12}\,(1-\eta)\;(2\xi+\eta)/4 \qquad &(a) \\ B_{12} &= J^*_{11} \; (1-\xi)\;(2\eta-\xi)\;/4 + J^*_{12}\,(1+\eta)\;(2\xi-\eta)/4 \qquad &(b) \\ B_{13} &= J^*_{11} \; (1+\xi)\;(2\eta+\xi)\;/4 + J^*_{12}\,(1+\eta)\;(2\xi+\eta)/4 \qquad &(c) \\ B_{14} &= J^*_{11} \; (1+\xi)\;(2\eta-\xi)\;/4 + J^*_{12}\,(1-\eta)\;(2\xi-\eta)/4 \qquad &(d) \end{aligned} \qquad (6.32)$$

etc.

Equations (6.28) and (6.32) can be used to evaluate the element matrices in (6.21). The resulting integrals are too complicated to evaluate explicitly and therefore numerical integration must be employed.

6.5.3 Slope matrices

Differentiating equation (6.26) with respect to x and y at any point within the element gives:

$$\left\{ \begin{matrix} \frac{\partial W}{\partial x} \\ \\ \frac{\partial W}{\partial y} \end{matrix} \right\} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots\cdots & \frac{\partial N_8}{\partial x} \\ \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots\cdots & \frac{\partial N_8}{\partial y} \end{bmatrix} \left\{ \begin{matrix} W_1 \\ W_2 \\ \vdots \\ W_8 \end{matrix} \right\} \quad \text{i.e.} \quad \{W'\} = \left[N'_W \right] \{W\}. \tag{6.33}$$

This relation can be evaluated by using the operating matrix B given by equations (6.31) and (6.32). Then

$$\left\{ \begin{matrix} \frac{\partial W}{\partial x} \\ \\ \frac{\partial W}{\partial y} \end{matrix} \right\} = \begin{bmatrix} B_{11} & B_{12} & \cdots\cdots & B_{17} & B_{18} \\ \\ B_{21} & B_{22} & \cdots\cdots & B_{27} & B_{28} \end{bmatrix} \left\{ \begin{matrix} W_1 \\ W_2 \\ \vdots \\ W_8 \end{matrix} \right\}_e \tag{6.34}$$

on the element boundary s_n, equation (6.17) can be written for each side as:

$$\frac{\partial W}{\partial s} = \left[L_W \right]_i \left\{ \begin{matrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \end{matrix} \right\}_i \tag{6.35}$$

for i = 1, 2, 3, 4. Where (i) is the element side and the corresponding direction cosines matrix is given by:

$$\left[L_W \right]_i = \left(- \sin \beta : \cos\beta \right)_i \tag{6.36}$$

For each element side, β is the angle between the normal to the boundary s_n and the x-axis (Fig. 6.3). In general β is variable along curved element boundaries and therefore should be calculated numerically. A computer subroutine is written (section 7.7) which calculates β at different integration points on the element boundary.

Substituting from (6.34) into (6.35) yields:

$$\frac{\partial W}{\partial s} = \left(-\operatorname{Sin}\beta \quad \operatorname{Cos}\beta\right)_i \begin{bmatrix} B_{11} & B_{12} & \ldots\ldots & B_{17} & B_{18} \\ B_{21} & B_{22} & \ldots\ldots & B_{27} & B_{28} \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ \vdots \\ W_7 \\ W_8 \end{Bmatrix} \qquad (6.37)$$

i.e. $\frac{\partial W}{\partial s} = \left[Y \right]_i \; \{W\}_e$ for i = 1, 2, 3, 4

where

$$\left[Y \right]_i = \left[Y_1 \;\; Y_2 \ldots\ldots\ldots Y_7 \;\; Y_8 \right]_i \qquad (a)$$

in which (6.38)

$$Y_1 = (-B_{11}\operatorname{Sin}\beta + B_{21}\operatorname{Cos}\beta)_i, \text{ etc.} \qquad (b)$$

6.5.4 Shear force intensity matrix

The shear force intensities Q_x and Q_y may be expressed in terms of nodal bending moments. Differentiating the matrix relation (6.27) within the element yields:

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \left[\begin{array}{ccc|c|ccc} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_1}{\partial y} & \ldots\ldots\ldots & \frac{\partial N_8}{\partial x} & 0 & \frac{\partial N_8}{\partial y} \\ \hline 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \ldots\ldots\ldots & 0 & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} \end{array} \right] \begin{bmatrix} M_{x1} \\ M_{y1} \\ M_{xy_1} \\ \vdots \\ M_{x8} \\ M_{y8} \\ M_{xy8} \end{bmatrix} \qquad (6.39)$$

substituting the components of the operating matrix B , (6.31) in the above matrix relation yields:

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \left[N'_M \right]_{\xi,\eta} \{M\}_e \tag{6.40}$$

where

$$\left[N'_M \right]_{\xi,\eta} = \left[\begin{array}{c|c|c|c|c|c|c} B_{11} & 0 & B_{21} & \cdots\cdots\cdots & B_{18} & 0 & B_{28} \\ \hline 0 & B_{21} & B_{11} & \cdots\cdots\cdots & 0 & B_{28} & B_{18} \end{array} \right]_{(\xi,\eta)}$$

6.5.5 <u>Normal twisting moment along each element side</u>

The normal twisting moment M_{ns} in terms of the natural coordinates ξ and η and the nodal moments is given by equation (6.19). The direction cosine matrix $[L]$ is taken from relation (3.26), and is given by:

$$[L] = \left[-\cos\beta \quad \sin\beta \;\vdots\; \cos\beta \quad \sin\beta \;\vdots\; \cos^2\beta \quad -\sin^2\beta \right]_i \tag{6.41}$$

for i = 1, 2, 3, 4

The components of $[L]$ are evaluated numerically at each integration point along the element boundaries. The twisting moment M_{ns} can then be represented as:

$$M_{ns} = \left[L_{11} N_1 \vdots L_{12} N_1 \vdots L_{13} N_1 \vdots \ldots\ldots \vdots L_{11} N_8 \vdots L_{12} N_8 \vdots L_{13} N_8 \right] \begin{bmatrix} M_{x_1} \\ M_{y_1} \\ M_{xy_1} \\ \vdots \\ M_{x_8} \\ M_{y_8} \\ M_{xy_8} \end{bmatrix} \quad (6.42)$$

i.e. $M_{ns} = \left[L_{ns}\right]_{\xi,\eta} \{M\}_e$ with ξ or $\eta = \pm 1$

6.5.6 Mixed element matrices and load vector

Having established the Jacobian transformation in the matrice $\left[N'_W\right]$ and $\left[N'_M\right]$, we can proceed to the integration of equations (6.21). As in the usual numerical integration method dxdy is replaced by $|J_{(\xi,\eta)}|$ $d\xi\, d\eta$ where

$$|J_{(\xi,\eta)}| = \det (J) \quad (6.43)$$

then for thin plates we have:

$$\left[g\right] = \int_{-1}^{1}\int_{-1}^{1} \left[N_M\right]^t \left[D\right]^{-1}\left[N_M\right] \det J \; d\xi\, d\eta \quad (a)$$

$$(6.44)$$

$$\left[h\right] = \int_{-1}^{1}\int_{-1}^{1} \left[N'_M\right]^t \left[N'_W\right] \det J \; d\xi d\eta - \int_{-1}^{1} \left[L_{ns}\right]^t \left[Y\right] ds \quad (b)$$

$$[m] = \int_{-1}^{1} \int_{-1}^{1} \rho h \left[N_W\right]^t \left[N_W\right] \det J \; d\xi \, d\eta \quad (c)$$

(6.44)

$$\{r\} = \int_{-1}^{1} \int_{-1}^{1} \left[N_W\right]^t p(x,y,t) \det J \; d\xi d\eta \quad (d)$$

The line integral in equation (6.44b) is to be evaluated along each element side. Therefore, ds is replaced by:

$$ds = \frac{\partial s}{\partial \xi} \, d\xi \qquad (a) \qquad (\eta = \text{const})$$

or (6.45)

$$ds = \frac{\partial s}{\partial \eta} \, d\eta \qquad (b) \qquad (\xi = \text{const})$$

$\frac{ds}{d\xi}$ and $\frac{ds}{d\eta}$ may be determined from the following relations:

$$\frac{ds}{d\xi} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2} \; , \qquad \frac{ds}{d\eta} = \sqrt{\left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dy}{d\eta}\right)^2} \qquad (6.46)$$

where the components $\frac{dx}{d\xi}$, etc. are evaluated using equation (6.25). If the element edge is a straight line of length L then

$$ds = \frac{L}{2} \, d\xi \qquad (6.47)$$

If the load p(x,y,t) is not constant over the area of the plate, then it is assumed that within an element, the distribution is given by:

$$p(x,y,t) = \left[N_p\right] \{p\}_e \qquad (6.22)$$

The interpolation function $\left[N_p\right] = \left[N_W\right]$ and $\{p\}_e^t = \left[p_1, p_2, \ldots, p_7, p_8\right]$ is a vector with nodal pressures. Substituting from (6.22) in (6.44d) yields:

$$\{r\} = \int_{-1}^{1} \int_{-1}^{1} \left[N_W\right]^t \left[N_p\right] \{p\}_e \ \det J \ d\xi \, d\eta \qquad (6.48)$$

Now the components of the element mixed matrices and load vector can be given as:

$$\left[g\right] = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{17} & g_{18} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{81} & g_{82} & \cdots & g_{87} & g_{88} \end{bmatrix} \qquad (6.49)$$

A typical submatrix (g_{ij}) linking nodes i and j is given by the expression

$$\left[g_{ij}\right] = \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix} N_i \ C_{11} \ N_j & N_i \ C_{12} \ N_j & N_i \ C_{13} \ N_j \\ N_i \ C_{21} \ N_j & N_i \ C_{22} \ N_j & N_i \ C_{23} \ N_j \\ N_i \ C_{31} \ N_j & N_i \ C_{32} \ N_j & N_i \ C_{33} \ N_j \end{bmatrix} \det J \ d\xi \, d\eta \qquad (6.50)$$

For i, J = 1, 2, 3, 4, 5, 6, 7, 8

and,

$$[h] = \begin{bmatrix} h_{11} & h_{12} & \cdots\cdots & h_{17} & h_{18} \\ & & & & \\ h_{81} & h_{82} & \cdots\cdots & h_{87} & h_{88} \end{bmatrix} \qquad (6.51)$$

where

$$h_{ij} = \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} B_{1_i} & B_{1_j} \\ B_{2_i} & B_{2_j} \\ B_{2_i} & B_{1_j} + B_{1_i} B_{2_j} \end{bmatrix} \det J \, d\xi d\eta - \int_{-1}^{1} \begin{bmatrix} L_{11} & N_i & Y_j \\ L_{12} & N_i & Y_j \\ L_{13} & N_i & Y_j \end{bmatrix} ds$$

$$\text{for } i, j = 1, 2, \ldots, 8 \qquad (6.52)$$

in which the line integral should be evaluated over the four sides of the element. The mass matrix is given by:

$$[m] = \int_{-1}^{1}\int_{-1}^{1} \rho h \begin{bmatrix} N_1^2 & N_1 N_2 & \cdots\cdots & N_1 N_8 \\ & N_2^2 & & \vdots \\ \text{Symm.} & & \ddots & N_8^2 \end{bmatrix} \det J \, d\xi d\eta \qquad (6.53)$$

where ρ is the mass of the plate per unit volume and h is the thickness.

The load vector for p = constant is:

$$\{r\} = \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_8 \end{Bmatrix} = \int_{-1}^{1} \int_{-1}^{1} \begin{Bmatrix} N_1 \\ N_2 \\ \vdots \\ N_8 \end{Bmatrix} p \det J \; d\xi \, d\eta \qquad (6.54)$$

However, if p is not constant then it is assumed that p varies parabolically within an element with $\left[N_p \right] = \left[N_W \right]$ we have

$$r_i = \int_{-1}^{1} \int_{-1}^{1} \left(\sum N_i \, N_j \, p_j \right) \det J \; d\xi \, d\eta \qquad (6.55)$$

for i. j = 1, 2, ... 8

We will now outline the Gaussian numerical integration procedure which will be used in the evluation of various matrices involved.

6.5.7. Approximate integration of element matrices. Gauss - quadrature rule

Exact integration of the matrices in euqations (6.44) to (6.55) is not generally possible because the composite function which forms the integrand cannot be expressed as a polynomial. This is due to the jaccbian matrix determinant det J, which is used in the transformation between the global coordinate system and the natural coordinate system and is usually a variable function. It is therefore necessary to resort to numerical integration procedures to evaluate the matrix coefficients. In particular, Gaussian integration techniques have been adopted because of their convenience and accuracy.

Consider, for example the numerical integration of

$$I = \int_{-1}^{1} f(\xi) \, d\xi \qquad -1 \leq \xi \leq +1 \tag{6.56}$$

Using Gaussian integration, the function $f(\xi)$ is evaluated at several sampling points with coordinates $\xi_i = a_i$ within the region of integration. Each value $f(\xi_i)$ is multiplied by the appropriate "weight" W_i and added. Thus, the integration becomes a summation of products, i.e.

$$I \simeq \sum_{i=1,n} f(\xi_i) \, W_i \tag{6.57}$$

where n is the number of points taken. The Gauss method locates the sampling points so that for a given number of them, greatest accuracy is obtained. Table (6.2) gives the appropriate Gauss quadrature coefficients for the first three orders (44). It should be noted that an n-point Gauss rule integrates a function of order (2n-1) or less, exactly.

In two dimensions the function f in the integrand is a function of two variables, i.e.

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi,\eta) \, d\xi \, d\eta \qquad \begin{matrix} -1 < \xi < +1 \\ -1 < \eta < +1 \end{matrix} \tag{6.58}$$

a product Gauss rule may be used. Therefore, the summation becomes:

$$I \approx \sum_i \sum_j W_i W_j \ f(\xi_i, \eta_j) \qquad (6.59)$$

The 3 x 3 point Gauss quadrature is used for the evaluation of the area integrals in equation (6.59). The location of these points is shown in figure (6.3).

TABLE 6.2 Gauss quadrature points

No. of points	Locations (ξ_i, η_i)	Associated Weights, W_i
1	0.0000000	2
2	±0.5773502691	1
3	±.774596669	$^5/_9$
	0.0000000	$^8/_9$

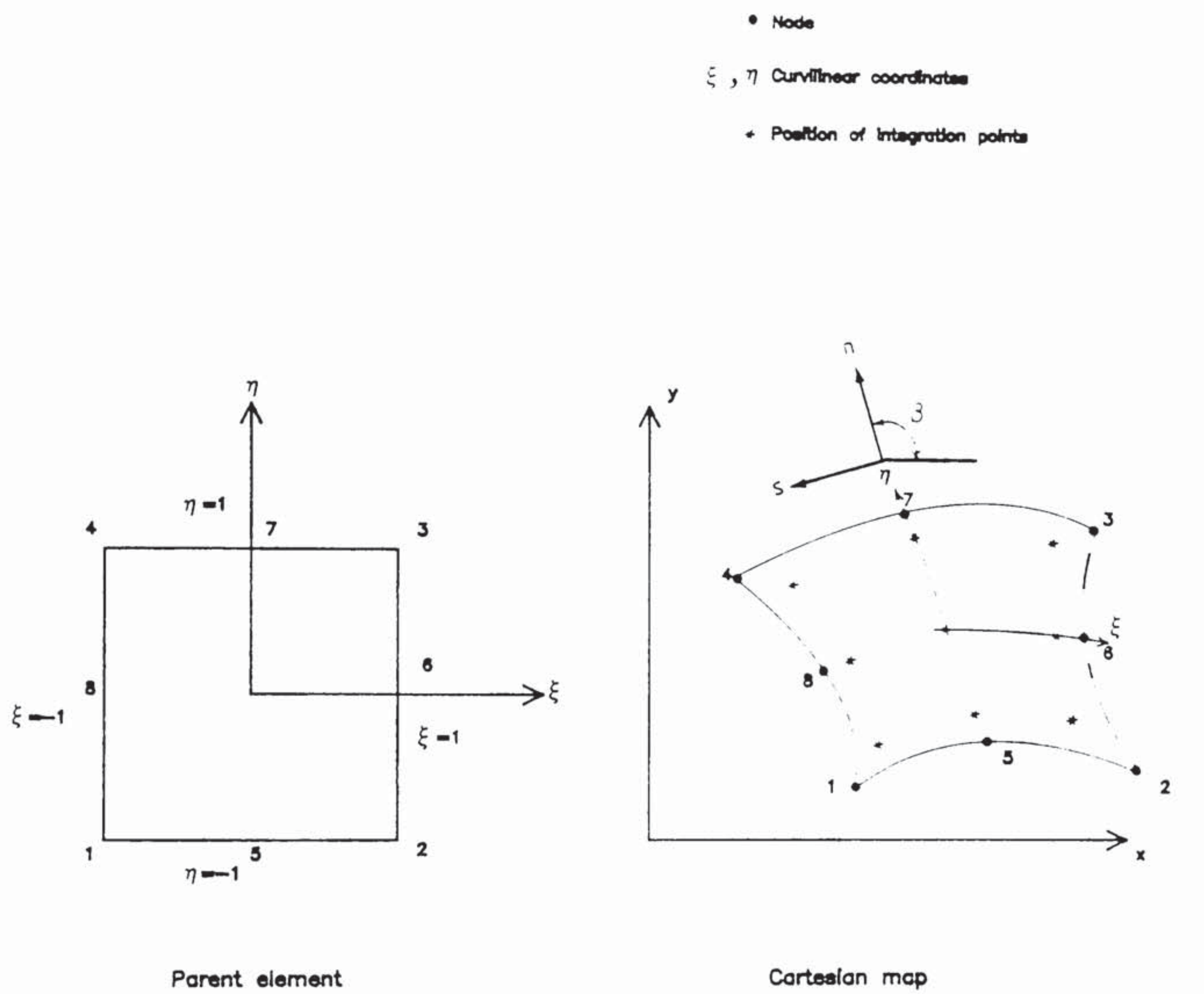


Fig 6.3 Isoparametric quadratic plate element.

6.6 ASSEMBLY OF THE OVERALL MATRICES AND LOAD VECTORS

In the preceding section the Reissner principle was applied to a single element and the individual element characteristics were established. The element matrices in equation (6.24) can be assembled in the usual manner, (64) to obtain the global matrices. Let the global deflections and moments be represented by:

$$\{W^*\} = \left[W_1^*, W_2^*, \ldots\ldots W_n^* \right]^t$$

$$\{M^*\} = \left[\left[M^*\right]_1, \left[M^*\right]_2, \ldots\ldots, \left[M^*\right]_n \right]^t$$

$$\text{with } \left[M^* \right]_i = \left[M_x \;\; M_y \;\; M_{xy} \right] \text{ node } i$$

and n representing the number of nodes. Equation (6.24) for the overall structure may now be written as:

$$\begin{aligned} -[G]\{M^*\} + [H]\{W^*\} &= \{0\} \\ [H]^t\{M^*\} + [C]\{\dot{W}^*\} + [M]\{\ddot{W}^*\} &= \{R\} \end{aligned} \qquad (6.60)$$

where the overall partitioned matrices can be represented as follows:

$$\left[G\right] = \sum_{e=1}^{N} \left[G_e\right] \qquad (a)$$

$$\left[H\right] = \sum_{e=1}^{N} \left[H_e\right] \qquad (b) \qquad\qquad (6.61)$$

$$\left[C\right], \left[M\right] = \sum_{e=1}^{N} [c]_e, [m]_e \qquad (c)$$

$$\{R\} = \sum_{e=1}^{N} \{r\}_e \qquad (d) \qquad\qquad N = \text{number of elements}$$

$\left[G_e\right]$, $\left[H_e\right]$, etc. have the same dimensions of $\left[G\right]$, $\left[H\right]$, ... but the only non-zero locations are those due to the coefficients of $[g]$, $[h]$, etc. for the eth element, globally located.

6.7 BOUNDARY CONDITIONS

Before proceeding to solve the problem, we must impose the kinematic (or essential) boundary conditions of equation (6.11) on the global system of equations. The constrained nodal deflections and/or moments are eliminated to yield non-singular matrices (by deleting the corresponding rows and columns). At boundary nodes where the constrained moments do not coincide with the x-y global axes, there will have to be a change of coordinates to normal and tangential components. The relation between the unknown moments for a typical boundary node (i) (Fig. 6.4) is given by:

$$\begin{Bmatrix} M_s \\ M_n \\ M_{ns} \end{Bmatrix} = \begin{bmatrix} \sin^2\beta & \cos^2\beta & -2\sin\beta\cos\beta \\ \cos^2\beta & \sin^2\beta & 2\sin\beta\cos\beta \\ -\cos\beta\sin\beta & \cos\beta\sin\beta & \cos^2\beta-\sin^2\beta \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \qquad (3.26)$$

$$\text{i.e.} \quad \{M'\}_i = \left[\, l_i \,\right] \{M^*\}_i$$

where the primed notation indicates the nodal components in the new normal axes. $\left[\, l_i \,\right]$ is the simple point transformation matrix and β is the angle between the normal of the true boundary at the ith node and the x-axis. Following a procedure similar to the one described by Mota Soares (7), the mixed-matrix governing equation is tranformed to the new set of coordinates, at element level, before the assembly process. Thus for an individual element we have:

$$\left[\, g \,\right]' = \left[\, L_K \right]^t \left[\, g \,\right] \left[\, L_K \right] \qquad (a)$$

and (6.62)

$$\left[\, h \,\right]' = \left[\, L_K \right]^t \left[\, h \,\right] \qquad (b)$$

where the matrix $\left[\, L_K \right]$ is (24 x 24) and has the following typical form:

$$
\left[\, L_K \,\right] = \begin{bmatrix} I & \cdots & \cdots & \cdots & 0 \\ \vdots & I & & 0 & \vdots \\ \vdots & & l_i & & \vdots \\ \vdots & 0 & & \ddots & 0 \\ 0 & & & & \\ 0 & \cdots\; 0 & \cdots & \cdots & I \end{bmatrix} \tag{6.63}
$$

in which I is the identity matrix of the same order as l_i. The prescribed values are summarized in the following table.

TABLE 6.3

Boundary conditions	Nodal variables - Thin Plates			
	W	M_s	M_n	M_{ns}
Simply-supported	0		0	
Clamped	0			
Free			0	
Symmetrique				0

For the rest of this chapter we will assume that all the boundary conditions have been applied and the mixed matrices correspond to free nodal deflections and moments.

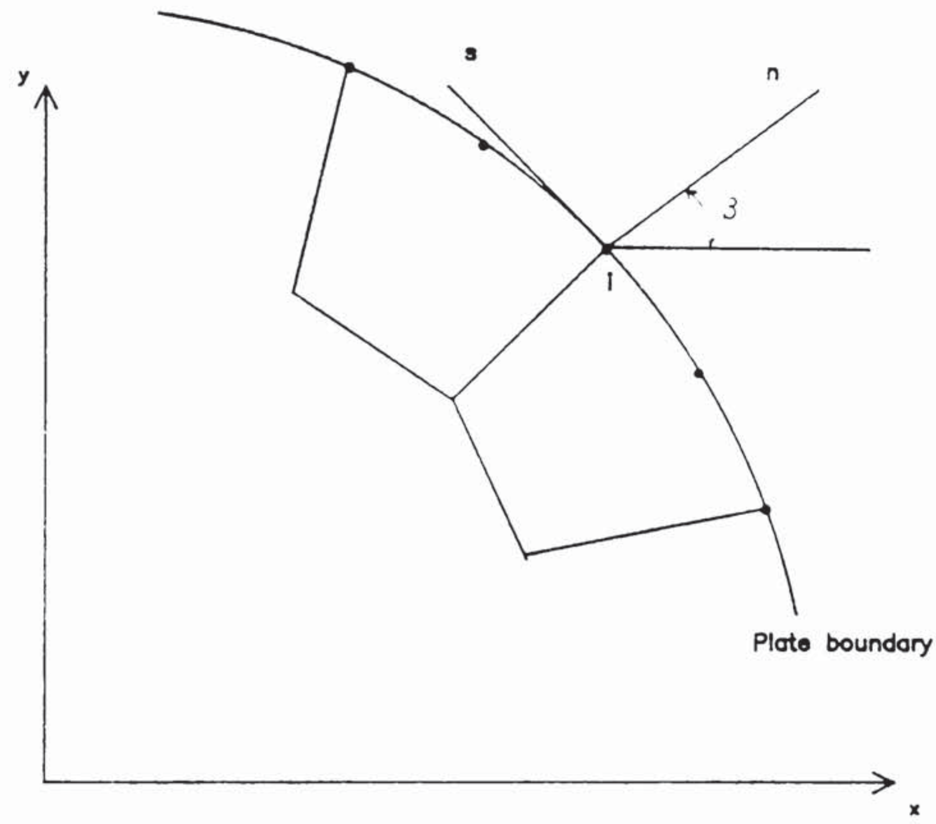


Fig 6.4 Typical boundary node.

6.8 MATRIX CONDENSATION OF THE MIXED GOVERNING EQUATIONS

The mixed matrix equations (6.60) should be trnaformed into an appropriate form prior to the solution procedure. In static analysis (7), the effect of inertia and damping is neglected and the following equation is obtained

$$\begin{bmatrix} -[G] & [H] \\ [H]^t & 0 \end{bmatrix} \begin{bmatrix} \{M^*\} \\ \{W^*\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{R\} \end{bmatrix} \tag{6.64}$$

For an effective solution the nodal freedoms are rearranged and equation (6.64) is re-written as:

$$\begin{bmatrix} K \end{bmatrix} \{\delta\} = \{F\} \tag{6.65}$$

in which

$$\{\delta\}^t = \left[(W^*, M_x^*, M_y^*, M_{xy}^*)_1, (W^*, M_x^*, M_y^*, M_{xy}^*)_2, \ldots (W^*, M_x^*, M_y^*, M_{xy}^*)_n \right] \tag{6.66}$$

$$\{F\}^t = \left[(R_1, 0, 0, 0), (R_2, 0, 0, 0), \ldots, (R_n, 0, 0, 0) \right] \tag{6.67}$$

the mixed matrix $\begin{bmatrix} K \end{bmatrix}$ is banded and non-positive definite. To avoid the zeros in the diag onal element, the Gauss elimination method, reference (65), with row interchanges is used in the solution of the equations. However, since the symmetry of the overall matrix is lost throughout the numerical process, it is required that the complete band form of the matrix $\begin{bmatrix} K \end{bmatrix}$ to be stored as a two-dimensional array.

In dynamic problems, the matrix condensation is carried out as follows:

The first equation in equation (6.60) is solved for {M*} to give:

$$\{M^*\} = [G]^{-1} [H] \{W^*\} \tag{6.68}$$

and the result is substituted in the second of equation (6.60), thus:

$$[K^*] \{W^*\} + [C] \{\dot{W}^*\} + [M] \{\ddot{W}^*\} = \{R_{(t)}\} \tag{6.69}$$

where

$$[K^*] = [H]^t [G]^{-1} [H] \tag{6.70}$$

is a real symmetric positive definite matrix. It should be noticed that the reduction of degrees of freedom (equation 6.68) to (6.70), is an exact operation and that the moments are calculated (equation 6.68) by a matrix transformation of the displacements. The solution of equation (6.69) is performed by one of the direct integration or mode superposition methods described in section (4.6).

For free harmonic vibrations (neglecting damping) equations (6.60) become:

$$-[G]\{\hat{M}^*\} + [H] \{\hat{W}^*\} = \{0\} \quad (a)$$

$$[H]^t\{\hat{M}^*\} - \omega^2 [M] \{\hat{W}^*\} = \{0\} \quad (b) \tag{6.71}$$

which can be easily transformed into the standard eigenvalue problems (see section 4.4.2). In terms of $\{\hat{W}^*\}$, the eigenvalue problem becomes:

$$[K^*] \{\hat{W}^*\} - \omega^2 [M] \{\hat{W}^*\} = \{0\} \tag{6.72}$$

and in terms of $\{\hat{M}^*\}$ eigenvectors, we will obtain the following equation:

$$\left[H \right] \left[M \right]^{-1} \left[H \right]^{t} \{\hat{M}^*\} - \omega^2 \left[G \right] \{\hat{M}^*\} = \{0\} \qquad (6.73)$$

The natural frequencies (ω) and the mode shapes ($\{\hat{W}^*\}$ or $\{\hat{M}^*\}$) are determined by solving the corresponding eigenvalue problem. A standard subroutine is used which is described in Ref. (66).

CHAPTER 7

COMPUTER ALGORITHMS AND PROGRAM STRUCTURE

7.1 INTRODUCTION

Three finite element programs have been written which use the 8-node plate isoparametric element described in Chapter 6.

These programs are:

(1) RFPLT1 For free vibration of thin plates.

(2) RFPLT2 For forced vibration of plates using the mode superposition method.

(3) RFPLT3 For forced vibration of plates using the direct integration method.

The memory capacity of the computer on which these programs have been implemented is 182 K. Due to a memory capacity constraint, the treatment of plate problems, using these programs is limited to systems having no more than (10-20) non-constrained nodes. This machine can be configured to have a maximum memory of .5M bytes. This would enable the maximum number of non-constrained nodes to be increased to (30-60) nodes.

The flowcharts for the three programs RFPLT1, RFPLT2 and RFPLT3 are presented in sections (7.3) to (7.14). The routines for the generation of element characteristics matrices are common to all three programs. These are presented in sections (7.7) to (7.10).

7.2 CLASSIFICATIONS OF THE SECTIONS OF THE PROGRAM

The program may be classified into the following main sections:

(i) Specification of the structural idealisation.
Data is input to the program in the following form:

(a) Nodal coordinates and element topology - This data is prepared using a semi-automatic mesh generation program (Ref. (67)) where the structure is divided into a few large zones and the fineness of element subdivision within each is specified. The initial data is input in the normal way and the subdivision proceeds automatically.

(b) Material properties

(c) The boundary conditions

(d) The loading to which the structure is subjected

(e) Modal damping ratios, initial conditions, time varying forces, and time integration constants.

Data in parts a, b and c are common to all three programs dealt with in this section. This data is prepared by the mesh generation program and is input into the main program via data files (see Appendix B). Data in parts d and e are input at the keyboard by the operator in response to the appropriate programmed input prompts.

(ii) Evaluation of element characteristics.

The numerically integrated, mixed matrices [g], [h] and [m] are formed for each element in turn with reference to the global coordinate system (x,y). The load vector {r} due to a distributed load, matrix relation (6.48) is also evaluated numerically.

(iii) Assembly of the element matrices.

The element matrices are assembled into the overall structural matrices. The equivalent stiffness matrix is then calculated, from equation (6.70).

(iv) Solution of the eigenvalue-eigenvector problem.

The standard procedures Trans and Eigen in reference (66) yield the eigenvalues and eigenvectors which may be used for:

(a) Free vibration analysis

(b) Forced vibration analysis by mode superposition method.

(c) Construction of a complete damping matrix in the forced vibration analysis by direct integration method.

(v) Steady state and transient response solutions.

Either a mode superposition (Duhammel integration) or a direct integration (Wilson θ) method is used to calculate the response of the plate subjected to time varying forces.

(vi) Output - The free vibration results are output as a table consisting of numbers for frequencies and mode shapes. The response output from programs RFPLT2 and RFPLT3 include time history plots for displacements and bending moments.

7.3 SUBPROGRAM Feinpt

This subprogram reads the input data file, generated by a pre-program for mesh generation. The variables are read in, in the following order:

(a) Control data

Njb	: =	Number of jobs
N1	: =	Number of elements
N	: =	Number of nodes
Cw	: =	Number of nodes with constrained deflections
Cx	: =	Number of nodes with constrained M_x
Cy	: =	Number of nodes with constrained M_y
Cxy	: =	Number of nodes with constrained M_{xy}
Nmat	: =	Number of materials
Nskew	: =	Number of nodes with specified local coordinate direction

(b) Geometrical data

The subdivision of the structure into quadrilateral elements is defined by two sets of information:

(i) Nodal data - specifies the position of each node

$X_{(I)}$ x-coordinate of node I

$Y_{(I)}$ y-coordinate of node I

(ii) Element data - each element is identified by the nodal connection array and a material property set number. The nodal connections are represented by the array N (I,J), for I = 1, 2, ..., N1 and J = 1, 2, 3, ..., 9.

(c) Specified boundary conditions

Any one or more degrees of freedom (W, M_x, M_y, M_{xy}) may be specified as zero at a nodal point. Data for this section is read in the following order:

(i) Sequence of node numbers with specified W

(ii) Sequence of node numbers with specified M_x

(iii) Sequence of node numbers with specified M_y

(iv) Sequence of node numbers with specified M_{xy}

By using the information in part (c), an array Ndc (I,J), for I = 1, 2, ..., N and J = 1, 2, 3, 4 is constructed which specifies the free and restrained nodal degrees of freedom (moments, deflections). This array is used in the assembly process where the rows and columns corresponding to constrained degrees of freedom are determined and thus deleted. Figure (7.1) represents the nodal constraint array (Ndc(*)) for a quarter of a simply-supported plate. One element idealisation is used and zeros indicate the constrained degrees of freedom.

Node Number	M_x	M_y	M_{xy}	W
1	0	0	1	0
2	0	2	3	0
3	0	4	0	0
4	5	0	6	0
5	7	8	0	1
6	9	0	0	0
7	10	11	0	2
8	12	13	0	3

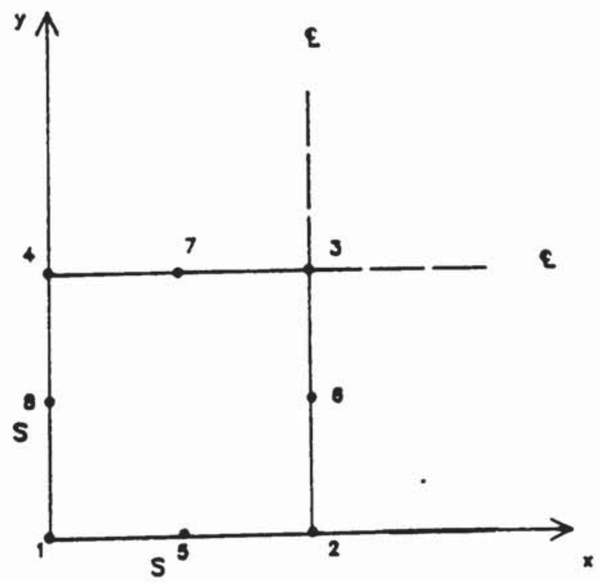


Fig. (7.1) Nodal connection array for a quarter of a SSSS plate.

Sequence of nodes with constrained W, 1, 2, 3, 4, 6

Sequence of nodes with constrained M_x, 1, 2, 3

Sequence of nodes with constrained M_y, 1, 4, 6

Sequence of nodes with constrained M_{xy}, 3, 5, 8, 7, 6

7.4 SUBPROGRAM Cmatrx

This subprogram reads the material properties of the structure from the input data file. The properties are specified for each set of elements of different materials. Each set is identified with a material property set number (Mat). Data is read in the following order:

(i) Th (Mat) : = Group material thickness

(ii) D (Mat) : = Group material density

(iii) E_x, ν_{xy}, G_{xy}, E_y, ν_{yx} : = Group material constants

7.5 SUBPROGRAM Excitn

This subprogram permits the user to define the force function from the keyboard.

7.6 SUBPROGRAM Rspipt

This subprogram is used to input the data required for the response calculations. Data at this stage is input by the operator at the keyboard. The input variables are:

Time	Time required for the response calculations
T	The value of theta in Wilson integration scheme
Delta	Time incremental
AØ (*)	Integration constants used in Wilson θ method
D(*)	Vector of initial displacements
0 f(*)	Vector of initial velocities
Nmod	Number of damped modes
Dr(*)	Array of damping ratios
Wp	Node number to calculate displacements for
Mp	Node number to calculate the moments for
J2	Code 1 - for M_x, 2 - for M_y, 3 - for M_{xy}

Subprograms Excitn and Rspipt are called in by programs RFPLT2 and RFPLT3.

7.7 SUBPROGRAM Qaux

Description: This subprogram is called in to perform the following operations:

(a) Calculates the shape functions $N_{1(\xi,\eta)}, \ldots, N_{8(\xi,\eta)}$ at the Gauss point within an element.

(b) Calculates the Jacobian J, its determinant and the inverse of the Jacobian, equation (6.30) at the Gauss point.

(c) Calculates shape function derivatives $\frac{\partial N_1}{\partial \xi}, \ldots, \frac{\partial N_8}{\partial \eta}$

(d) Calculates the angle β in equation (6.41), and ds in equation (6.45) along each element side by calling four subroutines Side 1, Side 2, Side 3 and Side 4 respectively.

The subprogram Qaux can be considered as a standard routine, which with little change, may be used in developing of other two dimensional isoparametric elements.

Variables list:

L_1, L_2	Natural coordinates of the sampling point (ξ_p, η_p)
Sf(*)	Shape function array
Pm(*)	Shape function derivative array
X(*), Y(*)	Nodal cartesian coordinate array
Z	Current element number
U	Determinant of Jacobian J
N(*)	Nodal connection array
Be	Angle between normal to a boundary with the +ve x-axis direction

Ds — Defined according to equation (6.45)

Jo — Integer indicating which operations to be carried:

1 - step (d) on Side 1 of the element ($\eta = -1$)

2 - step (d) on Side 2 of the element ($\xi = -1$)

3 - step (d) on Side 3 of the element ($\eta = -1$)

4 - step (d) on Side 4 of the element ($\xi = -1$)

5 - steps (a) and (b)

6 - steps (a), (b) and (c)

The variables are passed to and from the subprogram via the parameter list.Fig (7.2) shows the flow diagram for subprogram Qaux.

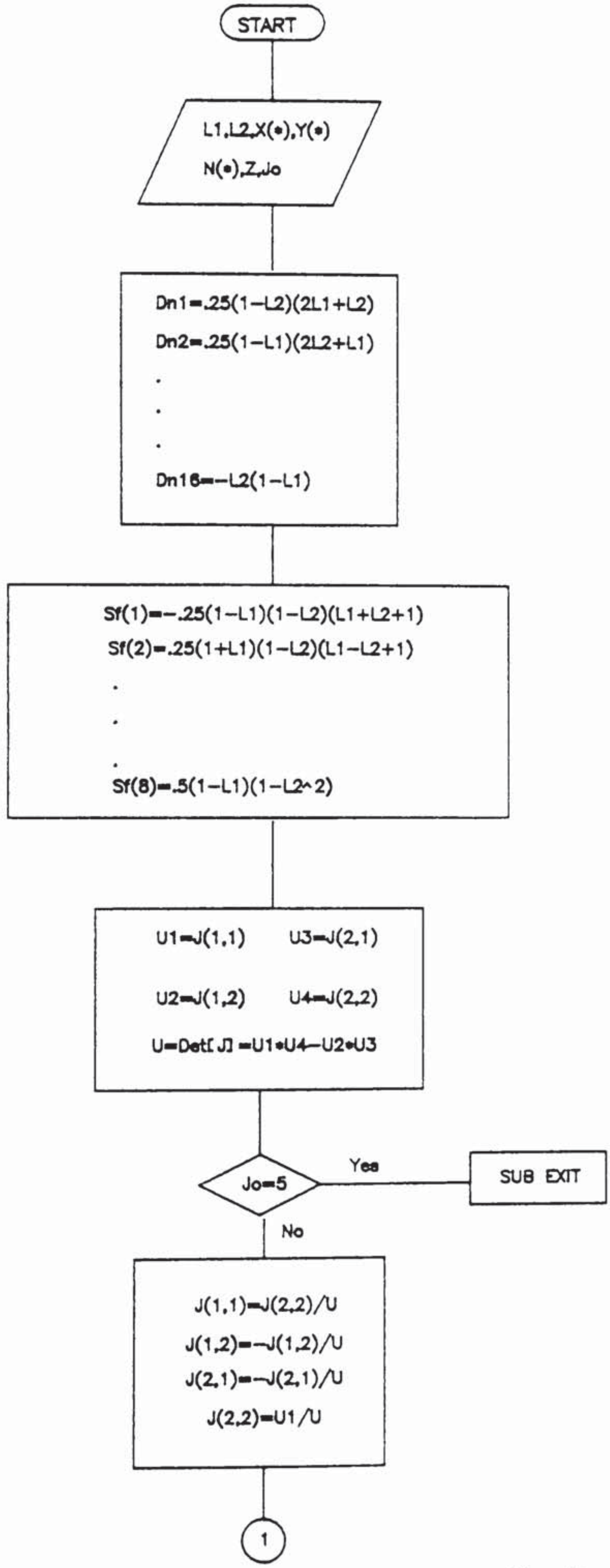
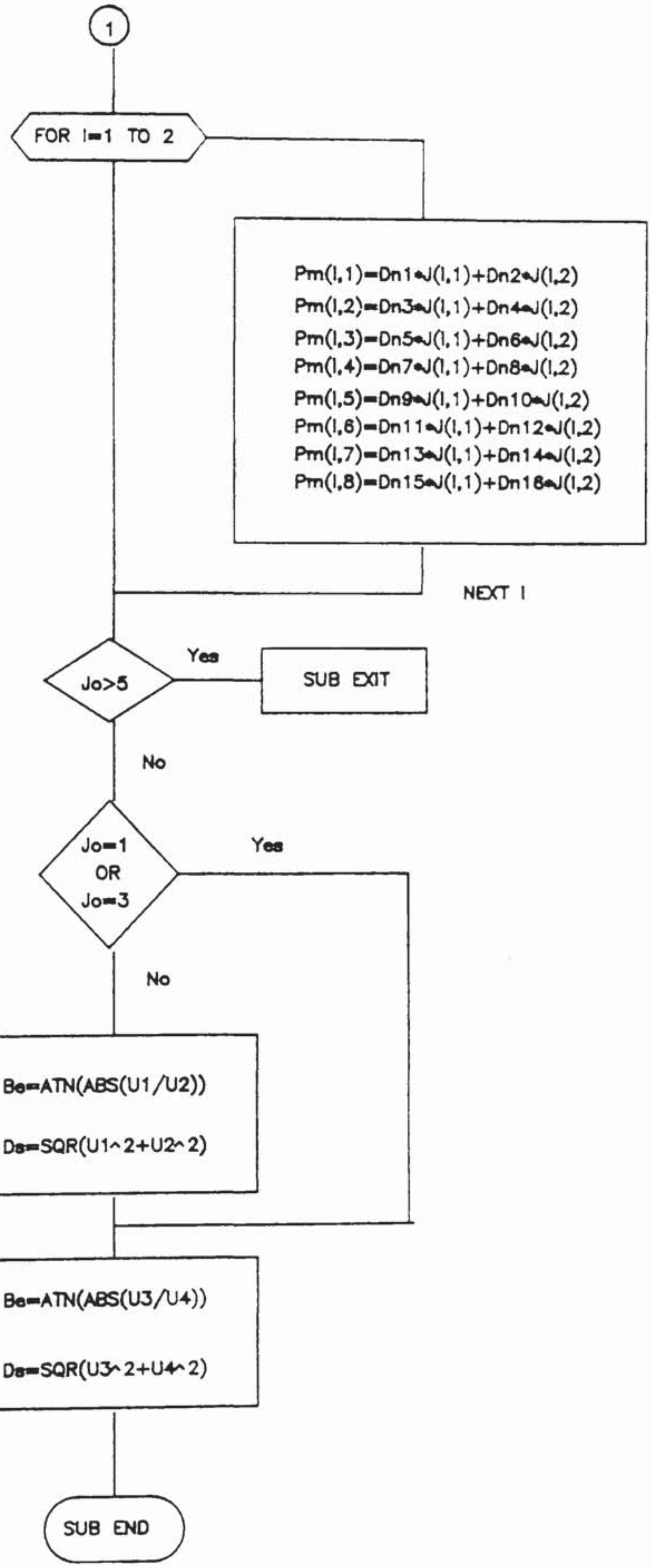


Fig 7.2 Flow diagram for SUB Qaux(L1,L2,X(•),Y(•),U,Pm(•),Sf(•),
Be,Ds,INTEGER Z,N(•),Jo)


1
FOR I=1 TO 2
Pm(I,1)=Dn1*J(I,1)+Dn2*J(I,2)
Pm(I,2)=Dn3*J(I,1)+Dn4*J(I,2)
Pm(I,3)=Dn5*J(I,1)+Dn6*J(I,2)
Pm(I,4)=Dn7*J(I,1)+Dn8*J(I,2)
Pm(I,5)=Dn9*J(I,1)+Dn10*J(I,2)
Pm(I,6)=Dn11*J(I,1)+Dn12*J(I,2)
Pm(I,7)=Dn13*J(I,1)+Dn14*J(I,2)
Pm(I,8)=Dn15*J(I,1)+Dn16*J(I,2)
NEXT I
Jo>5
Yes
SUB EXIT
No
Jo=1
OR
Jo=3
Yes
No
Be=ATN(ABS(U1/U2))
Ds=SQR(U1^2+U2^2)
Be=ATN(ABS(U3/U4))
Ds=SQR(U3^2+U4^2)
SUB END

7.8 ALGORITHMS FOR THE GENERATION OF ELEMENT MATRICES AND LOAD VECTOR

The coefficients of the element matrices $\left[h \right]$, $\left[g \right]$, $\left[m \right]$ and the equivalent nodal force vector {r} due to a distributed load are evaluated numerically using the Gauss quadrature rule, see section (6.5.7).

The subprogram Qaux, described by the flow diagram of Fig. (7.2) is called up in the element loop of the program to evaluate the variables required in the numerical integration process. For the integration of the partitioning matrix $[g]$ equation (6.49), we only need to integrate 36 different coefficients, related by the array Sfm(*), see figure (7.7). The coefficients are later multiplied by the constant values of the compliance matrix, array Cons(*), and located in $[g]$ in accordance with equation (6.50) by tne subprogram Geform. The contributions of the double integral in equation (6.52) to the mixed matrix $[h]$ are evaluated by calling up subprogram Heform. The line integral contributions are calculated in the subprogram Mnsws and added to construct the complete $[h]$ matrix. Subprograms Meform and Loadap are called up to calculate the element mass matrix and load vector respectively.

7.8.1 Subprogram Mnsws

This subprogram evaluates the contributions of the line integral $\int_{-1}^{1} \left[L_{ns} \right]^t \left[Y \right] ds$, to element mixed matrix $[h]$ (equation 6.52).

Variables list:

Zn(*) Gauss points & weights array

C(*)	L_{ns} array in Eqn. (6.42)
D(*)	Y array in Eqn. (6.38)
He(*)	Element mixed array $[h]$
X(*), Y(*)	Cartesian coordinates arrays
Bm(*)	Shape function derivative array
N(*)	Nodal connection array
Sf(*)	Shape function array
Z	Element counter
K	Element Side number (1,2,3 or 4)
Be	Angle
Ds	$(\frac{\partial s}{\partial \xi} d\xi)$ or $(\frac{\partial s}{\partial \eta} d\eta)$

Figure (7.3) shows the flow diagram for subprogram Mnsws.

7.8.2 Subprogram Heform

This subprogram calculates the coefficients of the mixed array $[h]$ due to double integral in equation (6.52).

Variable list:

T_1, T_2	weight coefficients
He(*)	Element mixed array $[h]$

Other variables are as defined in section 7.8.1.

Figure (7.4) shows the flow diagram for subprogram Heform.

7.8.3 Subprogram Geform

The coefficients of the submatrix $[g_{ij}]$ are multiplied by the constant values of the compliance matrix, array C(*) and located in [g] in accordance with Equation (6.50).

Variables list:

Ge(*)	Element mixed array [g]
A(*)	Array of shape function products
C(*)	Array of material constants
Matno	Number of material

Fig. (7.5) shows the flow diagram for this subprogram.

7.8.4 Subprogram Transf.

Description: This subprogram modifies the element mixed matrices [g] and [h] for those nodes on the boundary which require a coordinate transformation from the global x,y to a local n,s axes. Thus the boundary conditions for the bending and twisting moments may be applied directly in terms of normal and tangential components. The modification is carried out at element level. If for instance node 6 of element e requires modification then we have: (see section 6.7)

$$\left[g'_{ij}\right] = \left[g_{ij}\right]\left[1\right] \qquad \text{for } \begin{matrix} i = 1,2,3,4,5 \\ j = 6 \end{matrix}$$

$$\left[g'_{ij}\right] = \left[1\right]^t \left[g_{ij}\right] \qquad \text{for } \begin{matrix} i = 6 \\ j = 7,8 \end{matrix}$$

$$\left[g'_{66}\right] = \left[1\right]^t \left[g_{66}\right]\left[1\right]$$

and

$$\left[h'_{ij}\right] = \left[1\right]^t \left[h_{ij}\right] \qquad \text{for } \begin{matrix} i = 6 \\ j = 1,\ldots,8 \end{matrix}$$

$\left[1\right]$ is the direction cosine matrix given by equation (3.26), evaluated from the value of the angle which the normal n makes with the x axis. The matrices g_{ij} and h_{ij} are the partitioning matrices of element mixed matrices $\left[g\right]$ and $\left[h\right]$.

Figure (7.6) shows the flow diagram for this subprogram. Subroutine Cosd is called to calculate the direction cosine matrix $\left[1\right]$. The four subprograms Matmult, Matmult1, Matmult2 and Matmult3 are called up to evaluate the products $[1]^t [g_{ij}][1]$, $[1]^t [g_{ij}]$, $[g_{ij}][1]$ and $[1]^t [h_{ij}]$ respectively.

Variables list:

Ge(*)	Element array $[g]$
He(*)	Element mixed array $[h]$
Be	Angle (DEGREE)
W	Node number (1,2,3,4,5,6,7 or 8)

7.8.5 Subprogram Meform

The element consistent mass matrix equation (6.53), is formed by calling up subprogram Meform.

Variables list:

D(*)	Array of material densities
Th(*)	Material thicknesses
X(*), Y(*)	Nodal coordinates array
N(*)	Nodal connection array
Z	Element counter
Me(*)	Element mass matrix
U	Counter for numerical integration

Figure (7.8) shows the flow diagram for subprogram Meform.

7.8.6 Subprogram Loadap

Description: This subprogram evaluates the element load vector and adds contributions to global load vector from element load vector. Two types of loading may be accomodated. Firstly at each node a load in the z direction may be input. Secondly, a distributed load acting normal to the plate (i.e. in the z direction) may be applied. Such a loading is converted into equivalent nodal forces by use of the expression (6.55). Data is input at the keyboard.

Variables list:

Type	Specifies type of loading: 1 - for concentrated loads 2 - for uniformly distributed 3 - for general distributed loading
N(*)	Nodal connection array
Ndc(*)	Nodal constraint array - specifies free & constrained nodes

Nelemt	Number of elements
Sf(*)	Shape function array
U	Counter round the integration points
W(*)	Gauss points & weight factors
Re(*)	Element load vector
R(*)	Global load vector
N	Number of nodes with concentrated loading
S1	Node number
Val	Value of the concentrated loads
Nel	Number of elements with distributed loads
P	Load per unit area
El(*)	Array of element numbers with distributed loading
P(*)	Nodal pressure intensities array

Figure (7.9) shows the flow diagram for subprogram Loadap.

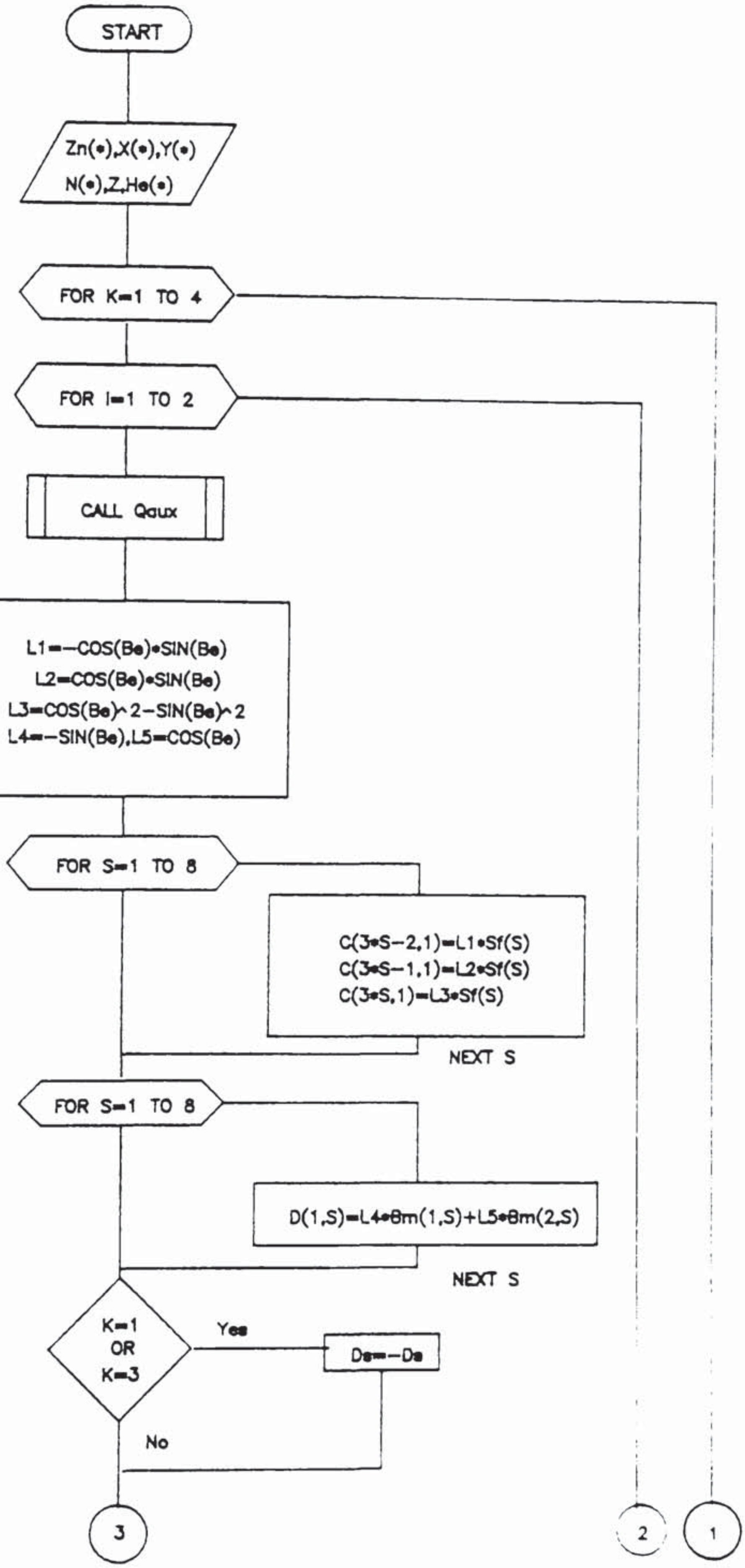
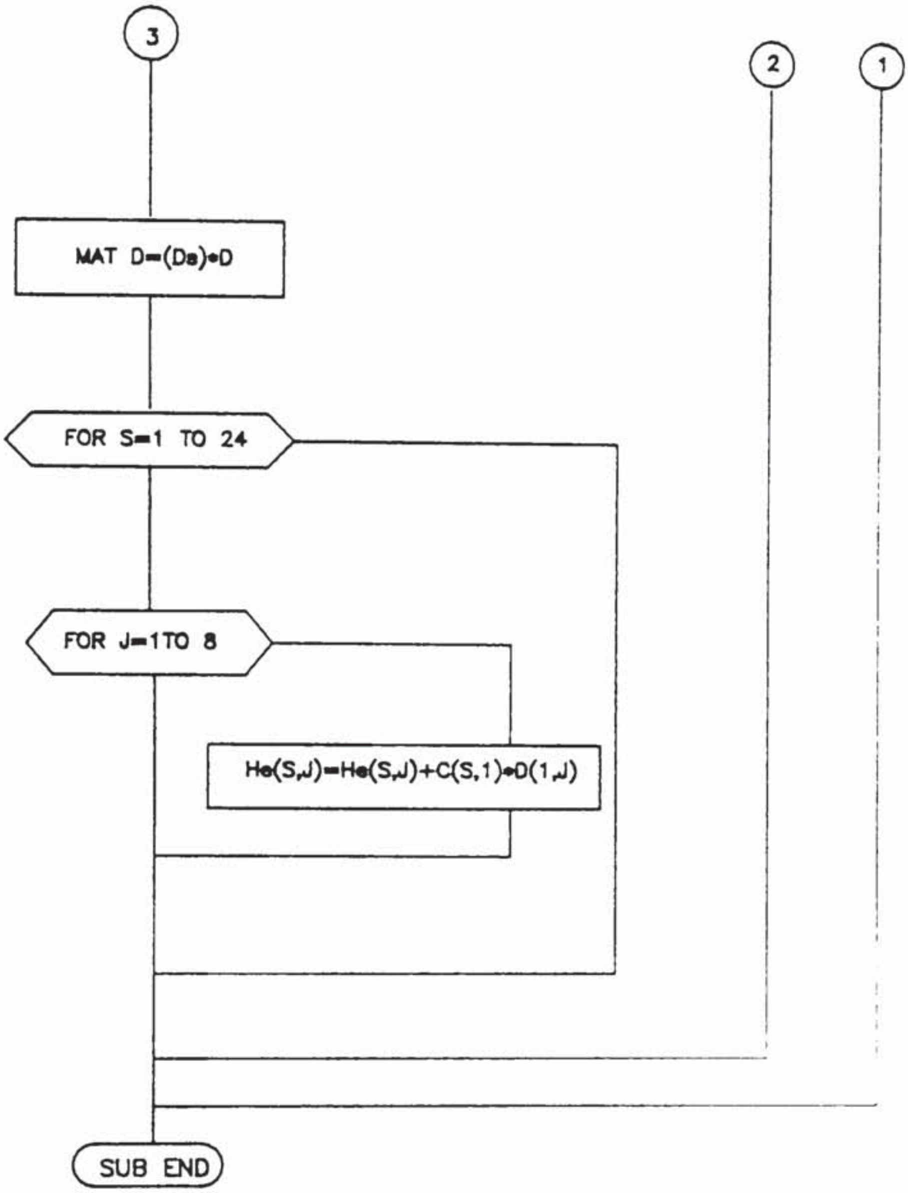


Fig 7.3 Flow diagram for SUB Mnews(He(•),X(•),Y(•),Bm(•),Sf(•),Be

Ds,INTEGER N(•),Z,K)


3
2
1
MAT D=(Ds)*D
FOR S=1 TO 24
FOR J=1TO 8
He(S,J)=He(S,J)+C(S,1)*D(1,J)
SUB END

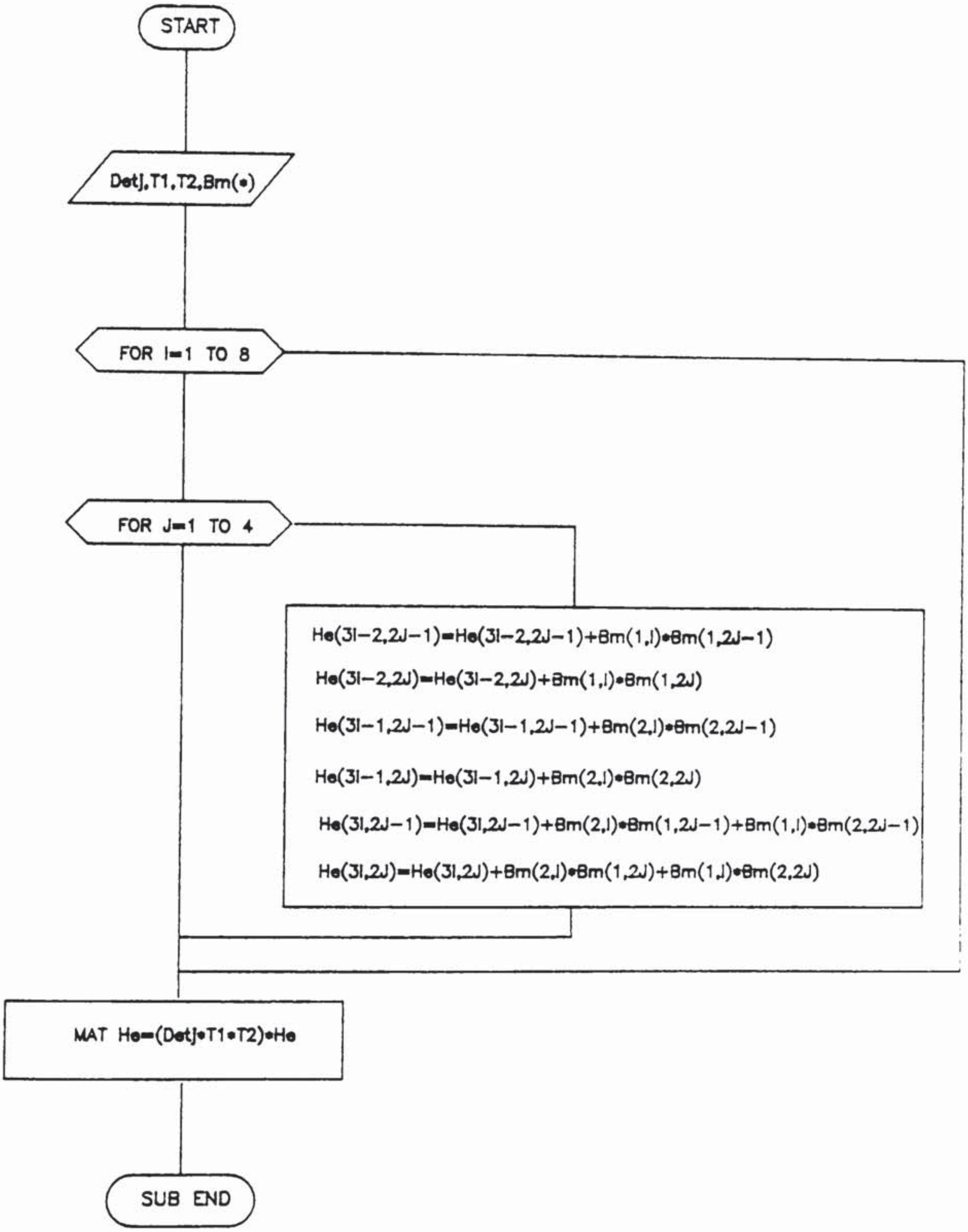


Fig 7.4 Flow diagram for SUB Heform(He(*),Bm(*),DetJ,T1,T2)

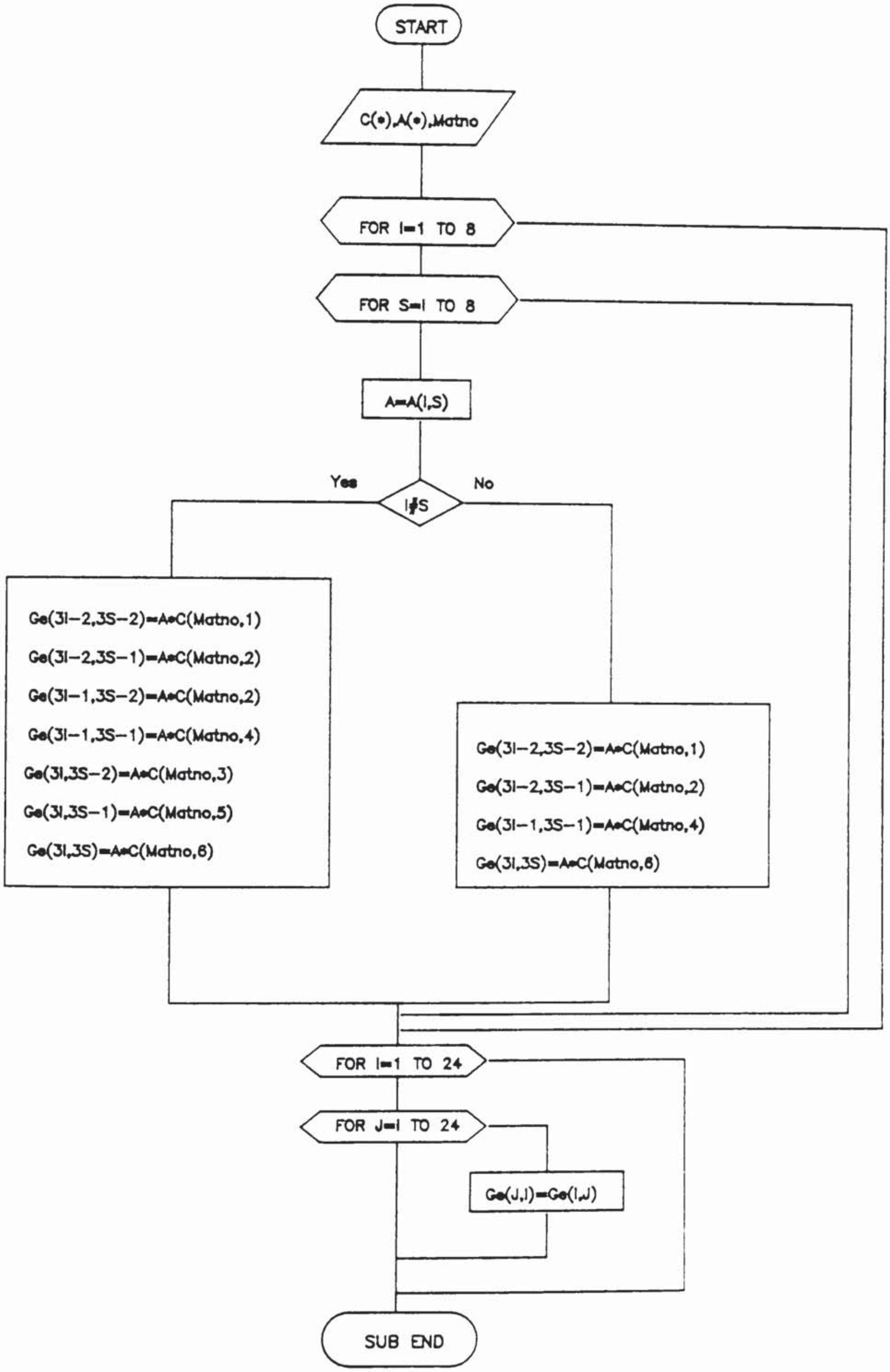


Fig 7.5 Flow diagram for SUB Geform(Ge(*),A(*),C(*),INTEGERMatno)

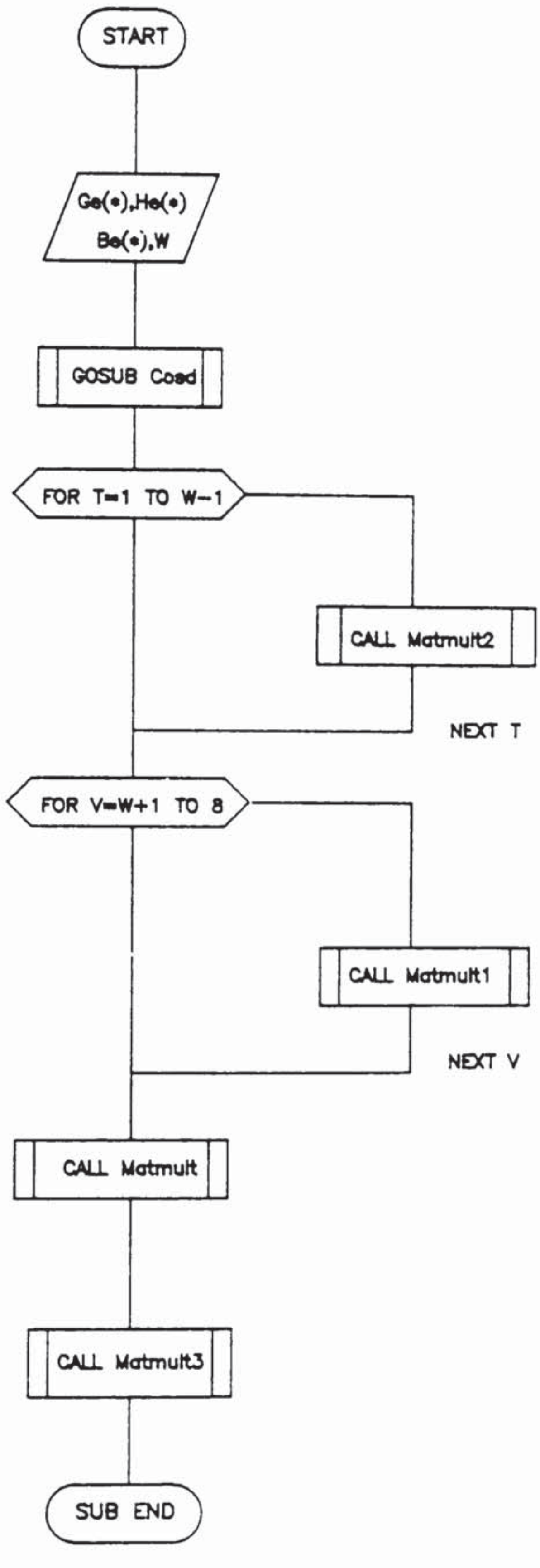


Fig 7.6 Flow diagram for SUB Transf(Ge(•),He(•),Be,W)

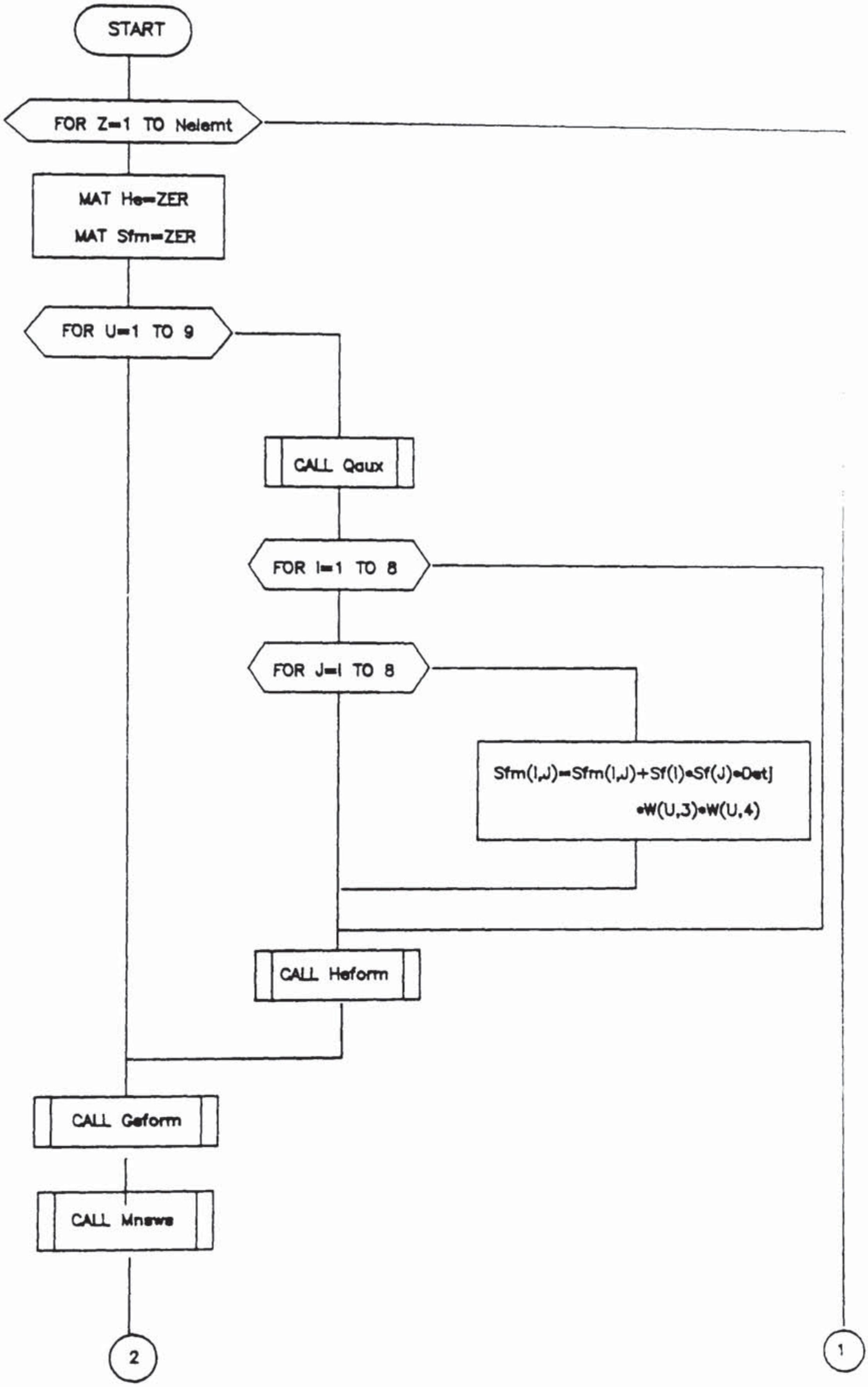
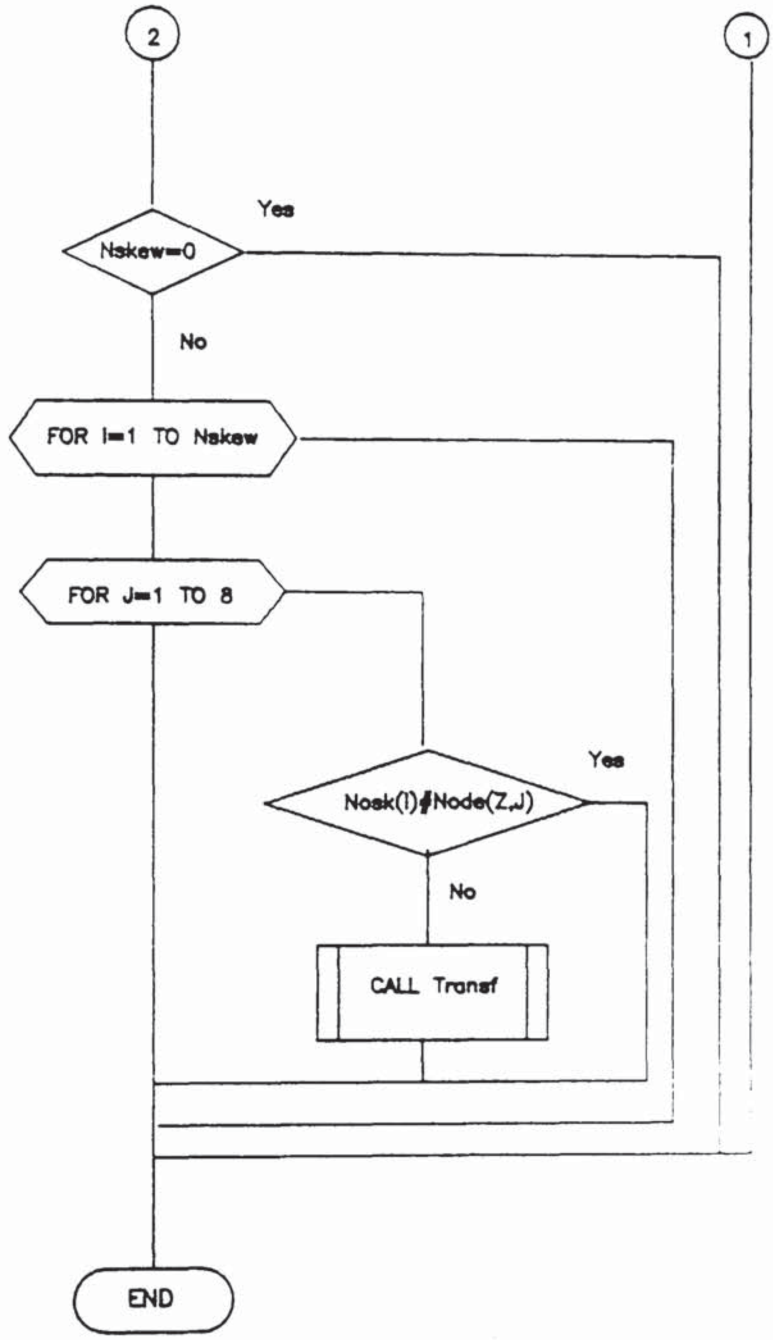


Fig 7.7 Flow diagram for the construction of element matrices:

[g] and [h]


2
1
Yes
Nskew=0
No
FOR I=1 TO Nskew
FOR J=1 TO 8
Yes
Nosk(I)≠Node(Z,J)
No
CALL Transf
END

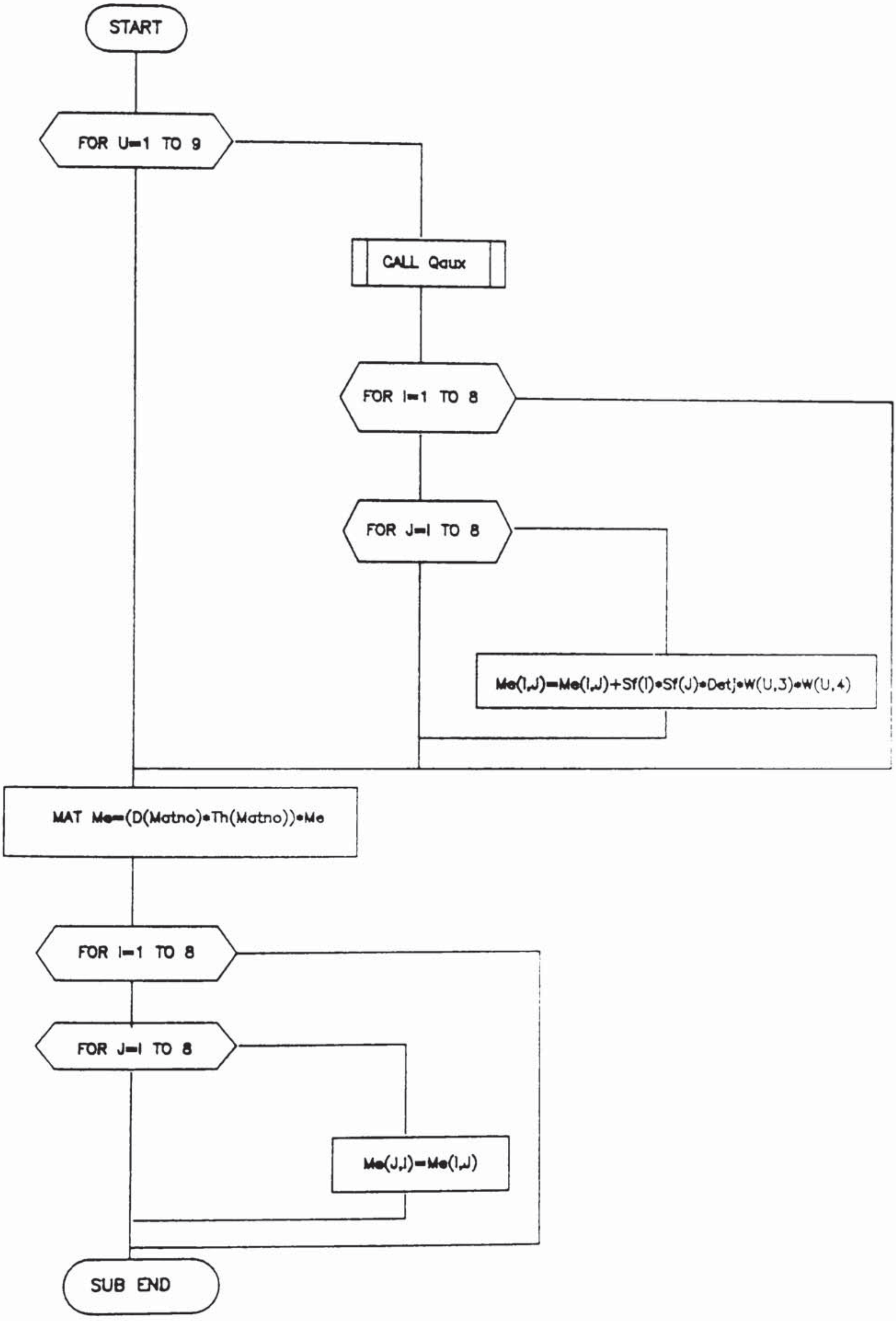


Fig 7.8 Flow diagram for SUB Meform(D(*),Th(*),Me(*),X(*),Y(*)
W(*),DetJ,Sf(*),Bm(*),INTEGER N(*),Matno,Z)

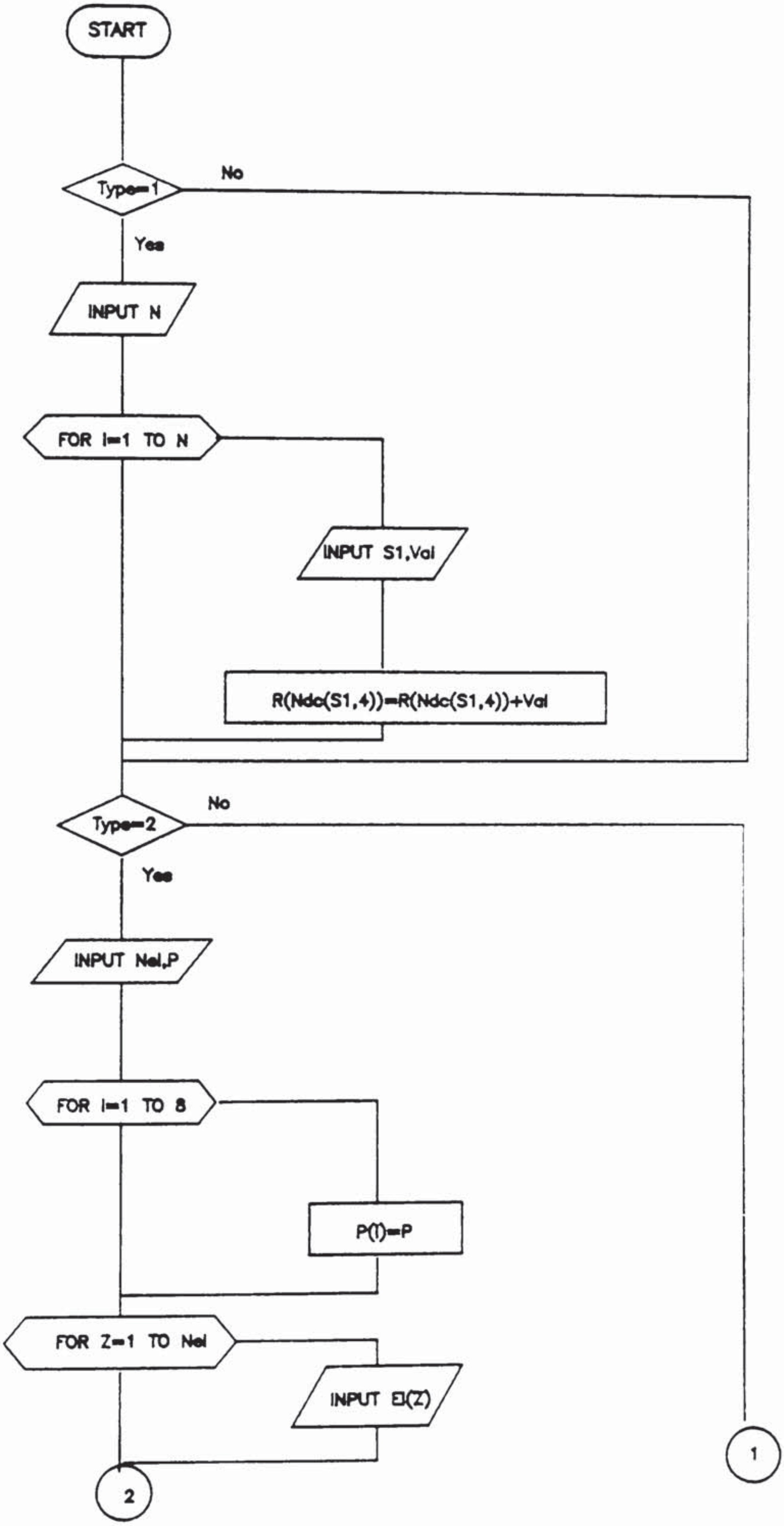
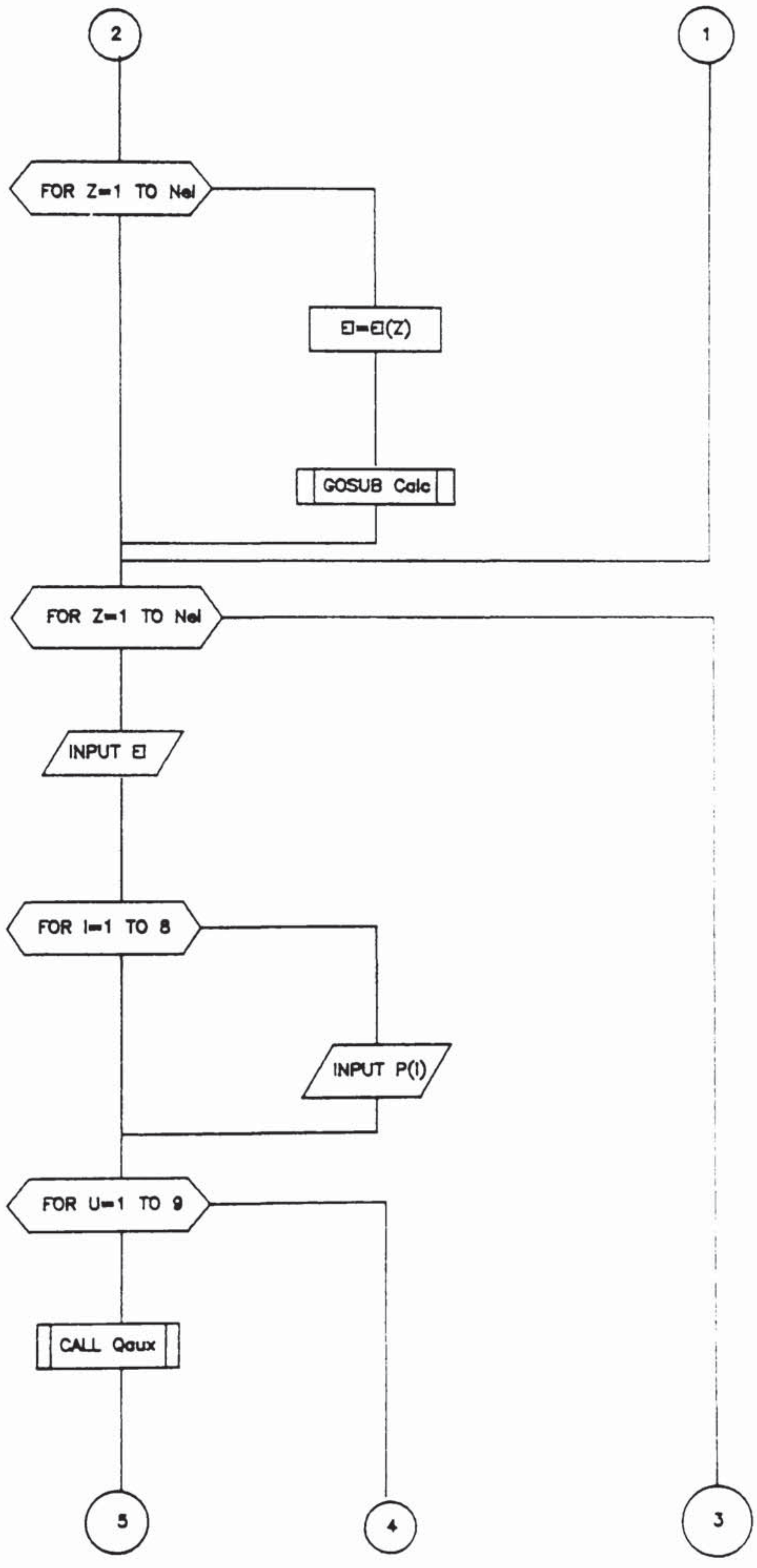
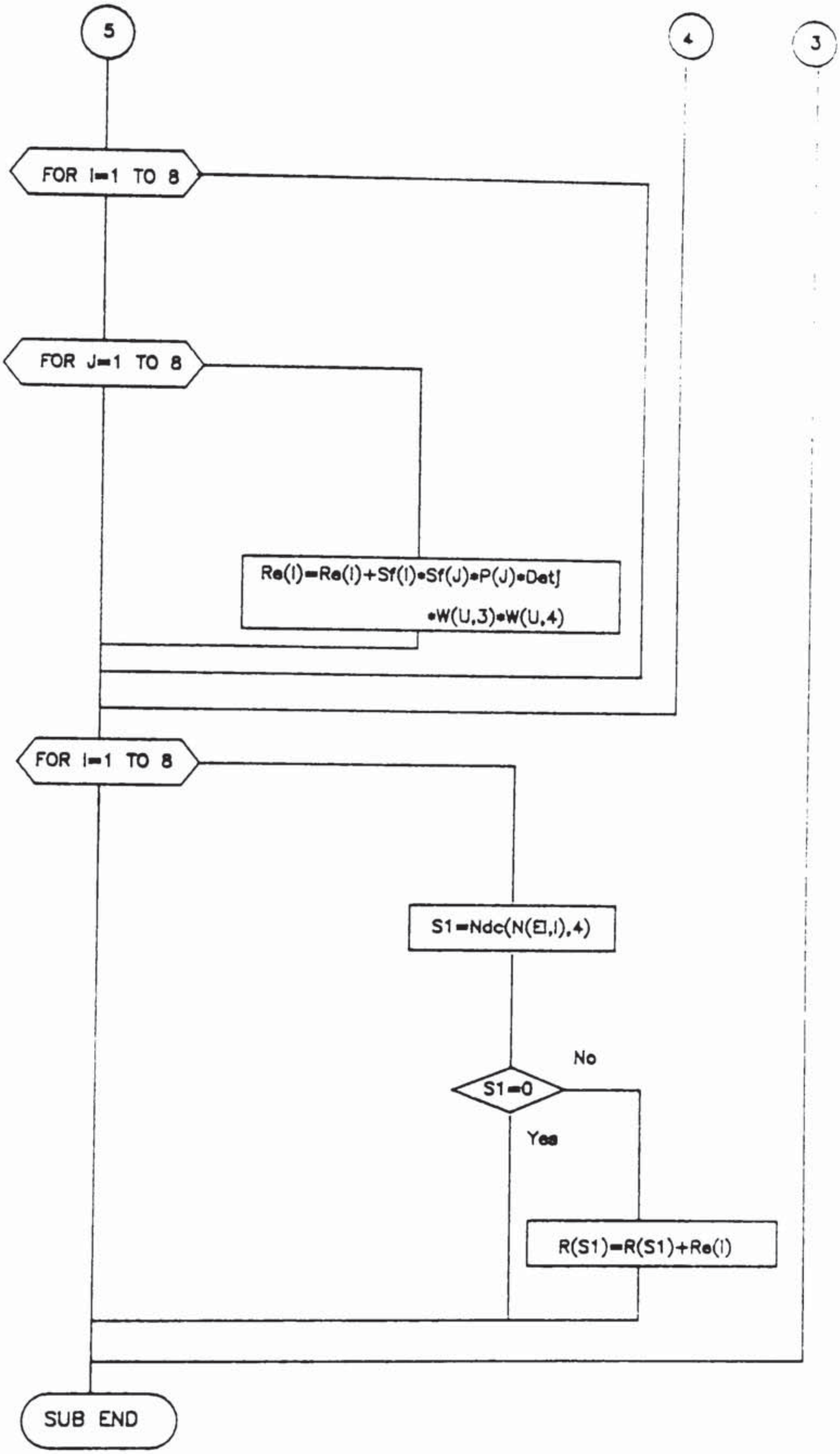


Fig7.9 SUB Loadap(R(*),W(*),X(*),Y(*),Detj,Bm(*),Sf(*),INTEGER N(*),Ndc(*),Nelemt,Nnode)


2
1
FOR Z=1 TO Nel
EI=EI(Z)
GOSUB Calc
FOR Z=1 TO Nel
INPUT EI
FOR I=1 TO 8
INPUT P(I)
FOR U=1 TO 9
CALL Qaux
5
4
3


5
4
3
FOR I=1 TO 8
FOR J=1 TO 8
Re(I)=Re(I)+Sf(I)*Sf(J)*P(J)*Detj
*W(U,3)*W(U,4)
FOR I=1 TO 8
S1=Ndc(N(El,I),4)
S1=0
No
Yes
R(S1)=R(S1)+Re(I)
SUB END

7.9 ALGORITHMS FOR THE ASSEMBLY OF THE OVERALL MATRICES

The overall matrices [G], [H] and [M] are assembled in full matrices by calling up the subprograms Ghasemb and Masemb respectively. Due to symmetry of coefficient matrices [M] and [G], only the lower half is used in the assembly process.

The variables involved in these subprograms are defined as follows:

K(*)	overall mixed matrix [G]
Ke(*)	element mixed matrix [g]
H(*)	overall mixed matrix [H]
He(*)	element mixed matrix [h]
Fm	number of stress degrees of freedom
Z	element counter
N(*)	nodal connection array
Ndc(*)	nodal constraint array
M(*)	element consistent mass matrix
Fw	number of displacement degrees of freedom

Figures (7.10) and (7.11) show the flow diagrams for subprograms Ghasemb and Masemb respectively.

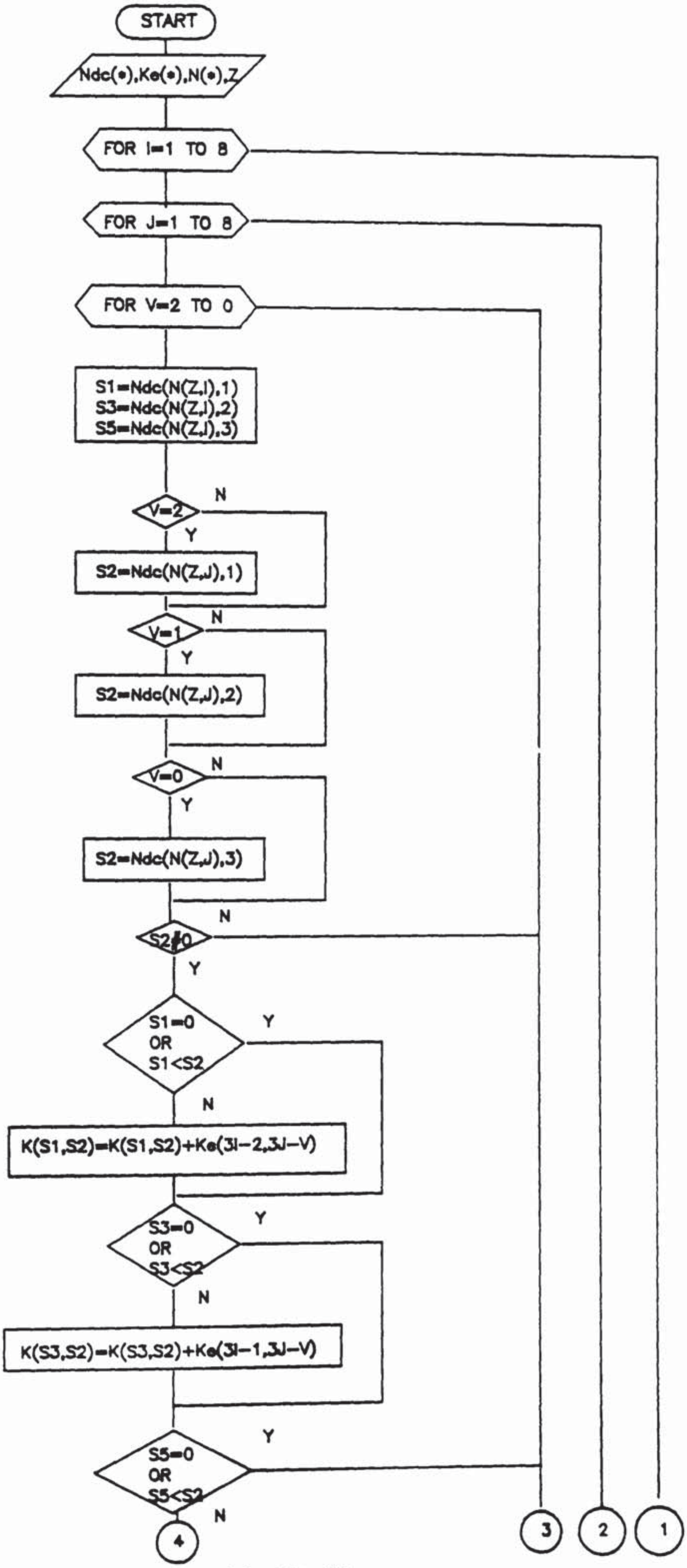
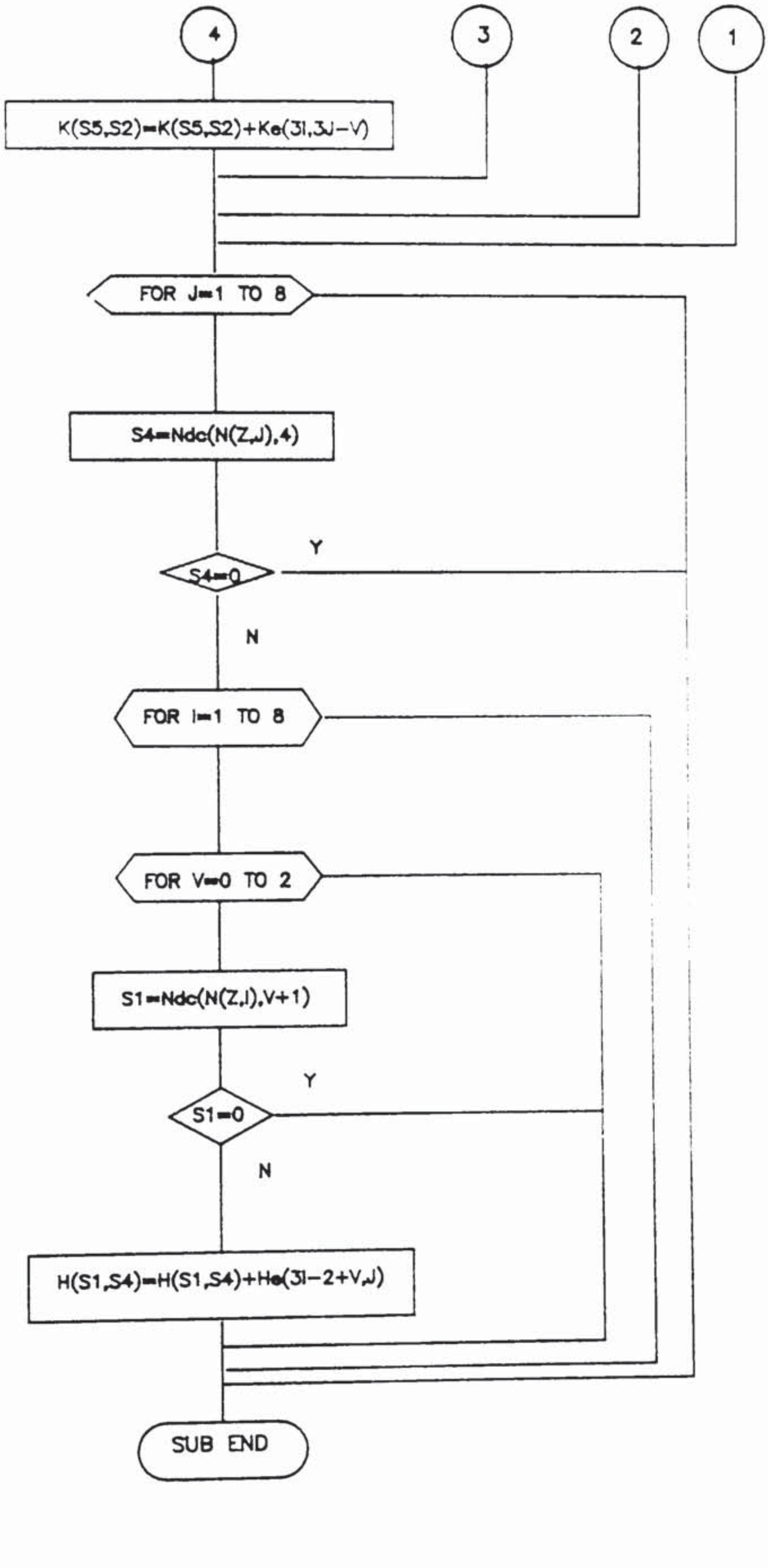


Fig 7.10 Flow diagram for SUB Ghasemb(K(•),Ke(•),H(•),He(•)
INTEGER Fm,Z,N(•),Ndc(•))


4
3
2
1
K(S5,S2)=K(S5,S2)+Ke(3I,3J-V)
FOR J=1 TO 8
S4=Ndc(N(Z,J),4)
S4=0
Y
N
FOR I=1 TO 8
FOR V=0 TO 2
S1=Ndc(N(Z,I),V+1)
S1=0
Y
N
H(S1,S4)=H(S1,S4)+He(3I-2+V,J)
SUB END

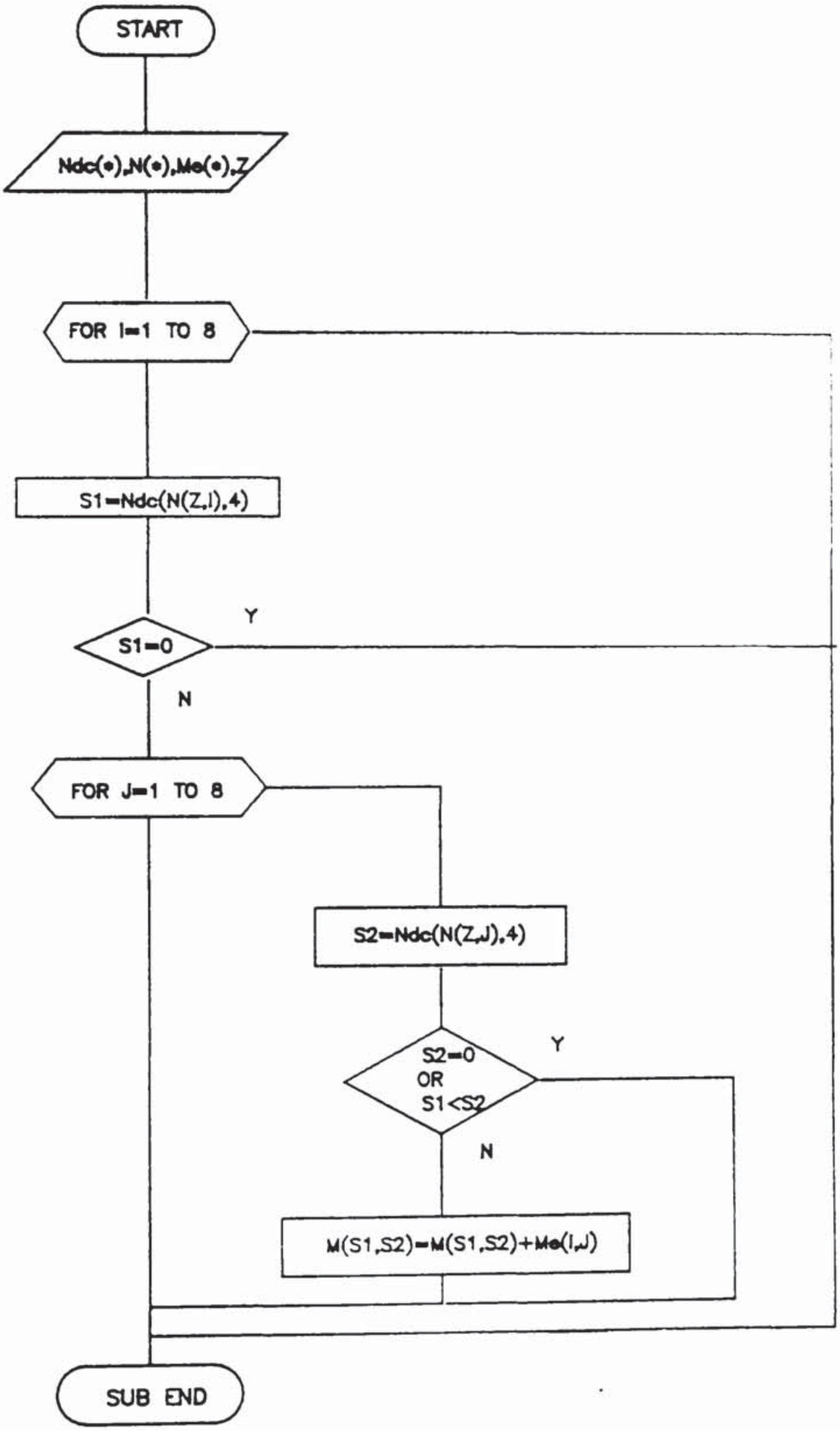


Fig 7.11 Flow diagram for SUB Masemb(M(•),Me(•),INTEGER Fw,Z,N(•)
Ndc(•))

7.10 ELIMINATION OF THE NODAL MOMENT DEGREES OF FREEDOM

The fully populated symmetric positive definite stiffness matrix K* , ($[K^*] = [H]^t [G]^{-1} [H]$) is obtained prior to the solution process. The product $[G]^{-1}[H]$ is calculated by calling up subprogram Eqsolv. This procedure was developed by Wilkinson (68) for the solution of symmetric positive definite equations by Choleski factorization method. The algorithm solves the system of equations $[A][X] = [B]$ where $[A]$ is a symmetric positive definite matrix of order N x N, and $[B]$ is a N x R matrix of R right end sides. The solution $[X]$ overwrites $[B]$. The stiffness matrix $[K^*]$ is then obtained by executing the product $[H]^t[X]$ i.e.

$$[K^*] = [H]^t [X]$$

7.11 SUBPROGRAM Dampmat

This subprogram computes an orthogonal damping matrix for the structure based on known modal damping ratios. Subprograms Trans and Eigen are called by Dampmat to evaluate the structure's normal modes and frequencies. The construction of the damping matrix is then carried out using a procedure described in Appendix A.

Variables list:

C(*)	Damping matrix
M(*)	Mass matrix
K(*)	Stiffness matrix
Vec(*)	Array of eigenvectors
Eval(*)	Array of eigenvalues
Zeta(*)	Array of damping ratios
P	Number of damped modes

Subprogram Dampmat is called by Program RFPLT3. Flow chart for this subprogram is shown in Fig. (7.12).

7.12 SUBPROGRAM Initil

This subprogram computes the initial acceleration vector from a knowledge of initial velocities and displacements. The initial acceleration vector is required when numerical integration is performed by Wilson θ method.

7.13 SUBPROGRAM Wilsnsol

This subprogram uses the Wilson method for numerical integration, described in section (4.6.1), to calculate the displacements at equal time intervals. Matrix inversion is carried out by Choleski factorization method (68). The subprogram Eqsolv1 is called before the solution procedure to triangularize the stiffness matrix according to $[K]=[L][D][L]^t$. The subprogram Eqsolv2 performs the back substitution process to calculate displacements at each time interval.

The variables in this subprogram are:

K(*), M(*), C(*)	Structural stiffness, mass and damping matrices
FØ (*)	Load vector
Ndc(*)	Nodal constraint array
N	Number of displacement degrees of freedom.
Npts	Number of time steps
H(*)	Mixed array $[G]^{-1}[H]$
P(*)	Vector of bending moments calculated from equation (6.68)
AØ (*)	Integration constants
D(*), D1(*), D2(*)	Vector of initial displacements velocities and accelerations

Fig. (7.13) shows the flow diagram for subprogram Wilsnsol.

7.14 SUBPROGRAM Duhammel

This subprogram is called by Program RFPLT2 to perform the numerical integration of the Duhammel integral (Eqn. 4.46). The order of the integration approximation being used is 2 (Trapezoidal rule). This procedure is explained in reference (42).

Listings of programs RFPLT1, RFPLT2 and RFPLT3 including the subprograms presented in this section are given in Appendix C.

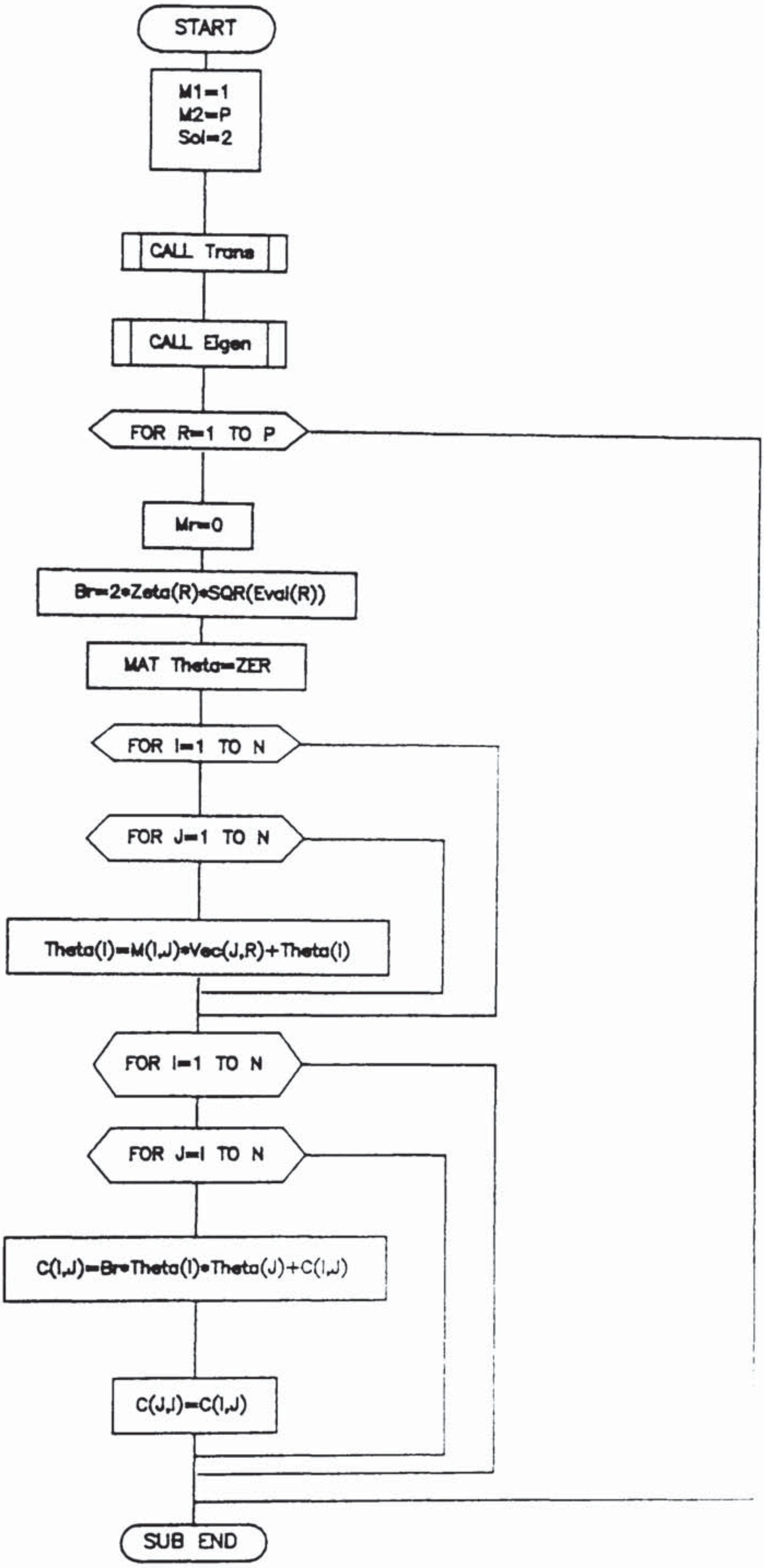


Fig 7.12 Flow diagram for SUB Dampmat(C(*),Vec(*),Eval(*),M(*),
K(*),Zeta(*),D(*),Offd(*),Offd2(*),Dl(*),INTEGER P,N,Type,Sol)

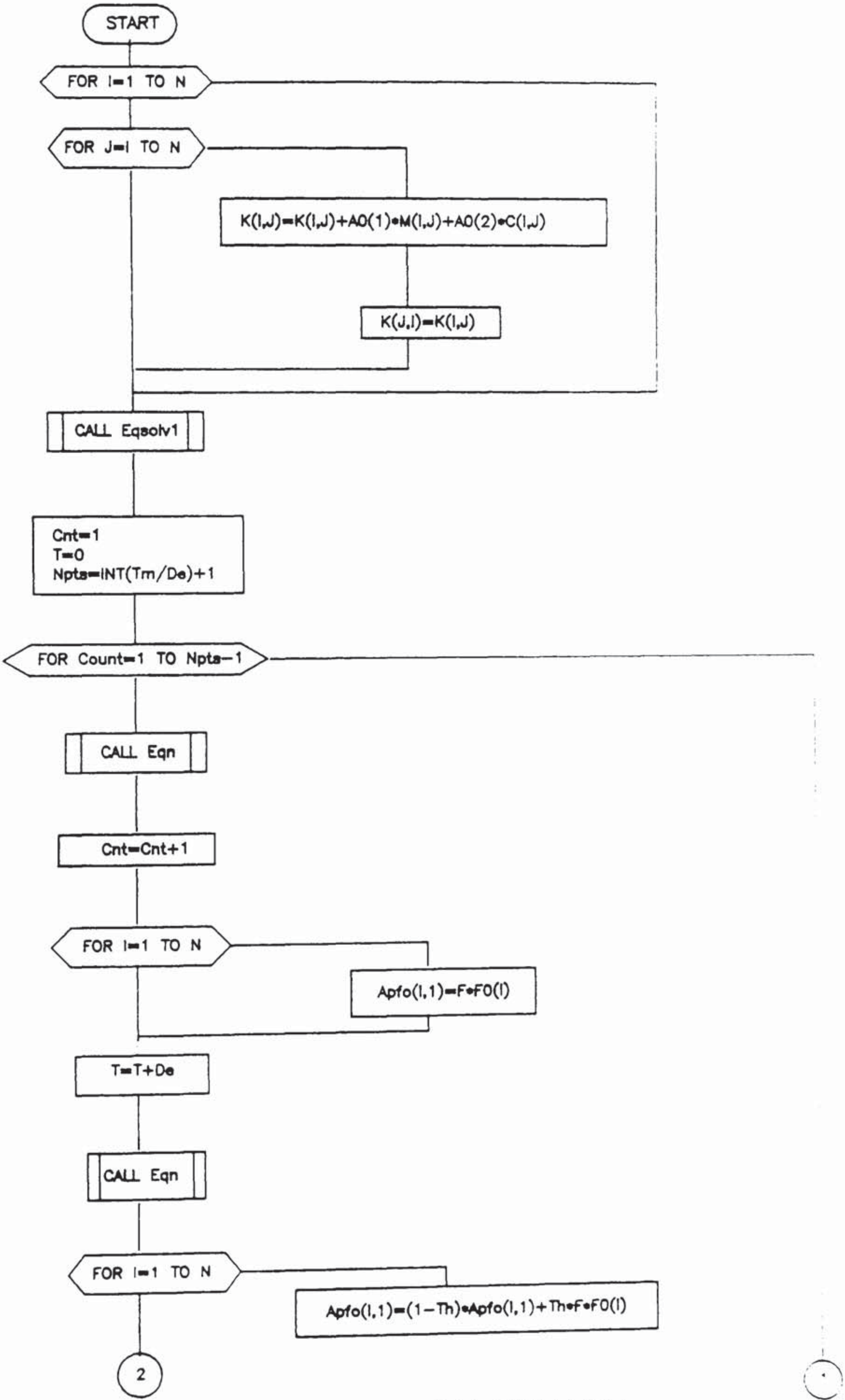
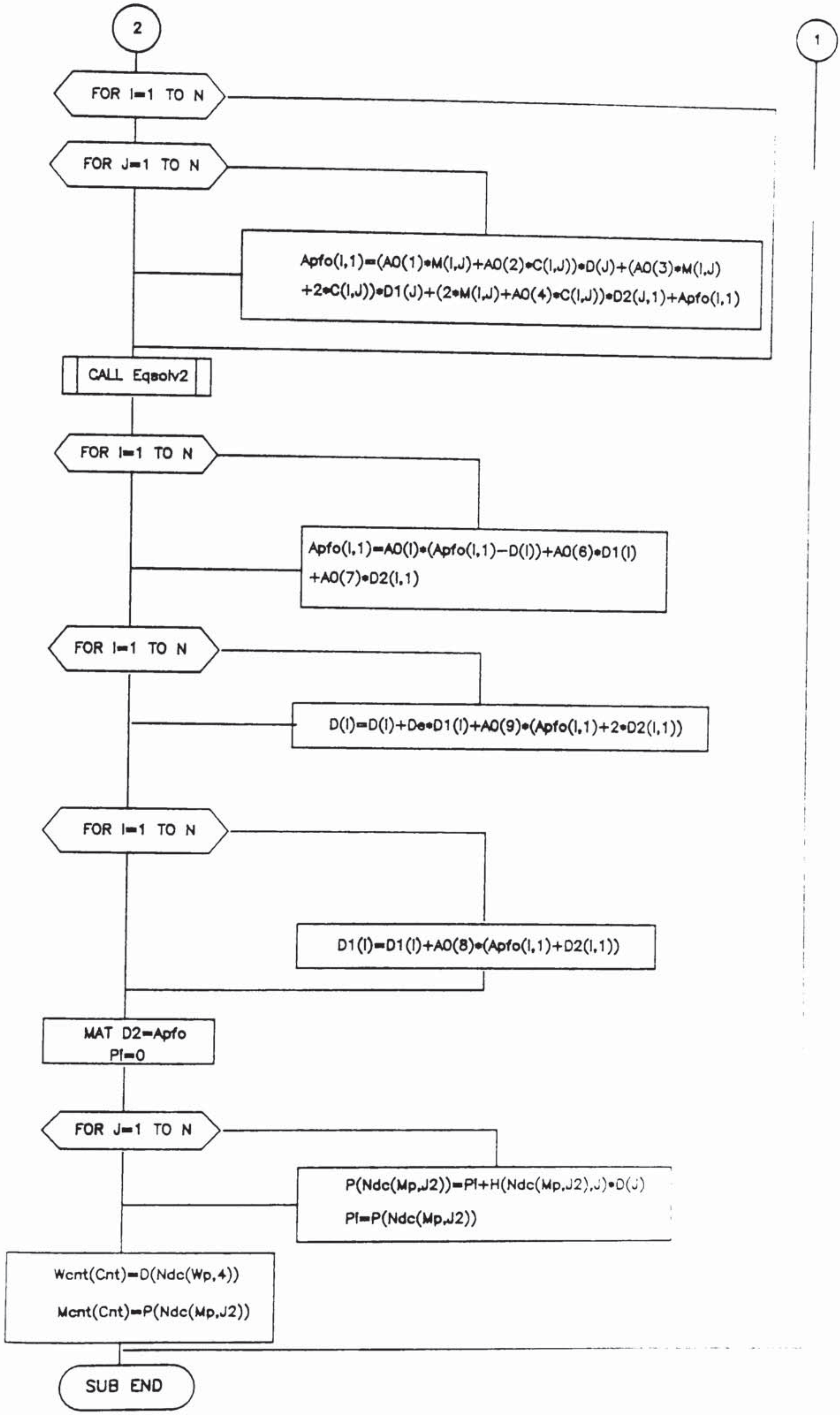


Fig 7.13 Flow diagram for SUB Wilsnsol(K(•),M(•),C(•),Apfo(•), F0(•),D(•),D1(•),D2(•),DI(•),A0(•),Tm,De,Th,K1,P(•),H(•),Wcnt(•) Mcnt(•),INTEGER Ndc(•),Mp,Wp,N,R,J2)


2
1
FOR I=1 TO N
FOR J=1 TO N
Apfo(I,1)=(A0(1)•M(I,J)+A0(2)•C(I,J))•D(J)+(A0(3)•M(I,J)
+2•C(I,J))•D1(J)+(2•M(I,J)+A0(4)•C(I,J))•D2(J,1)+Apfo(I,1)
CALL Eqsolv2
FOR I=1 TO N
Apfo(I,1)=A0(I)•(Apfo(I,1)−D(I))+A0(6)•D1(I)
+A0(7)•D2(I,1)
FOR I=1 TO N
D(I)=D(I)+De•D1(I)+A0(9)•(Apfo(I,1)+2•D2(I,1))
FOR I=1 TO N
D1(I)=D1(I)+A0(8)•(Apfo(I,1)+D2(I,1))
MAT D2=Apfo
Pl=0
FOR J=1 TO N
P(Ndc(Mp,J2))=Pl+H(Ndc(Mp,J2),J)•D(J)
Pl=P(Ndc(Mp,J2))
Wcnt(Cnt)=D(Ndc(Wp,4))
Mcnt(Cnt)=P(Ndc(Mp,J2))
SUB END

CHAPTER 8

APPLICATIONS OF MIXED BEAM & PLATE FINITE ELEMENTS IN FREE AND FORCED VIBRATION PROBLEMS

8.1 INTRODUCTION

In order to determine the convergence and accuracy of the developed finite element models, the numerical solutions so derived are compared with the available analytical and, other accepted, finite element model solutions. This comparison has been made with reference to displacement type finite element models for the following groups of problems:

1. Free vibration of beams.
2. Forced vibration of beams.
3. Free vibration of thin plates.
4. Forced vibration of thin plates.

Beam elements were shown in Table (6.1). Figure (8.1) shows the types of elements used in the solution of plate problems. QR4 and QR8 elements, represent the linear and quadratic mixed plate elements. QD4 and QD8 are the non-conforming displacement type element, using 12 and 24 term polynomials as the assumed displacement functions, (Ref (9).

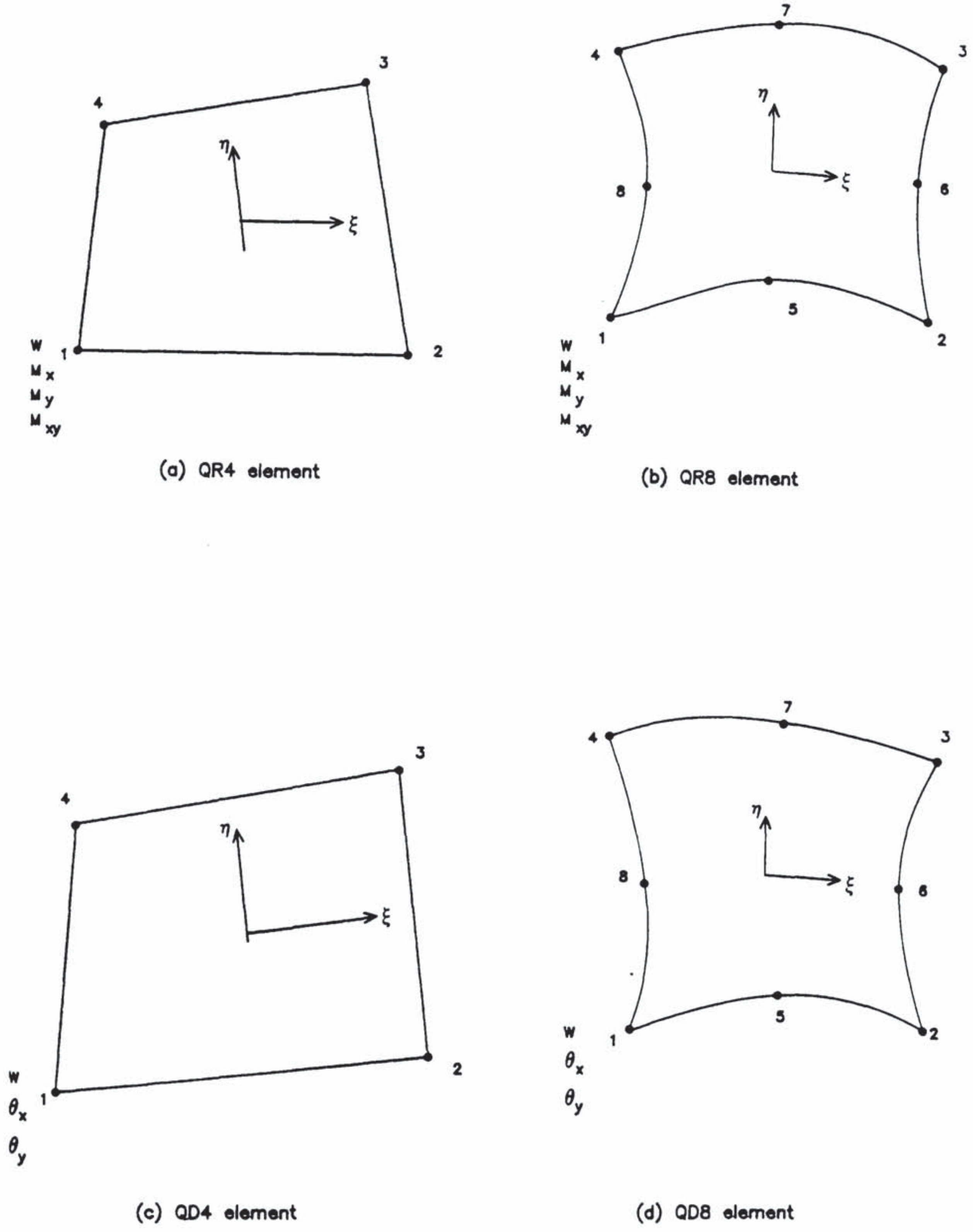


Fig 8.1 Types of plate bending elements.(a,b)—mixed elements
(c,d)—non conforming displacement elements.

8.2 NUMERICAL EXAMPLES ON FREE VIBRATION OF BEAMS

Several mixed beam elements were presented in Section 6.2 with various combinations of interpolations for the deflection w and bending moment M_x. In applying the elements to free vibration problems, it was found that the two elements MB7 and MB8 (defined in Section 6.2) produce erroneous results, and failed the convergency test. For these elements, the element mixed matrix $[h]$ in equation (6.10b) is found to be of the form

$$[h]_{MB7} = \begin{bmatrix} \frac{1}{l} & 0 & \frac{-1}{l} \\ \frac{-1}{l} & 0 & \frac{1}{l} \end{bmatrix} \qquad (a)$$

and (8.1)

$$[h]_{MB8} = \begin{bmatrix} \frac{1}{l} & \frac{-1}{l} \\ 0 & 0 \\ \frac{-1}{l} & \frac{1}{l} \end{bmatrix} \qquad (b)$$

The zero column and the zero row in the element matrices $[h]_{MB7}$ and $[h]_{MB8}$ reduce the rank of these matrices. Thus the element stiffness matrix derived from equation (8.2) will become deficient in rank.

$$[k] = [h]^t [g]^{-1} [h] \qquad (8.2)$$

The cause of failure can be attributed to the existence of the zero energy modes which do not correspond to the expected rigid body motion. The characteristics of such unwanted modes may be determined by carrying out an eigenvalue-eigenvector analysis on an individual unconstrained element.

The incorrect zero eigenvalue modes for element MB7 is shown in Figure (8.2a). For this mode the bending moment is constant throughout the element, whereas the displacement is varying.

Figure (8.2b) shows that a constant displacement and variable moment distributions are obtained for the zero eigenvalue mode of element MB8. It is anticipated that these modes can not be removed by applying the kinematic boundary conditions and therefore contribute to the misbehaviour of the aforementioned elements. The correct zero eigenvalues for the expected rigid body modes were obtained for other displacement-stress combinations. The zero energy mode for the element MB5 is shown in Figure (8.2c).

The results for elements other than MB7 and MB8 compare favourably with displacement type element. Figures (8.4) to (8.9) compare the accuracy of the mixed elements with the displacement element in predicting the fundamental natural frequency of the cantilever beam shown in Figure (8.3). The convergence curves correspond to degrees of freedom of the final eigenvalue problem and the total number of degrees of freedom of the models. It can be concluded that the natural frequencies predicted by the mixed element models are converging to the exact values. Also from talbes (8.1) and (8.2), it is observed that better accuracy in predicting the first 3 natural frequencies of the cantilever beam can be achieved by using fewer higher order elements than lower order elements.

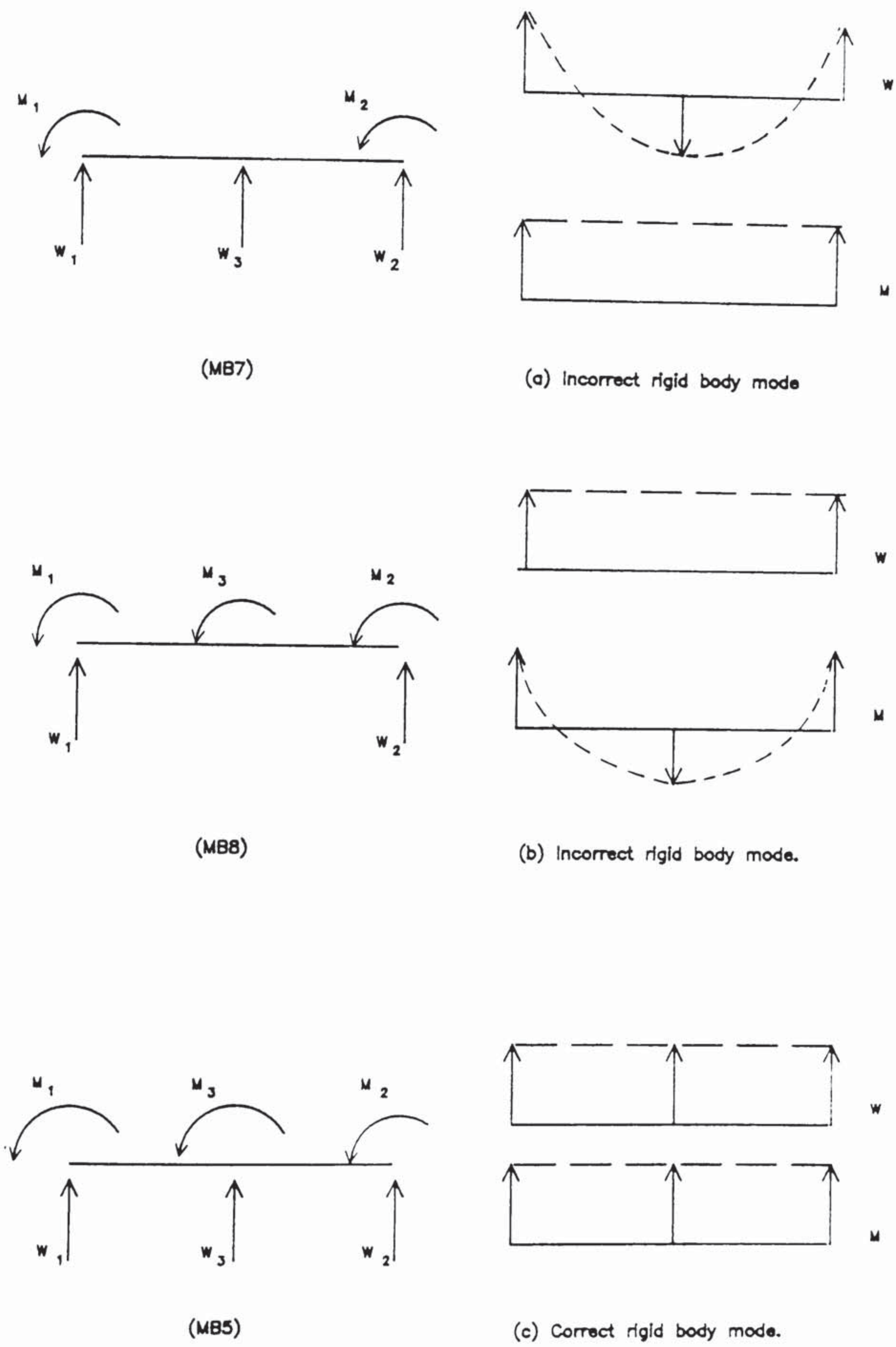
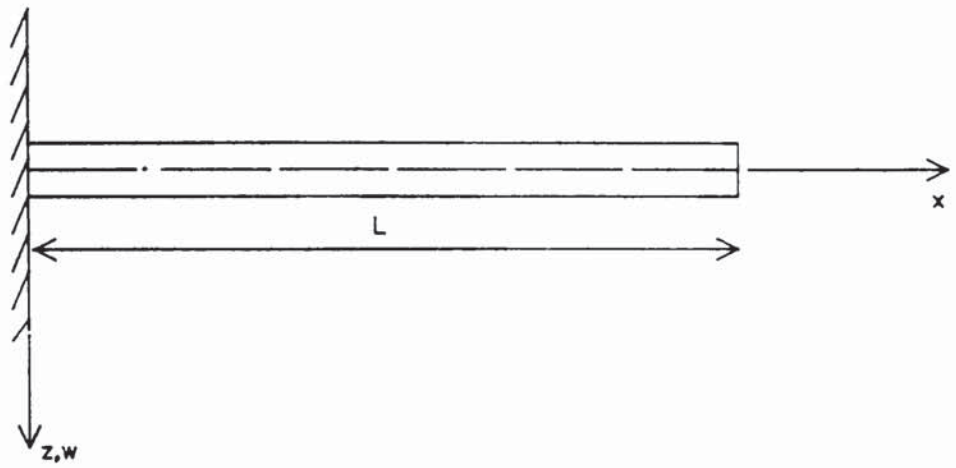


Fig 8.2 Zero energy modes in elements MB7(a),MB8(b) and MB5(c)



Beam properties:

L=80 Cm

ρ=7.8x10^-3 Kg/Cm^3

E=2.07x10^7 N/Cm^2

A=1 Cm^2

I=1/12 Cm^4

Exact natural frequencies(Rad/Sec):

ω_1=8.1689

ω_2=51.1979

ω_3=143.37

Fig 8.3 Cantilever beam used in free vibration tests.

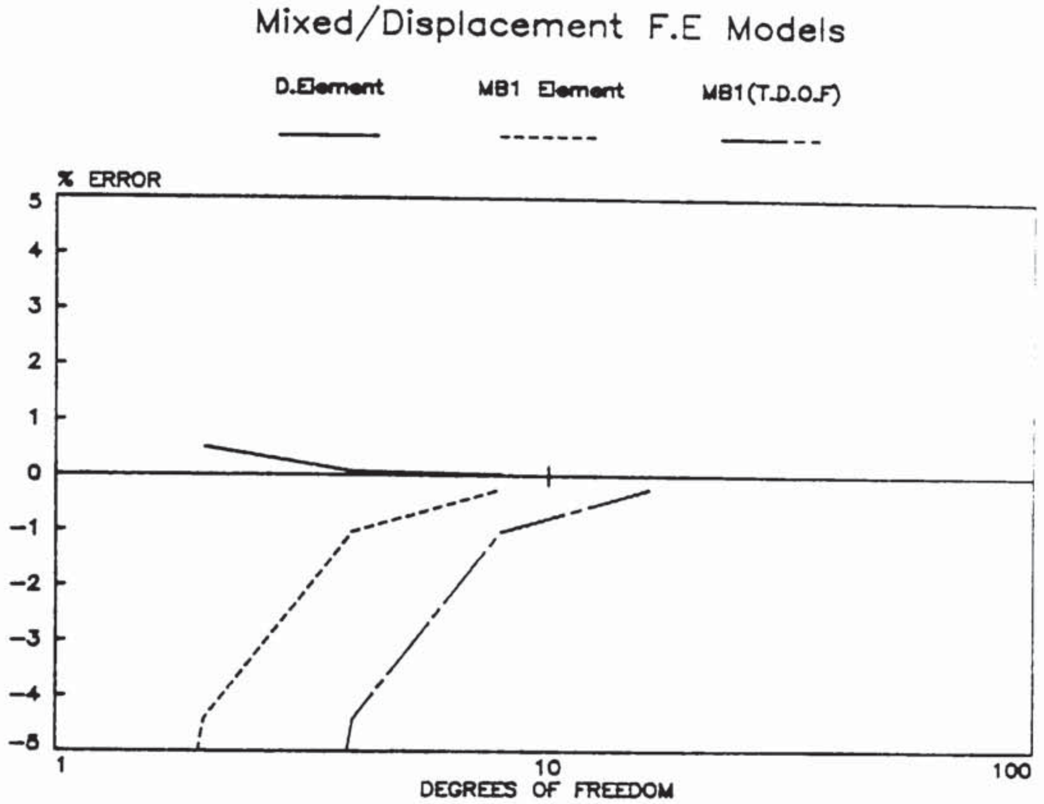


Fig 8.4 Prediction of 1st nat. freq. of CF beam :Elements D & MB1

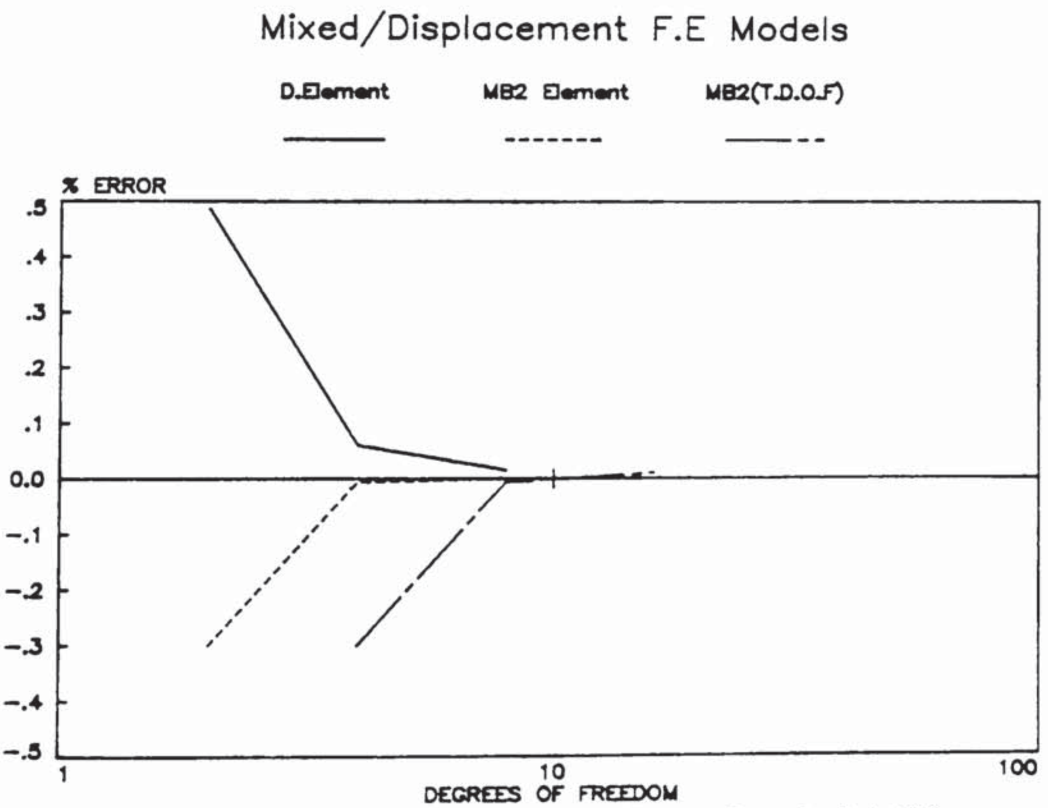


Fig 8.5 Prediction of 1st nat. freq. of CF beam:Elements D & MB2

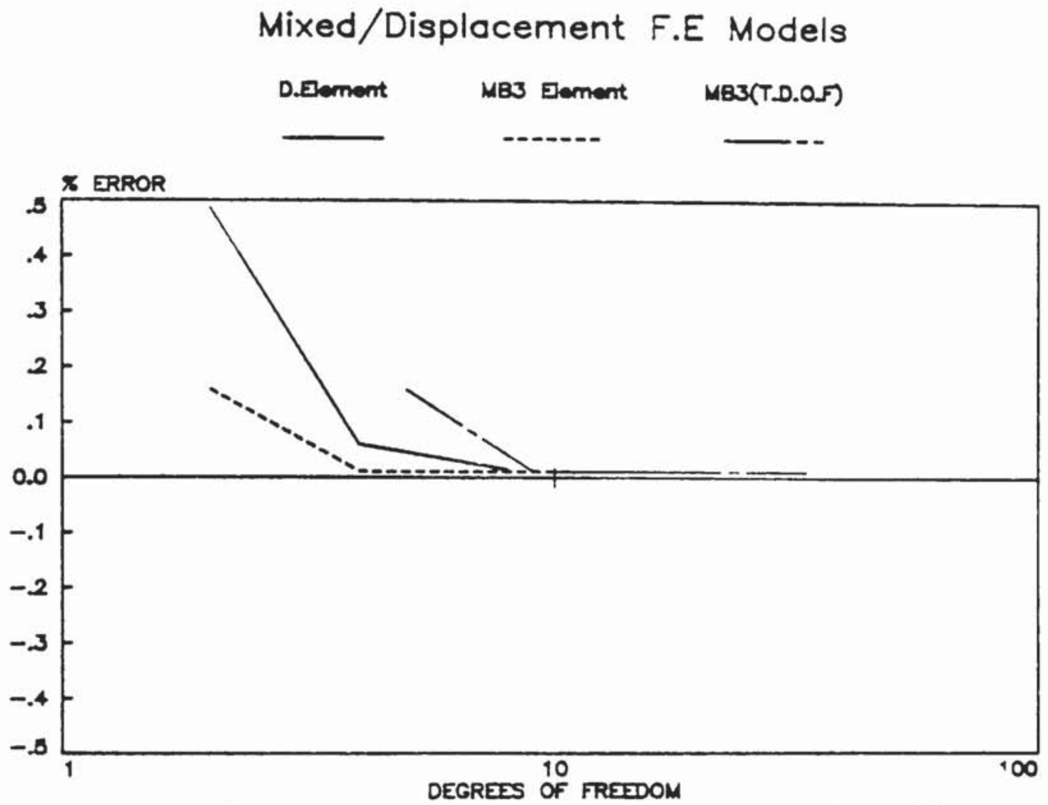


Fig8.6 Prediction of 1st nat. freq. of CF beam:Elements D & MB3

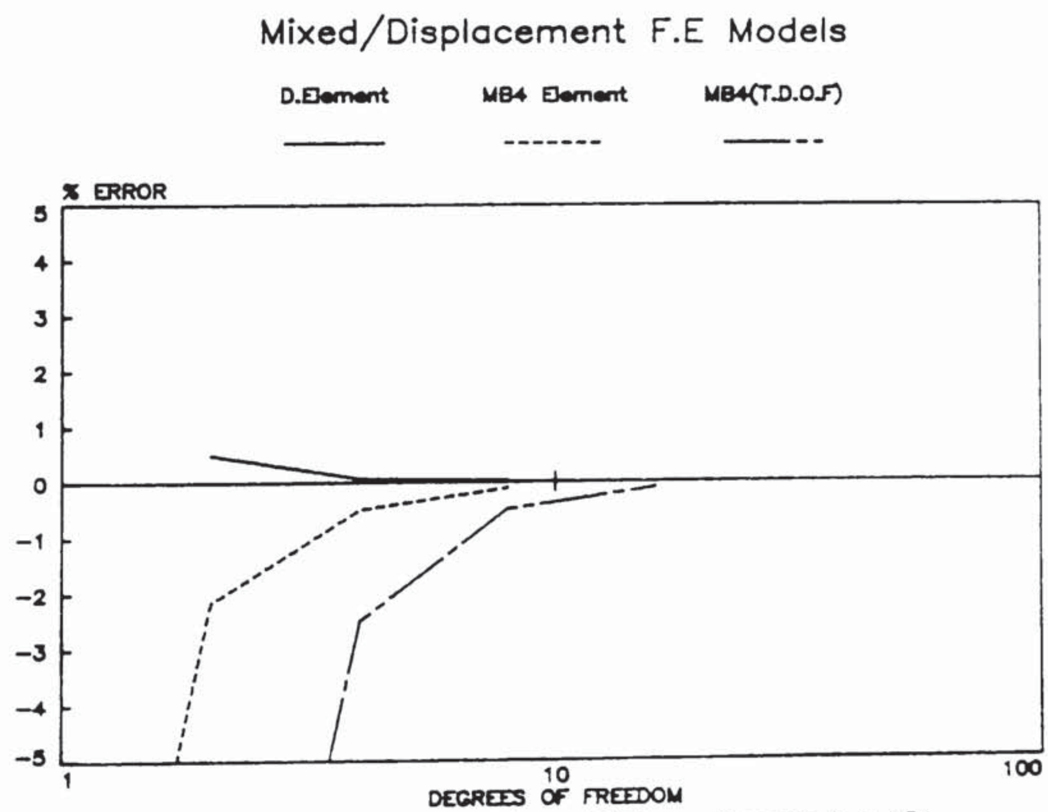


Fig8. 7 Prediction of 1st nat.freq. of CF beam:Elements D & MB4

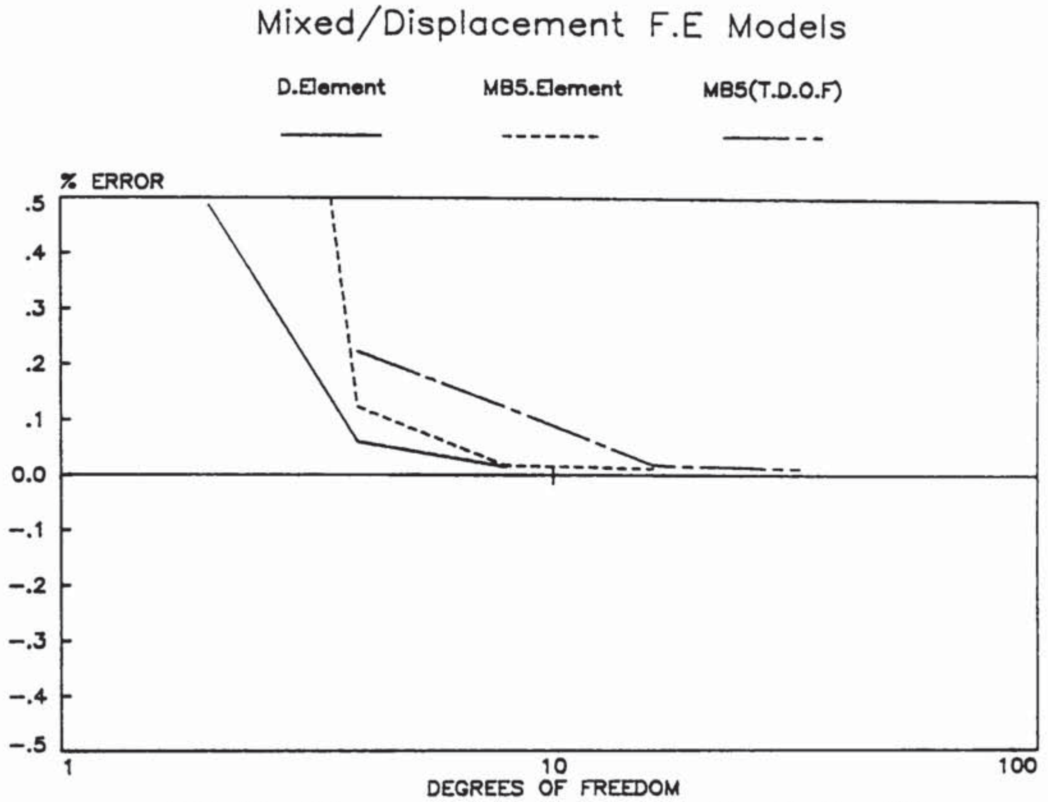


Fig8.8 Prediction of 1st nat.freq. of CF beam:Elements D & MB5

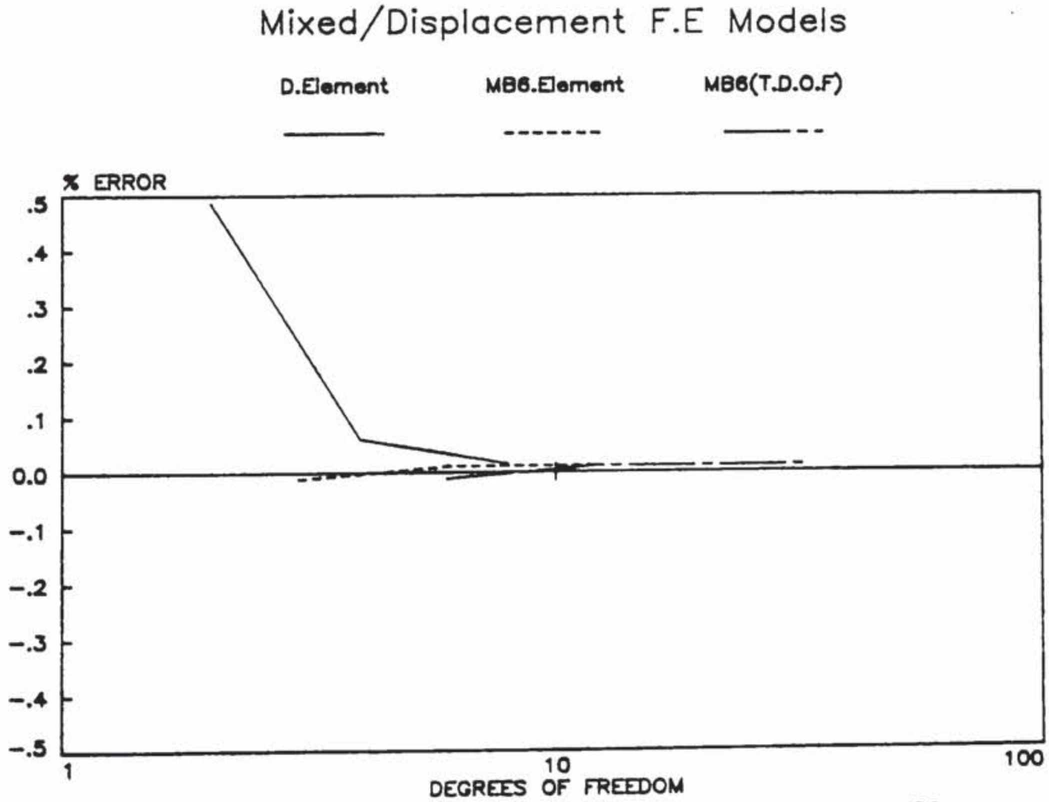


Fig8.9 Prediction of 1st nat.frequ. of CF beam:Elements D & MB6

Table 8.1

% Error in the first 3 natural frquencies of cantilever beam.

C1 elements

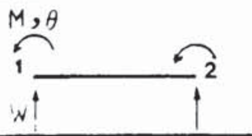
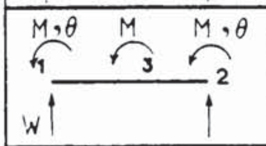
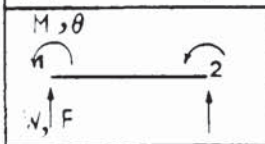
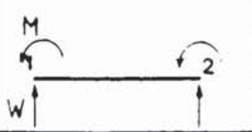
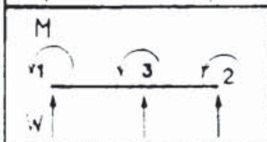
Type of Element	Number of Elements	Total Degrees of Freedom	Mode 1	Mode2	Mode3
	8	16	−.241	.70	3.291
	4	16	.01	.0626	−.079
	3	13	−.011	.0589	.06

Table 8.2

% Error in the first 3 natural frequencies of cantilever beam.

C0 elements

Type of Element	Number Of Elements	Total Degrees Of Freedom	Mode1	Mode2	Mode3
	8	16	−.1037	1.680	5.76
	4	16	.0177	.265	2.012
	2	12	.0107	.0328	2.1

8.3 NUMERICAL EXAMPLES ON FORCED VIBRATION OF BEAMS

In this section several examples of beam bending problems are solved using the developed finite element computer programs. These tests are aimed at illustrating the behaviour of mixed type beam elements in thesolution of forced vibration problems.

The results are compared with the analytical and displacement element solutions. Table (8.3) shows the types of problems which have been tackled. Unless otherwise specified, the constants used in these solutions are:

Beam: $E = 2.07 \times 10^7 \ ^N/_{cm^2}$, $\nu = 0.3$, $\rho = 7.8 \times 10^{-3} \ ^{Kg}/_{cm^3}$

$A = 100cm^2$, $I = \frac{1}{12} \ cm^4$, $L = 80 \ cm$

The numerical integration of the equations of motion is performed by Wilson θ method using a time step size of $\Delta t = .001$ sec.

8.3.1 Response of a cantilever to a transient force (half-sine pulse input)

The tip deflection and maximum bending moment at the root section of a uniform cantilever beam subjected, at the tip, to the transient force, shown in Table (8.3), are calculated. (T is the period of the fundamental mode of vibration of the cantilever). Figures (8.10) and (8.11) show the tip delfection and the root bending moment responses for models with two degrees of freedom. A two element idealisation is used for element MB4 and one element idealisation for the rest of the elements (D, MB5, MB2 and MB3). The results show that the displacements are predicted with good accuracy and the bending moments predicted by mixed elements are generally superior to those obtained from the displacement element (for the same number of degrees of freedom). However, for one mixed element, MB3, the displacement and moment predictions are similar to the displacement element, D, predictions.

The convergence of the results is studied by increasing the number of degrees of freedom. Figures (8.12) and (8.13) show the deflection and bending moment responses for the same cantilever with 4 degrees of freedom idealisations (only displacement d.o.f.). It is seen that the predictions from mixed elements (MB2) and MB3, converge more rapidly than those from MB4 and MB5. In this case, the total number of degrees of freedom (displacements and moments) used in the idealisations with mixed elements is 8.

8.3.2 Response of a cantilever to a ramp force input at the tip

The cantilever beam of the previous example is tested for a ramp force input at the tip. Bending moment response at the root of the beam and the tip deflection response are calculated for various finite element models with 2, 4 and 6 degrees of freedom. The results are shown in Figures (8.14) to (8.18). It is observed that the displacement solutions converge very rapidly towards the exact solution, for all types of finite element models (D, MB4, MB5, MB2 and MB3). In particular, Figure (8.14) shows that the displacement response prediction obtained by using element MB5 is much more accurate than those obtained by using other models including the displacement type element.

Figures 8.15, 8.17 and 8.18 show the bending moment response for finite element models with 2, 4 and 6 degrees of freedom respectively. It is interesting to notice that, for idealisations with 2 degrees of freedom, the predictions from mixed elements MB4 and MB5 (from C0 continuity class) are superior to solutions from other types of elements (Figure 8.15).

8.3.3 Response of a clamped-clamped beam to a step force input

Figures (8.19) to (8.21) show the deflection and bending moment responses of a clamped-clamped beam to a step force input applied at the middle of the beam. It is observed that the models exhibit good accuracy even with the lowest number of degrees of freedom. Besides, elements MB2 and MB3 of the C1 continuous class show similar accuracy to the displacement type element .

8.3.4 Response of a clamped-simply supported beam to half sine pulse input

Mid point deflection and bending moment responses for a CS beam subjected to half sine pulse input are shown in Figures (8.22) to (8.25). The convergence characteristics of various finite elements are studied by increasing the number of degrees of freedom from 3 in Figures (8.22), (8.23) to 7 in Figures (8.24) and (8.25). The results for the displacements show that for the same number of unknowns, the accuracy of the mixed elements matches that of the displacement type element. Bending moments, however, are calculated more accurately using the C1 continuous mixed elements (MB2, MB3). In addition, CØ continuous mixed elements are found to perform just as well, with the parabolic, MB5 element being superior to the linear MB4 element.

8.3.5 Response of a clamped-simply supported beam to a ramp force input. (Damping included)

Figures (8.26) and (8.27) show the mid point deflection and bending moment responses, respectively, for a damped CS beam of length, L = 40 cm. The results are obtained for idealisations with 3 degrees of freedom. Damping is introduced in mode 1 with $\zeta = .05$, modes 2 and 3 with $\zeta = .02$.

Both displacement and bending moment results show that very good accuracy has been obtained.

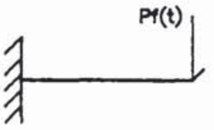
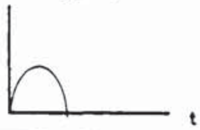
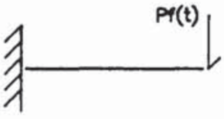
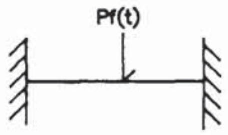
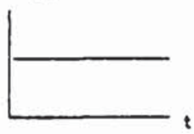
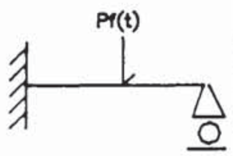
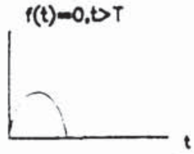
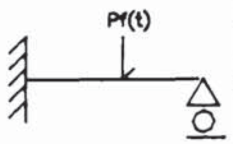
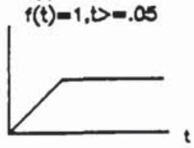
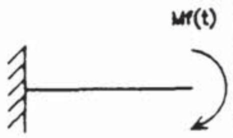
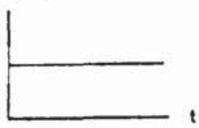
8.3.6 Response of a cantilever to a step moment input at the tip

Figures (8.28) to (8.33) show the deflection and bending moment responses of a cantilever subjected to a step moment input. The tests are performed in order to demonstrate the convergence of the results as the beam sub-divisions increases. The elements used in this example are from C1 continuous class (D, MB1, MB2, MB3). With these elements, the slope continuity between elements' joints is satisfied.

It can be seen that the mixed element MB1 has a weaker convergence rate than the rest of the elements used in this test.

From these applications, it is concluded that, in general, the elements developed for dynamic analysis of beams are capable of predicting the structure response to various force inputs with good accuracy. The rate of convergence in CØ continuous elements (MB4, MB5) is lower than C1 elements (MB2, MB3). Nevertheless, in view of the simplicity in formulation and programming, these elements offer some advantages over the more complex C1 continuous elements.

Table 8.3 Beam forced vibration problems.

Beam type	Type of force inputs.	Type of elements	Figure numbers
	f(t)=Sin(πt/T),t<=T f(t)=0,t>T 	D,MB2 MB3,MB4 MB5	(8.10) to (8.13)
	f(t)=20t,t<.05 f(t)=1,t>=.05 	D,MB2 MB3,MB4 MB5	(8.14) to (8.18)
	f(t)=1 	D,MB2 MB3,MB4 MB5	(8.19) to (8.21)
	f(t)=Sin(πt/T),t<=T f(t)=0,t>T 	D,MB2 MB3,MB4 MB5	(8.22) to (8.25)
	f(t)=20t,t<.05 f(t)=1,t>=.05 	D,MB2 MB3,MB4 MB5	(8.26) to (8.27)
	f(t)=1 	D,MB1 MB2,MB3	(8.28) to (8.33)

Addendum

Type of boundary cond'n	Type of element	Number of elements	Number of deg.of.freedom (displacement)	Number of deg.of.freedom (moments)
CLAMPED, FREE Figs 8.10,8.11, 8.14,8.15, 8.28,8.29	D	1	2	
	MB2	1	2	2
	MB3	1	2	2
	MB4	2	2	2
	MB5	1	2	2
CLAMPED, CLAMPED Figs 8.19,8.20	D	2	2	
	MB2	2	2	5
	MB3	2	2	6
	MB4	4	3	5
	MB5	2	3	5
CLAMPED, SIMPLY SUPP'TD Figs 8.22,8.23, 8.26,8.27	D	2	3	
	MB2	2	3	4
	MB3	2	3	4
	MB4	4	3	4
	MB5	2	3	4

BEAM ELEMENT CONFIGURATIONS IN FORCED VIBRATION PROBLEMS.

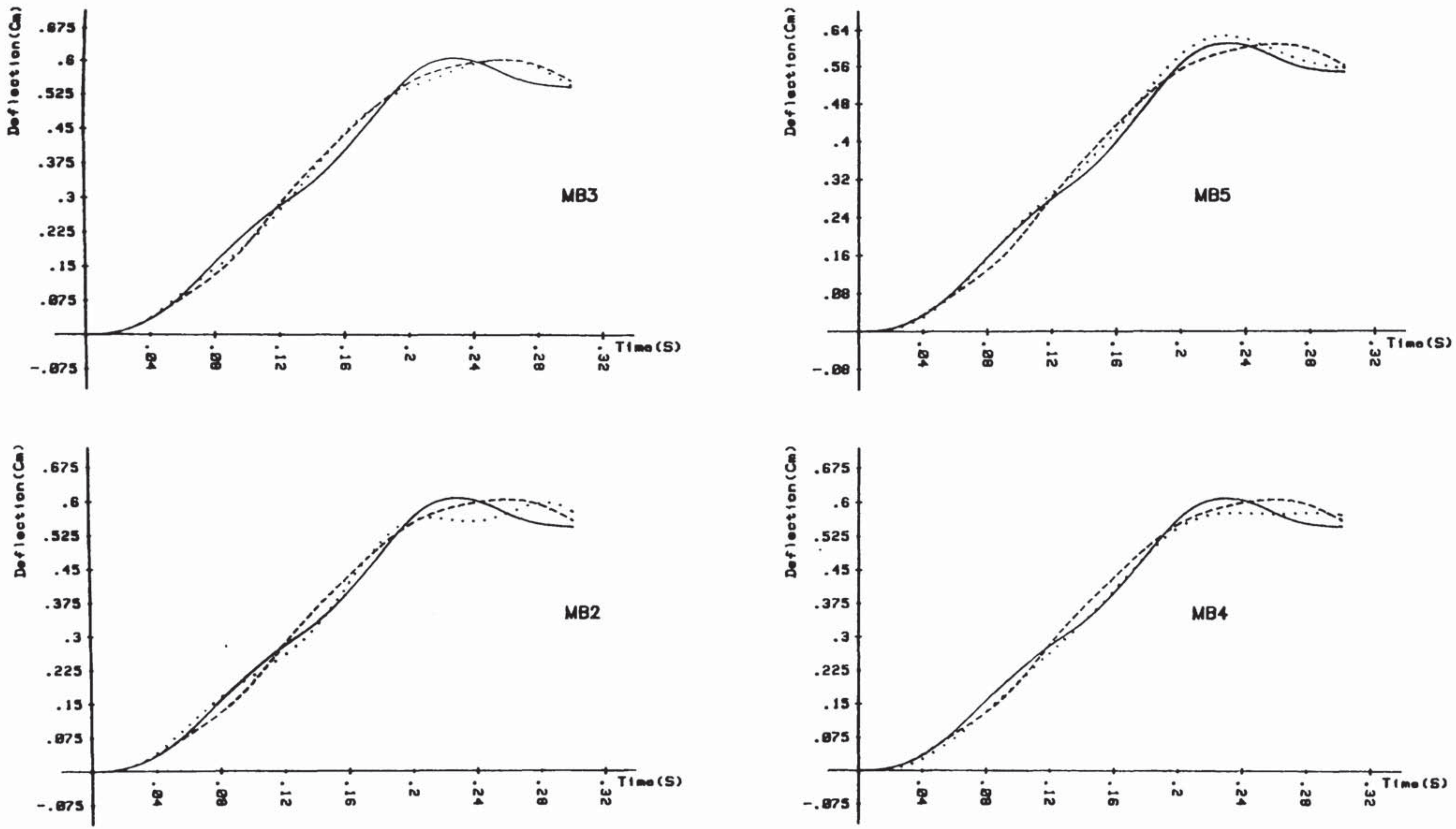


Fig 8.10 Tip displacement response of cantilever to half sine pulse input.
F.E.models with two d.o.f

——— Exact
– – – Displacement
·········· Mixed model

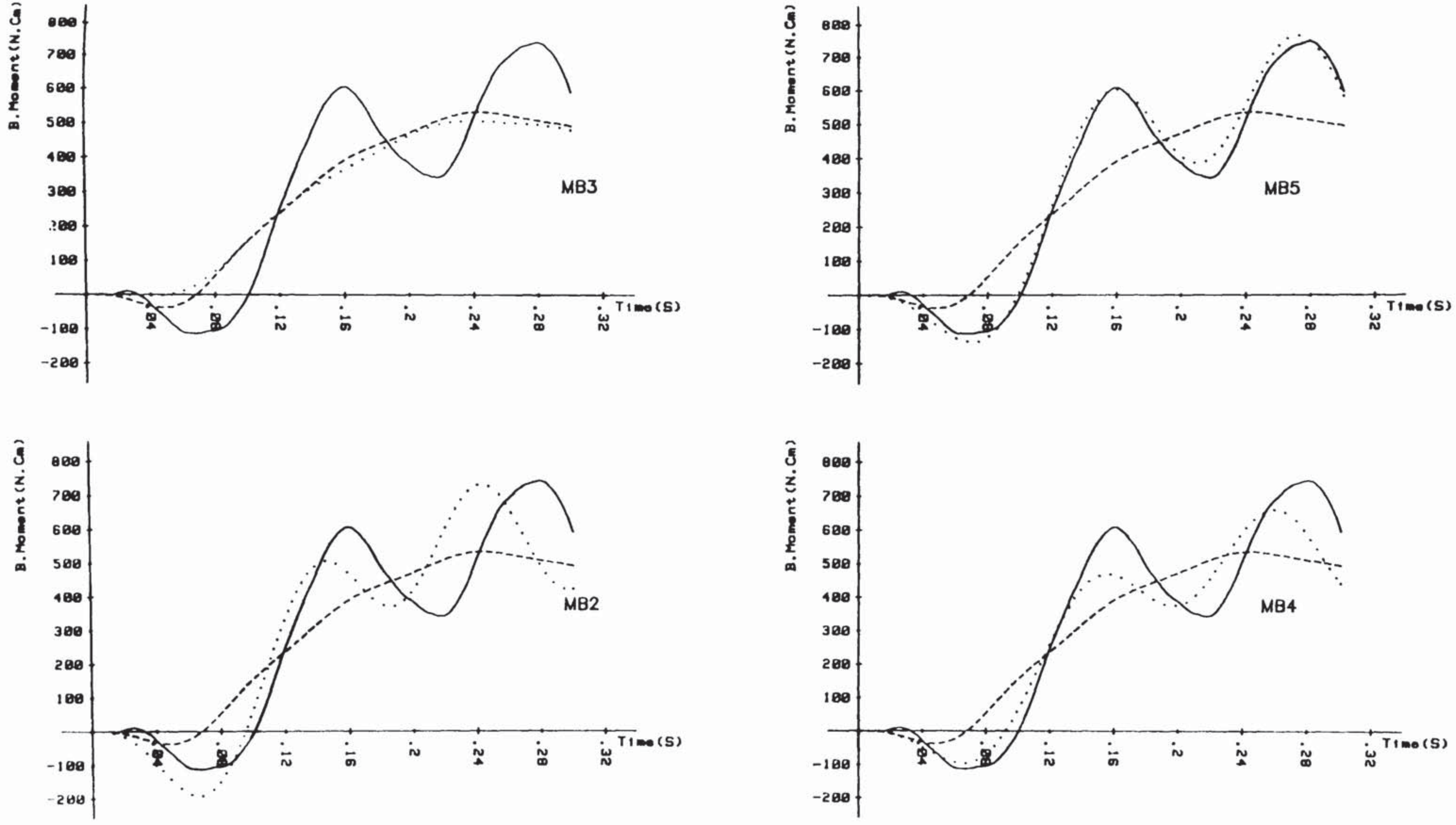


Fig 8.11 Bending moment response of cantilever to half sine puls input at x=0.0
F.E.models with two d.o.f

Exact
Displacement
Mixed model

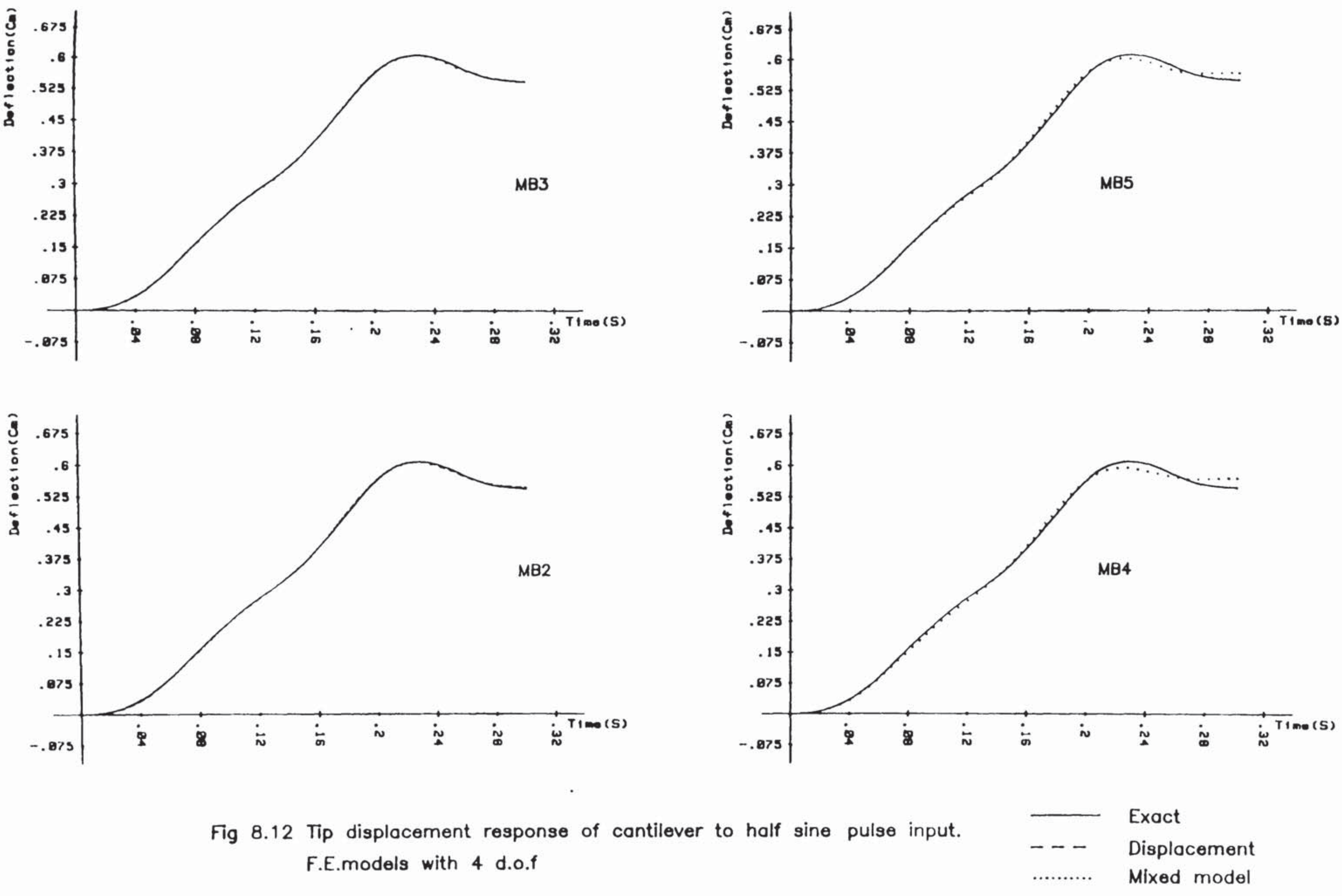


Fig 8.12 Tip displacement response of cantilever to half sine pulse input.
F.E.models with 4 d.o.f

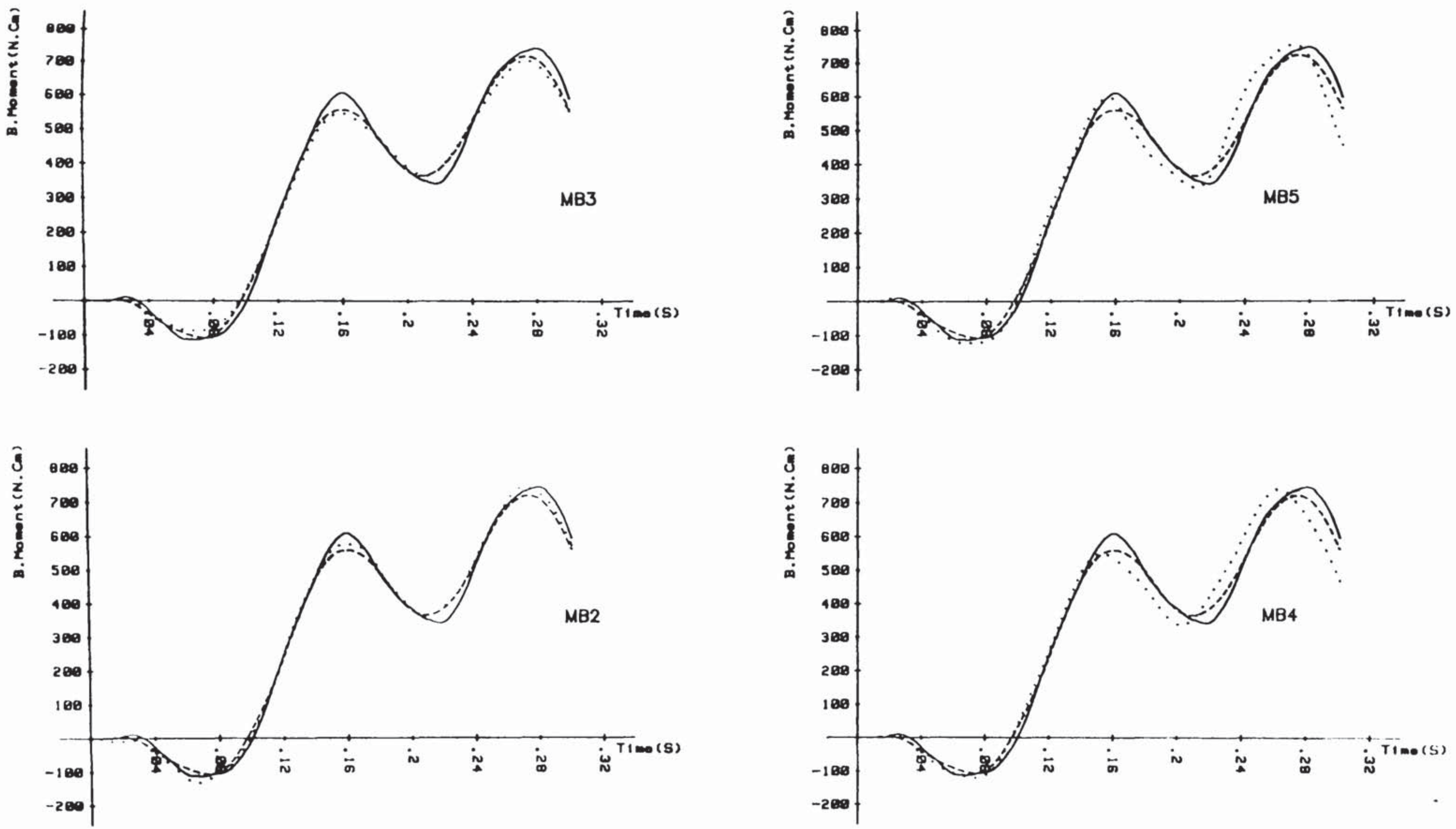


Fig 8.13 Bending moment response of cantilever to half sine puls input at x=0.0
F.E.models with 4 d.o.f

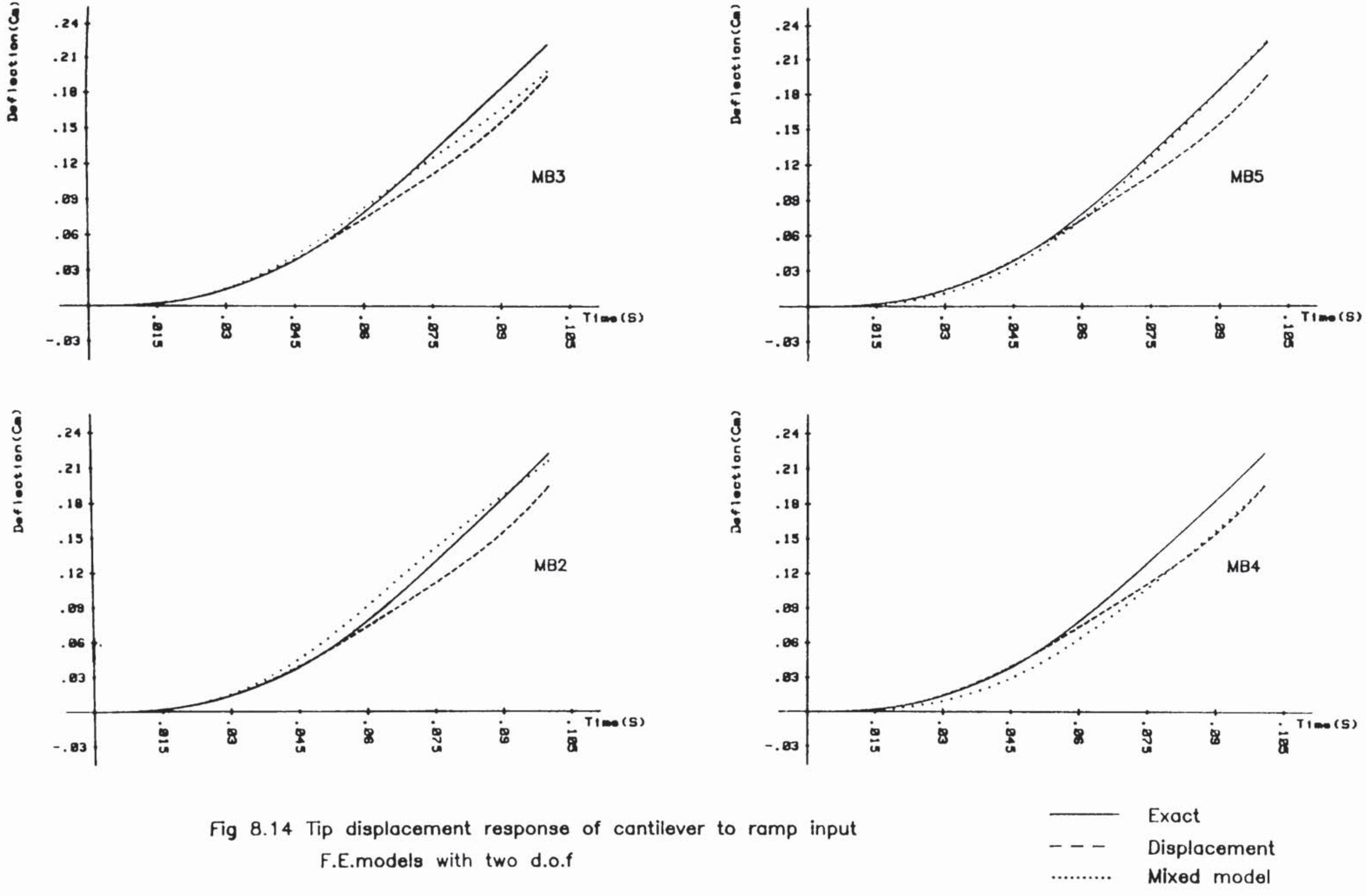


Fig 8.14 Tip displacement response of cantilever to ramp input F.E.models with two d.o.f

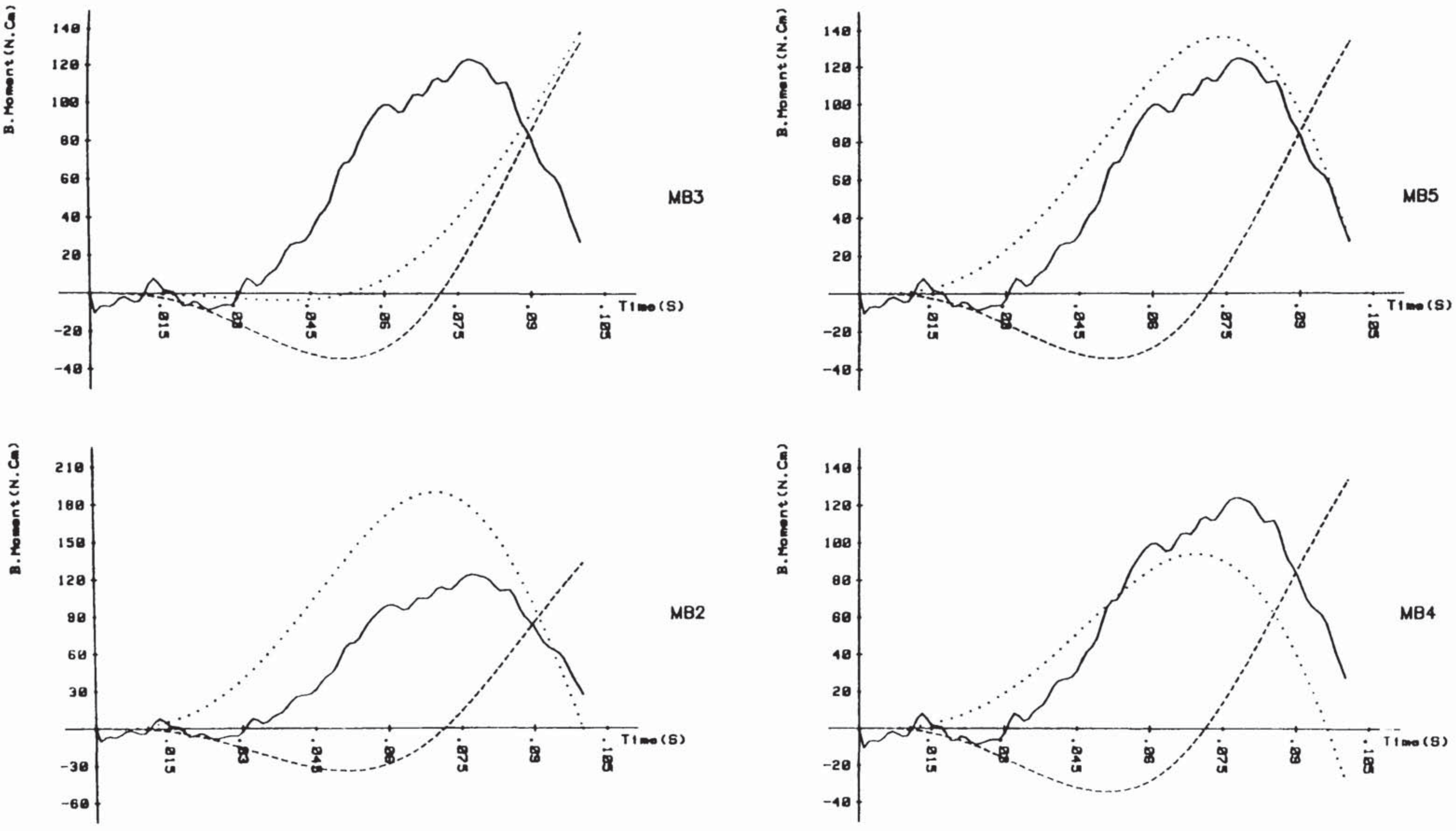


Fig 8.15 Bending moment response of cantilever to ramp force input , x=0.0
F.E.models with 2 d.o.f

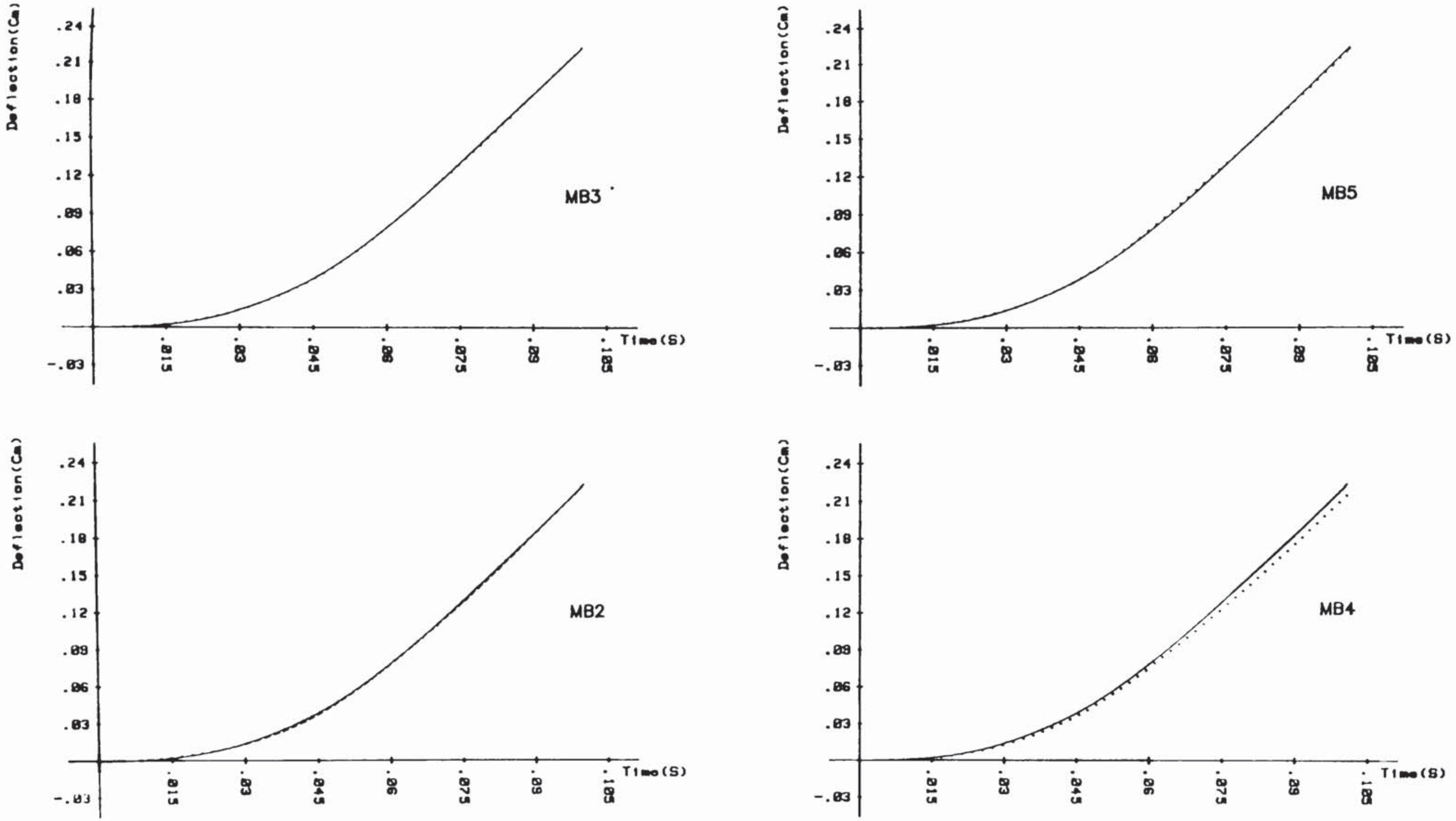


Fig 8.16 Tip displacement response of cantilever to ramp force input.
F.E.models with 4 d.o.f

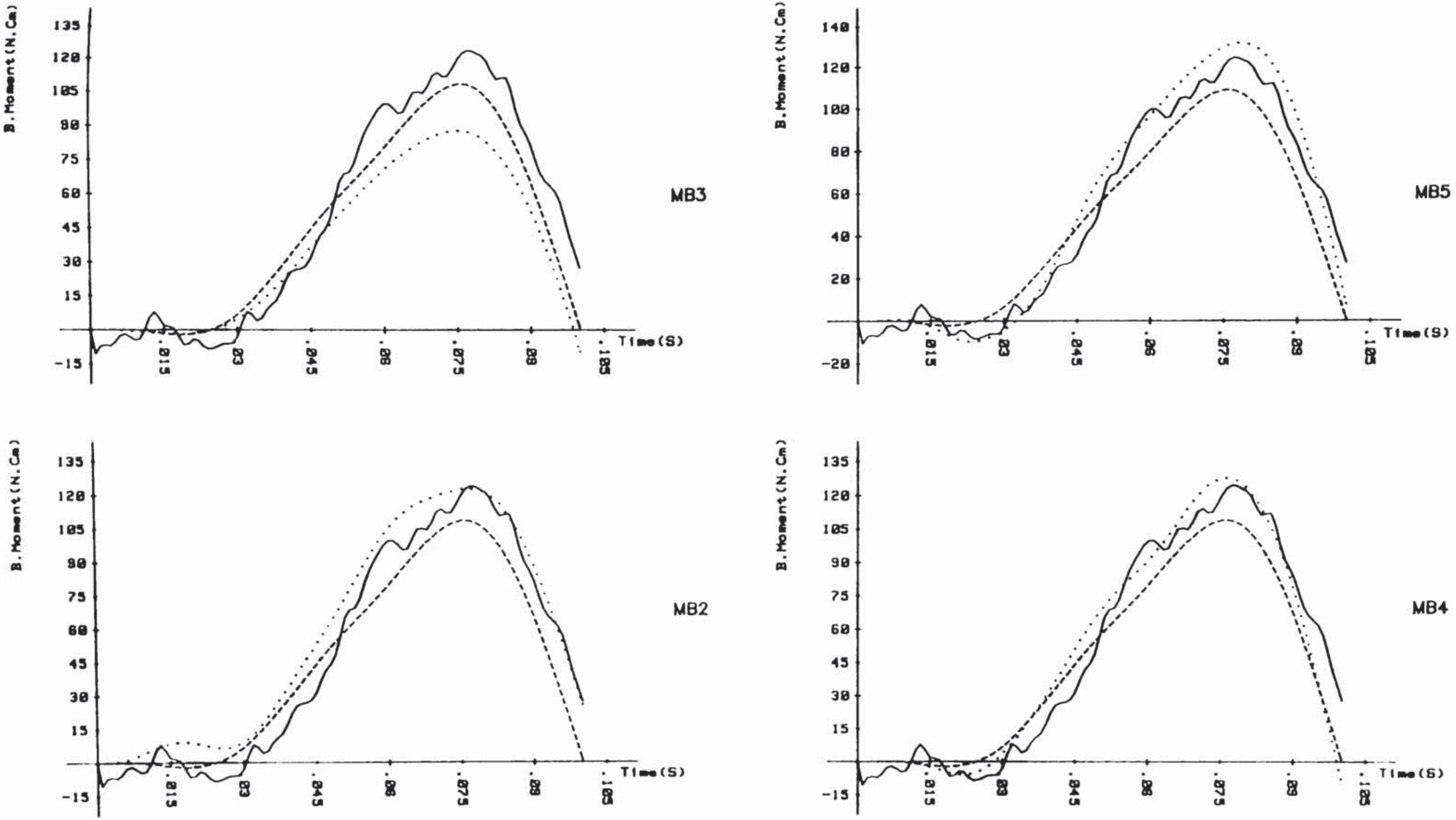


Fig 8.17 Bending moment response of cantilever to ramp force input , x=0.0
F.E.models with 4 d.o.f

——— Exact
– – – Displacement
········ Mixed model

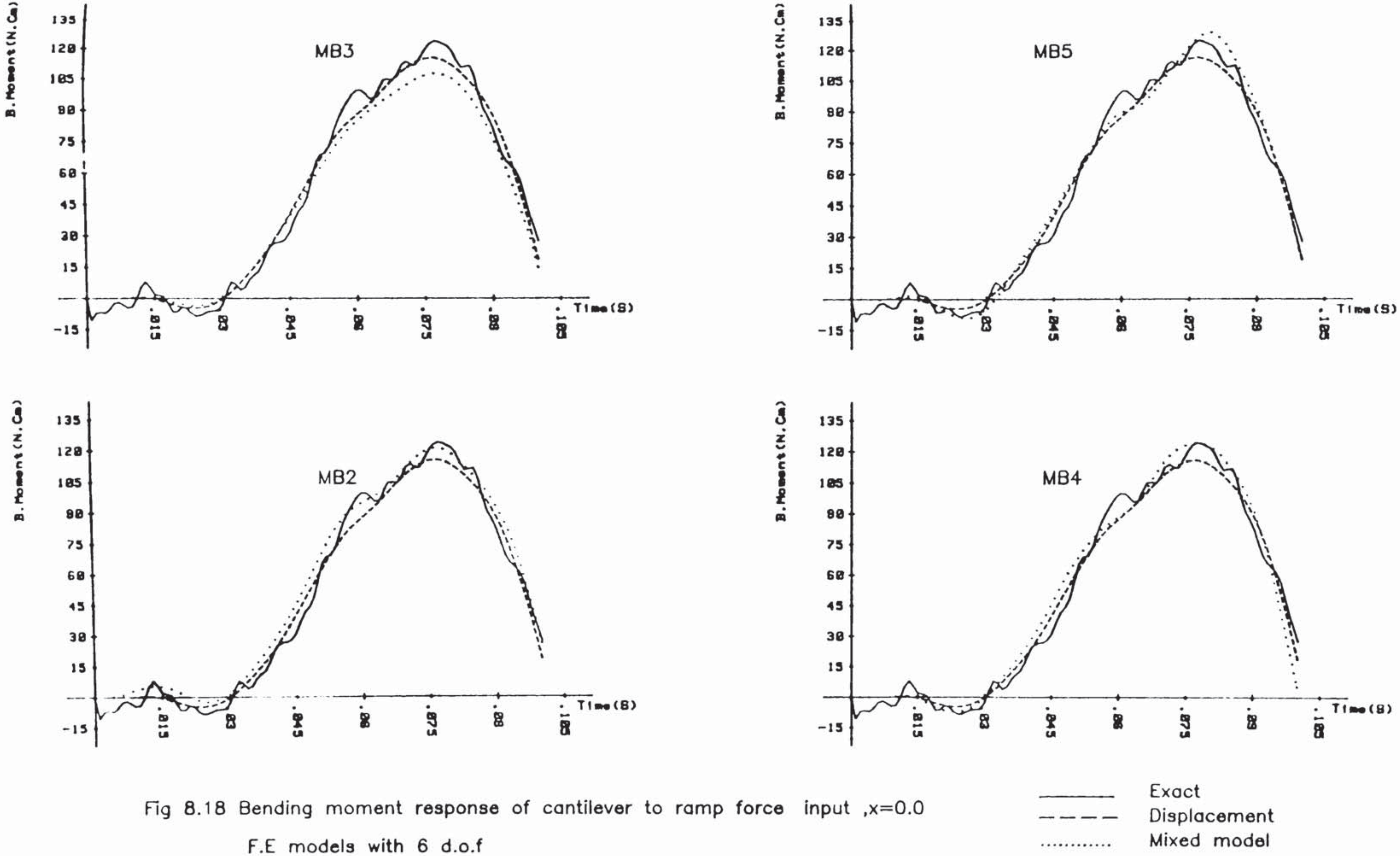


Fig 8.18 Bending moment response of cantilever to ramp force input ,x=0.0

F.E models with 6 d.o.f

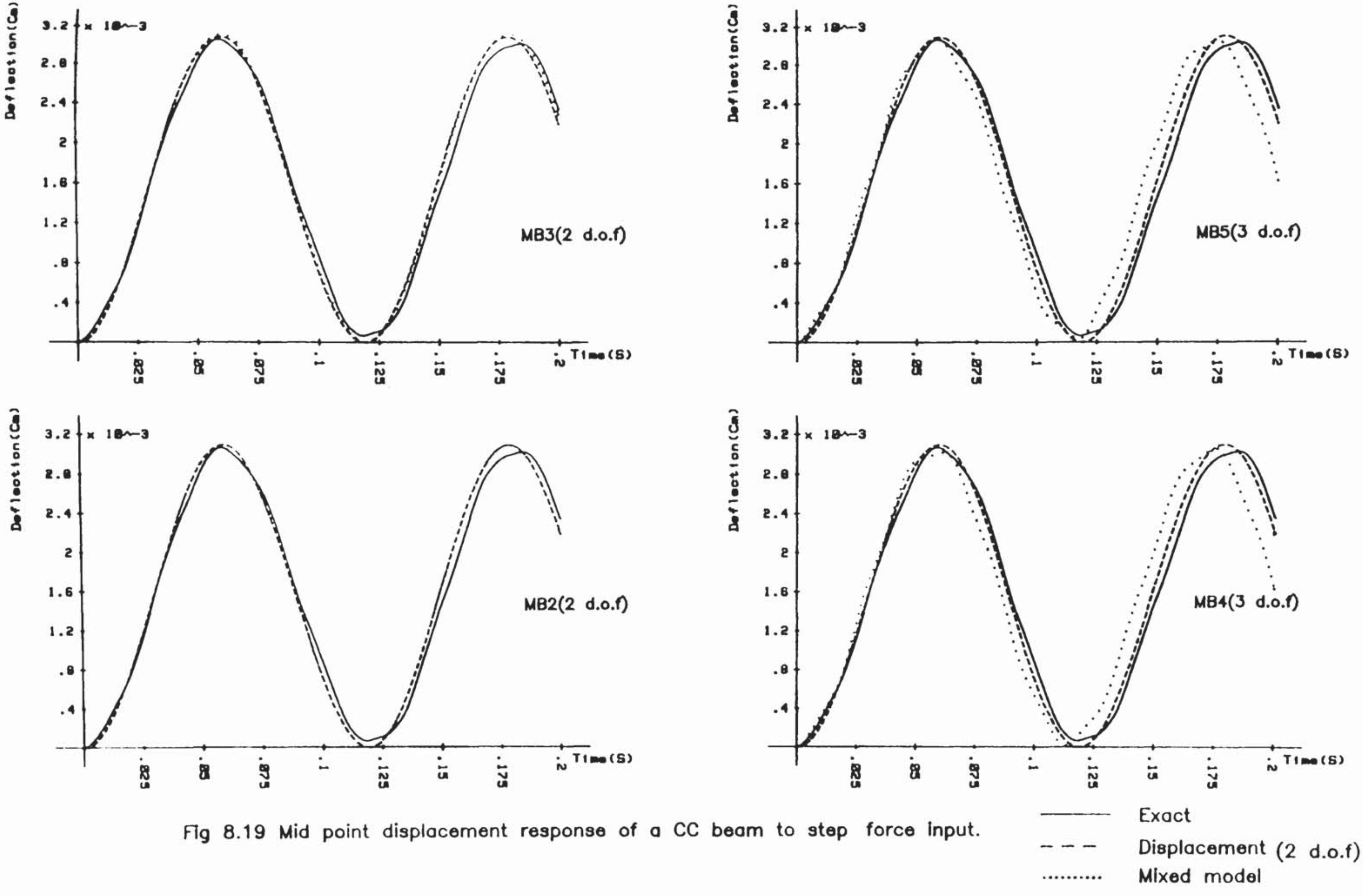


Fig 8.19 Mid point displacement response of a CC beam to step force input.

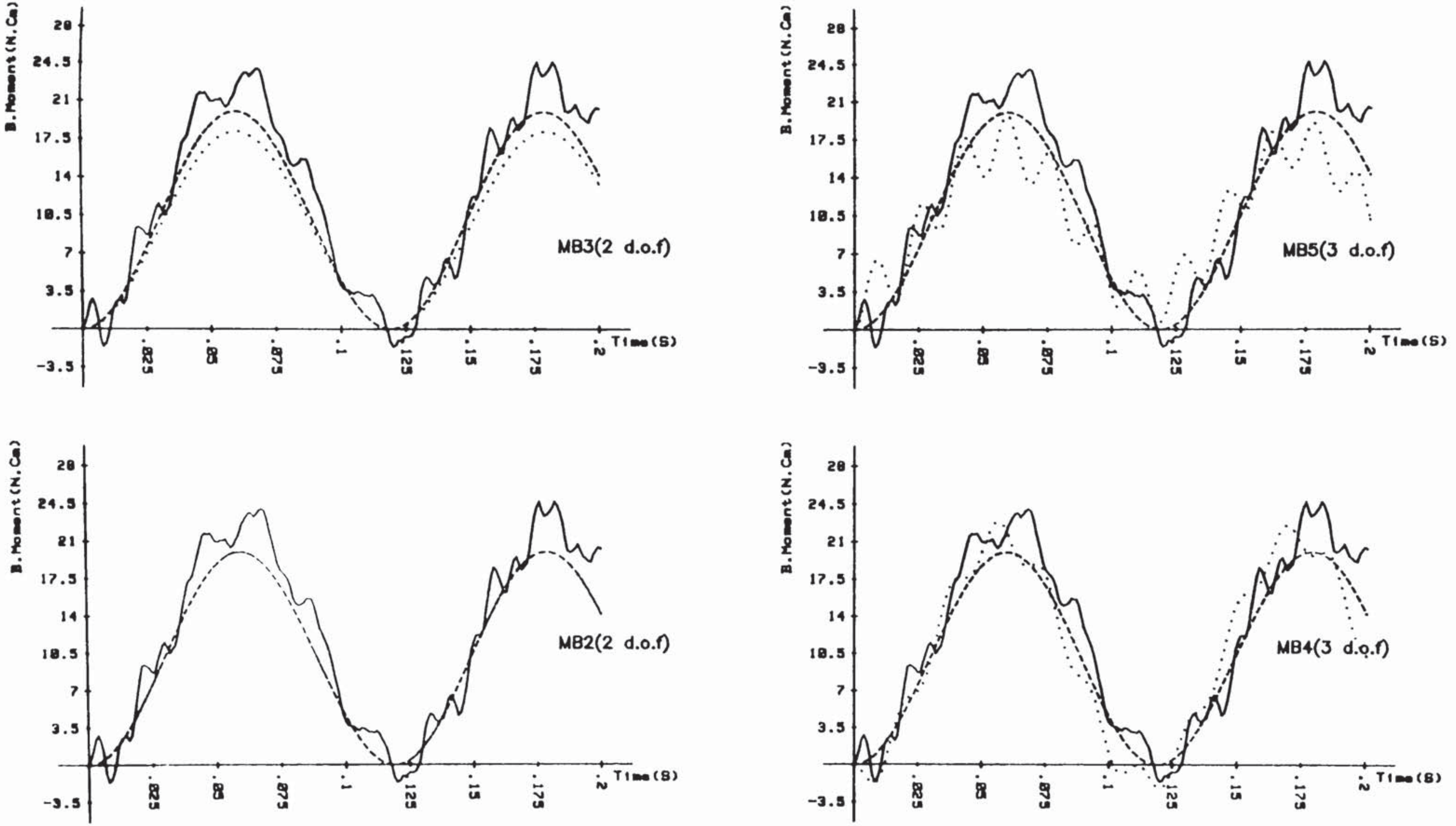


Fig 8.20 Bending moment response of a CC beam at x=0.0 to step force input.

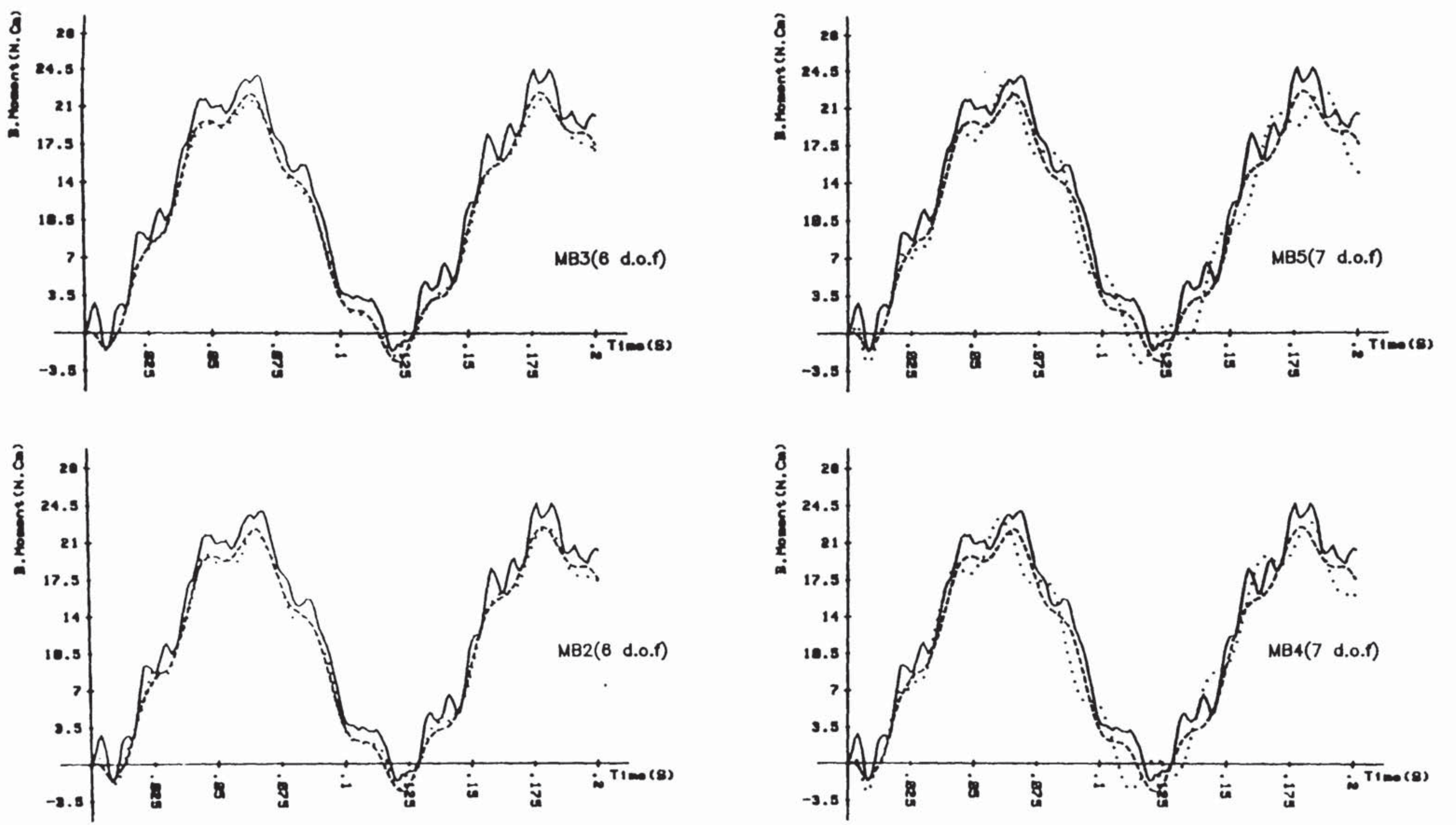


Fig 8.21 Bending moment response of a CC beam at x=0.0 to step force input.

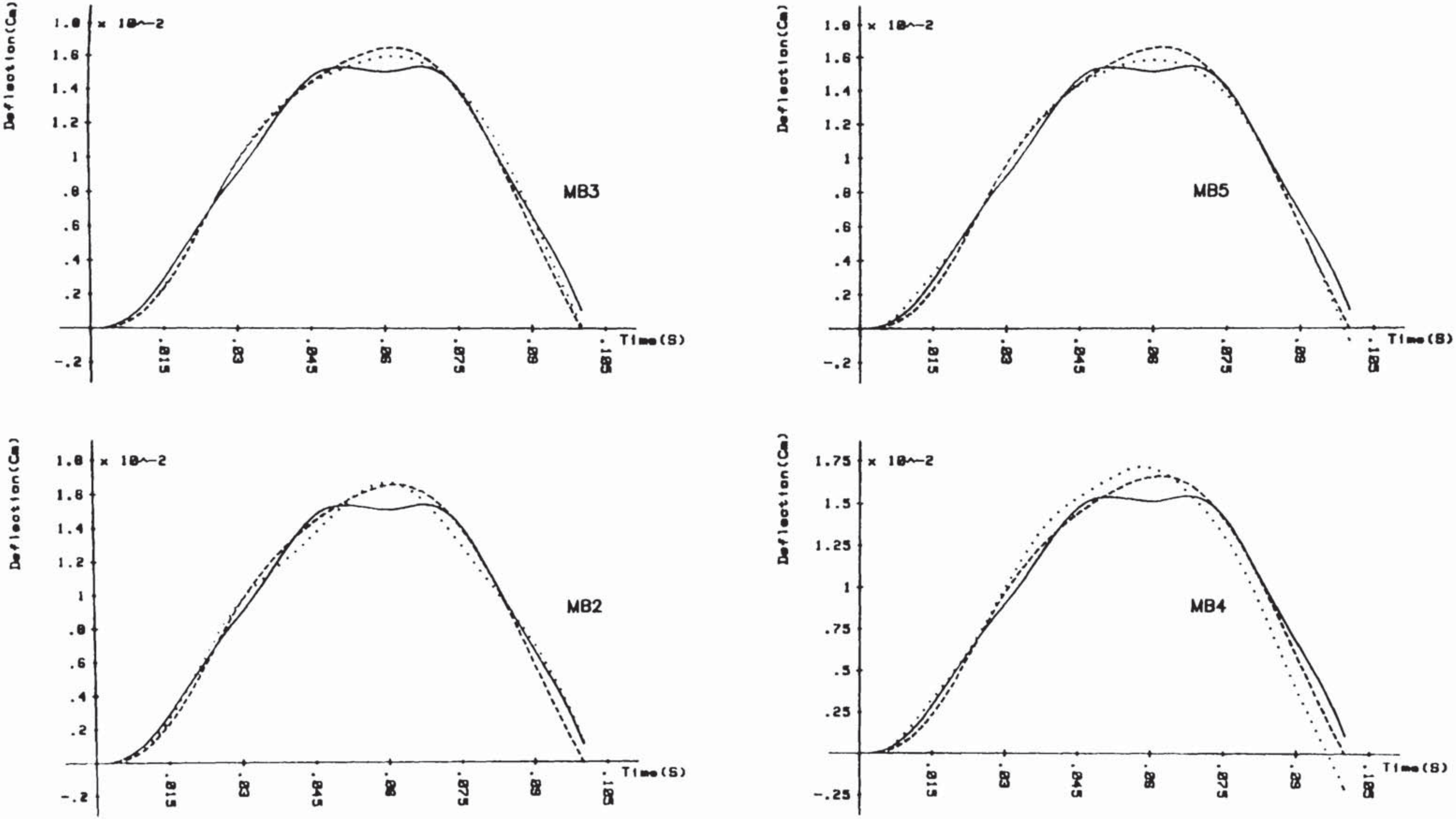


Fig 8.22 Mid point displacement response of CS beam to half sine pulse input.
F.E.models with 3 d.o.f

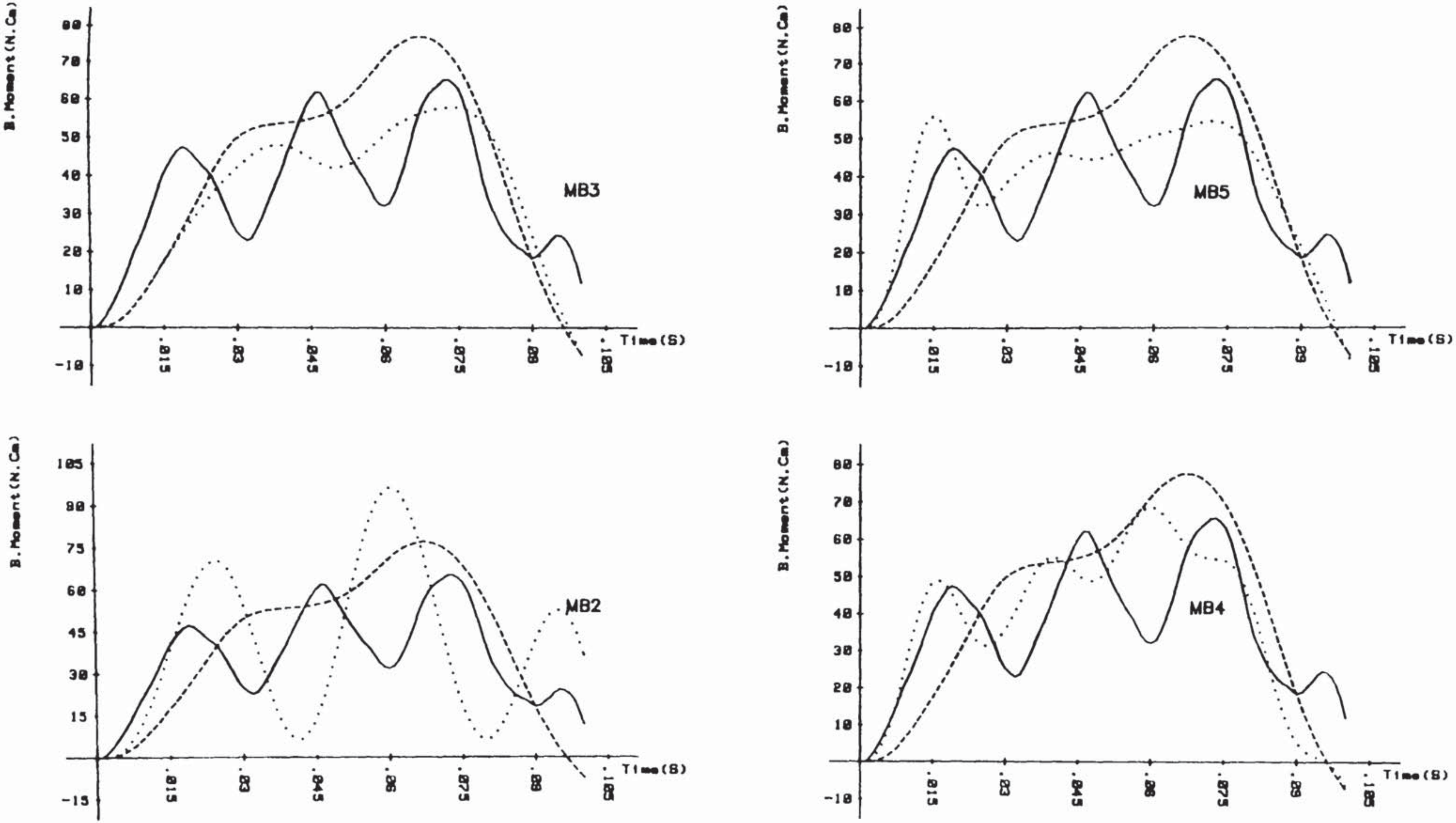


Fig 8.23 Mid point bending moment response of CS beam to sine pulse input.
F.E.models with 3 d.o.f

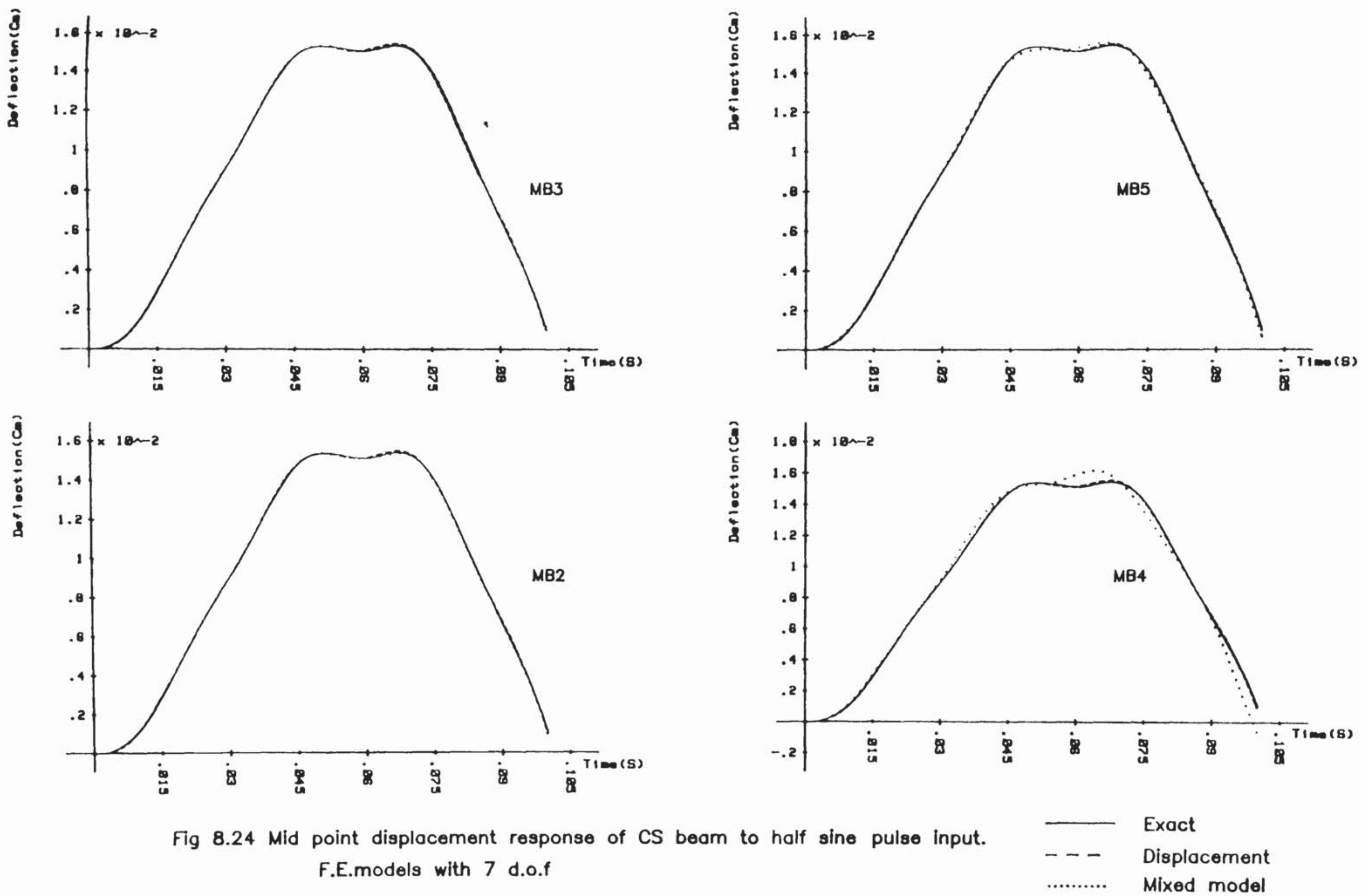


Fig 8.24 Mid point displacement response of CS beam to half sine pulse input.
F.E.models with 7 d.o.f

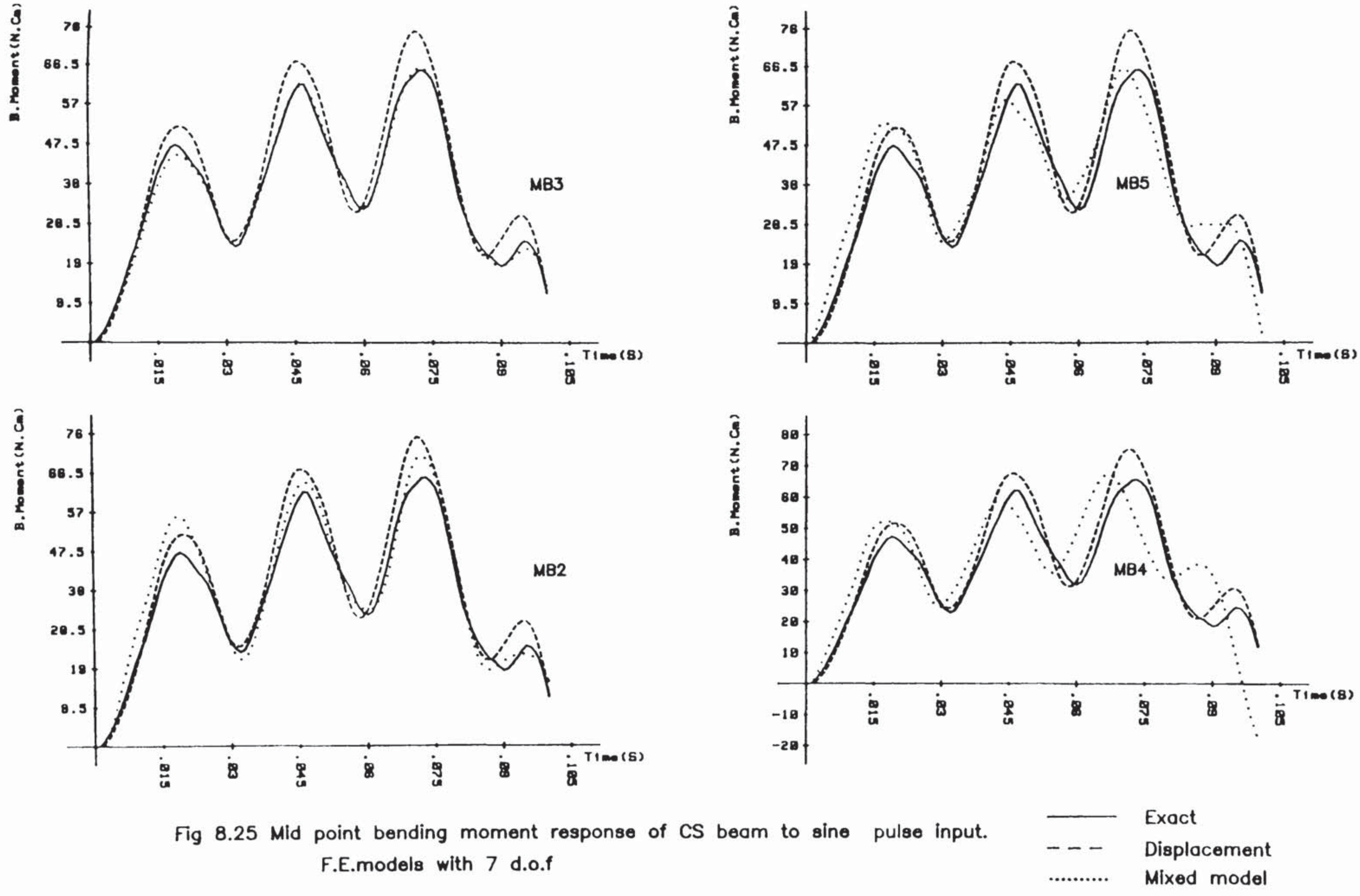


Fig 8.25 Mid point bending moment response of CS beam to sine pulse input. F.E.models with 7 d.o.f

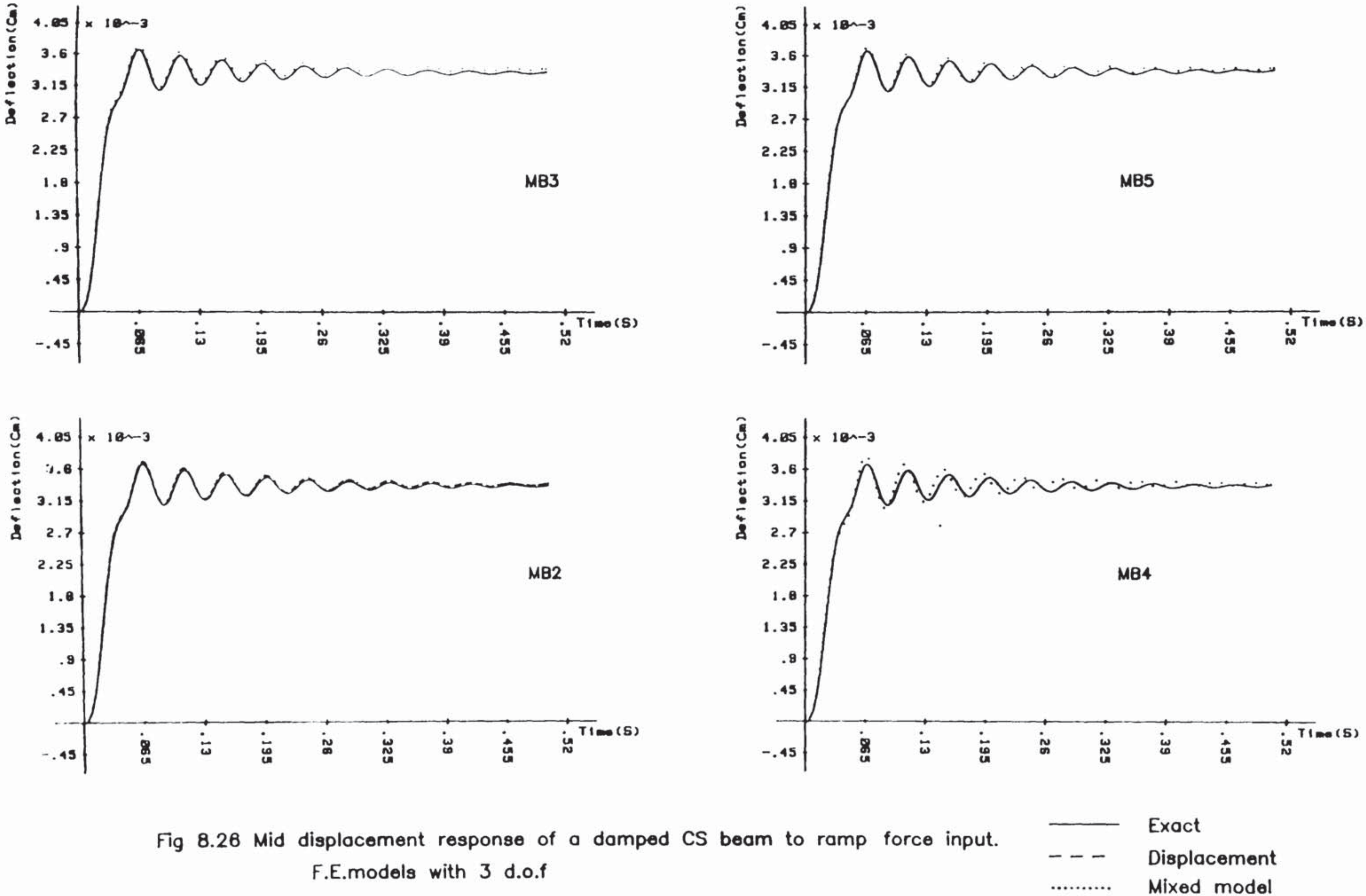


Fig 8.26 Mid displacement response of a damped CS beam to ramp force input.
F.E.models with 3 d.o.f

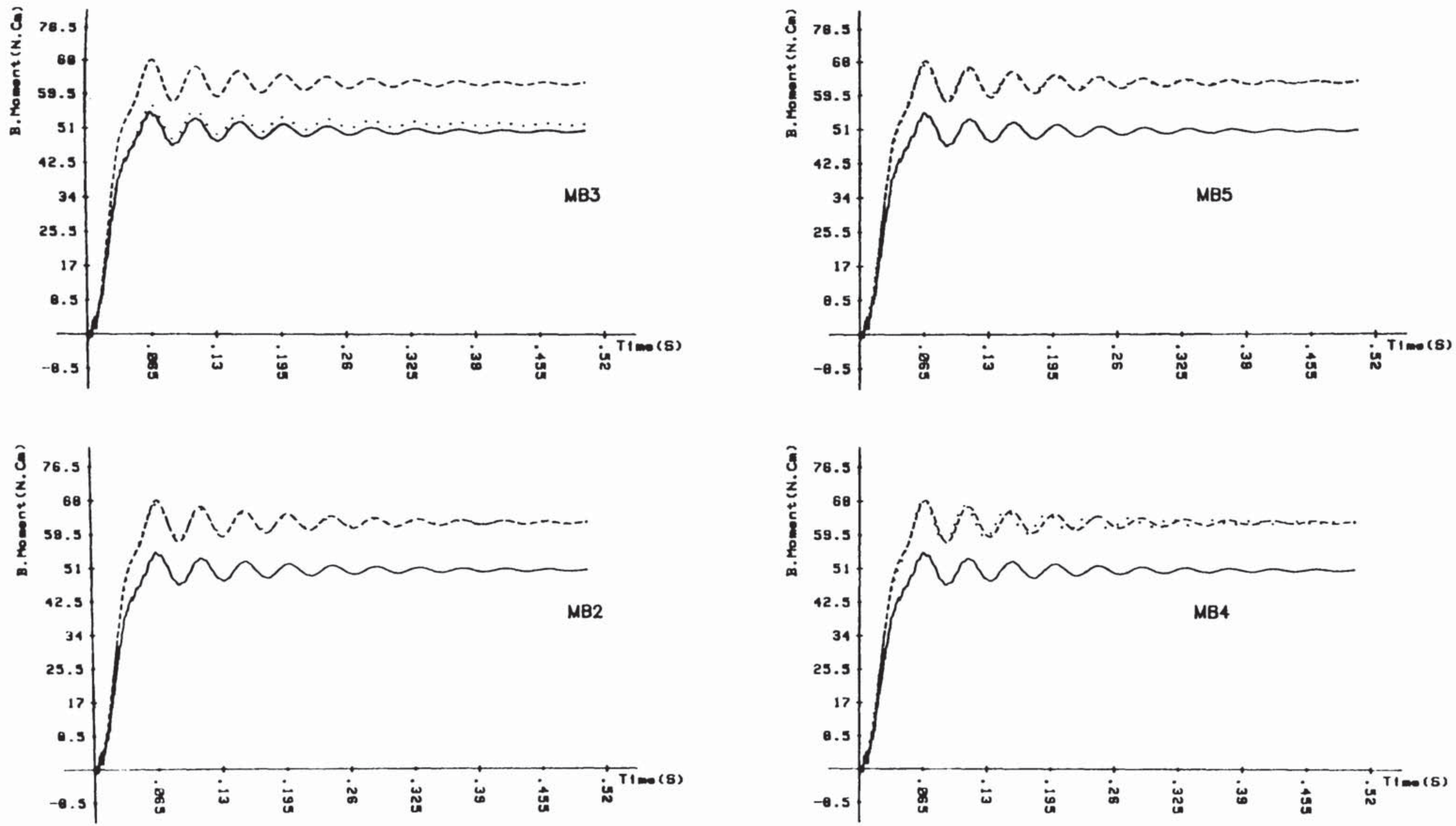


Fig 8.27 Mid point moment response of a damped CS beam to ramp force input.
F.E.models with 3 d.o.f

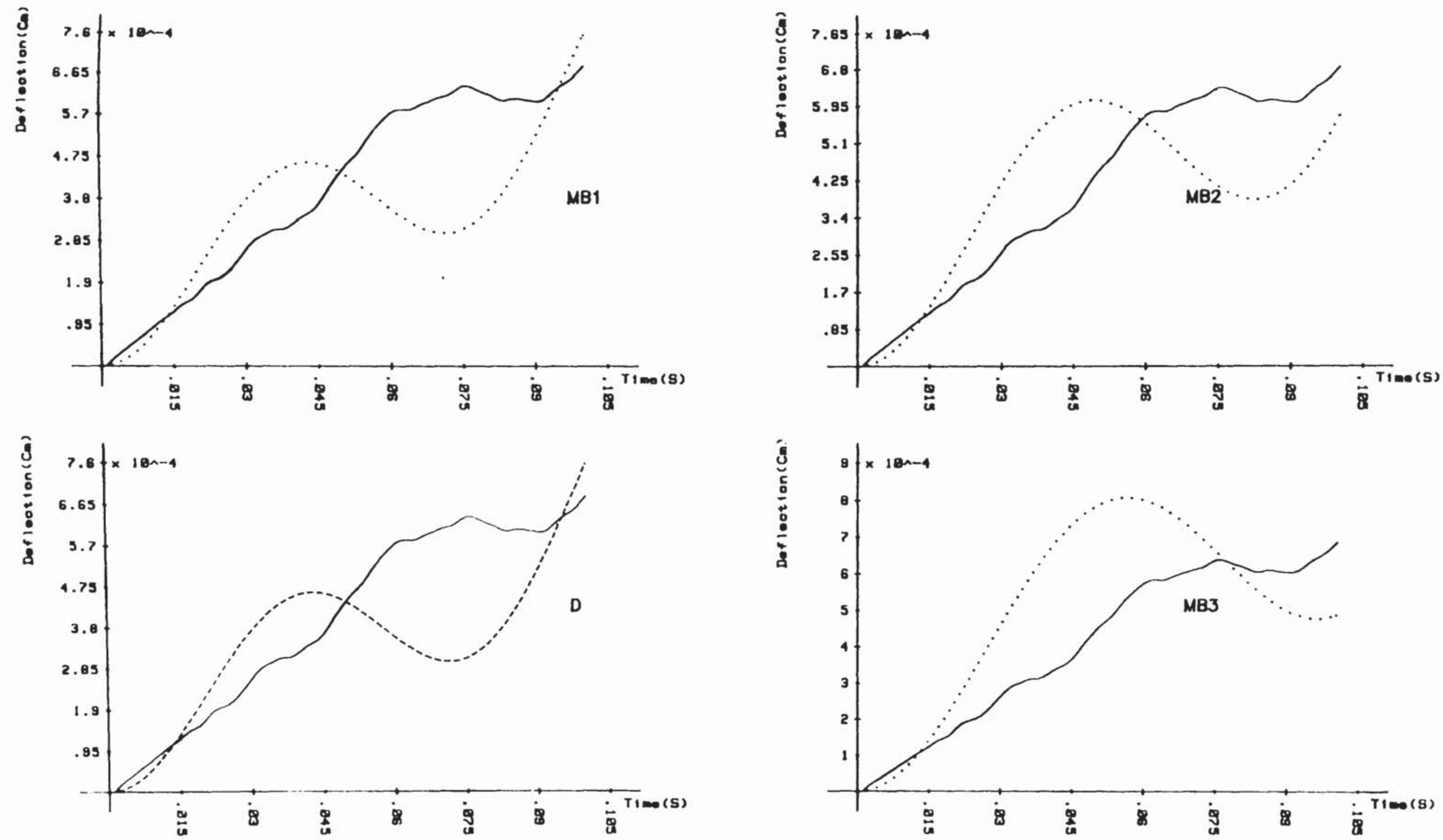


Fig 8.28 Tip deflection response of cantilever to step moment input.
F.E.models with 2 d.o.f

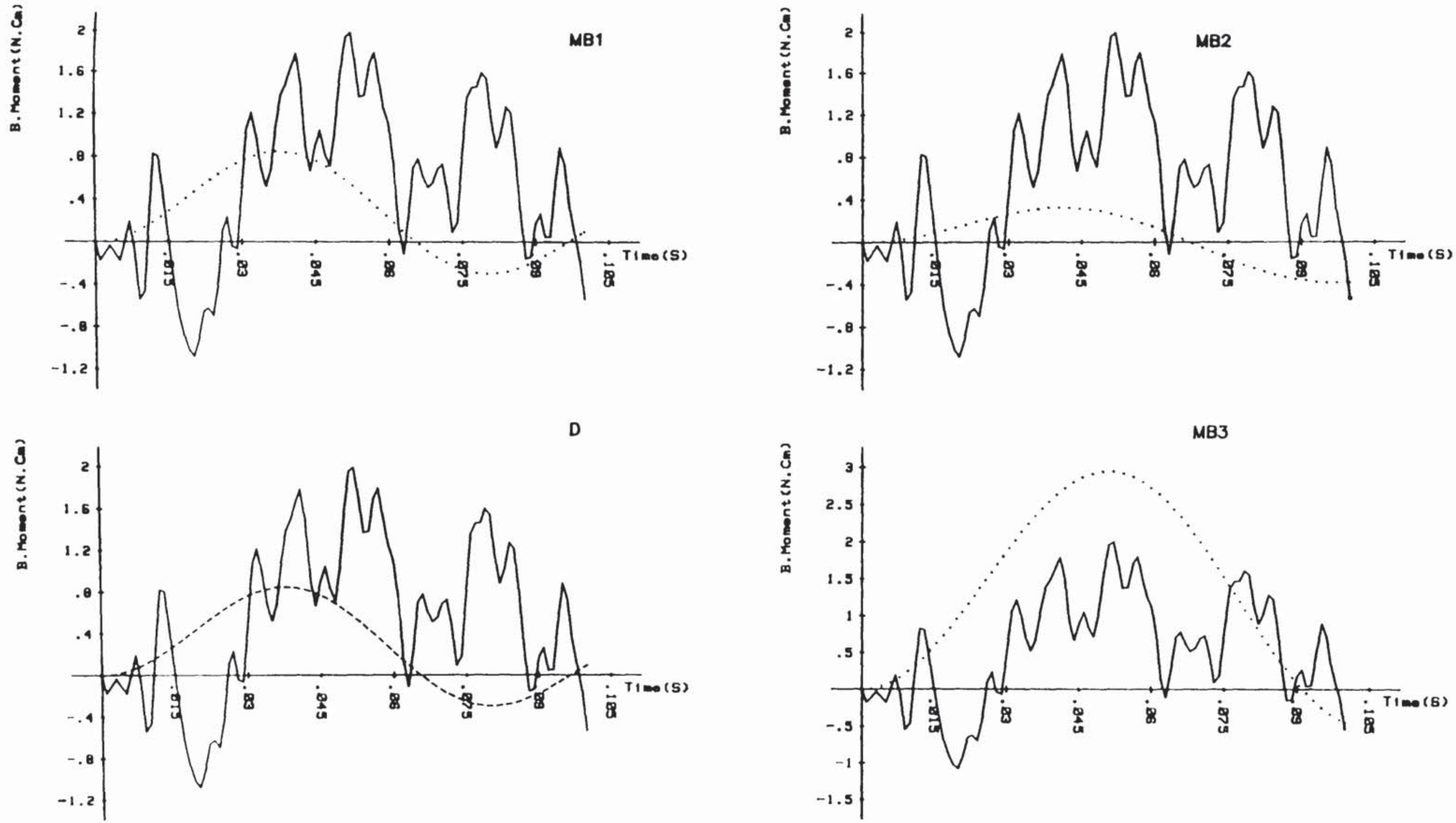


Fig 8.29 Bending moment response of cantilever at x=0.0 to step moment input.
F.E.models with 2 d.o.f

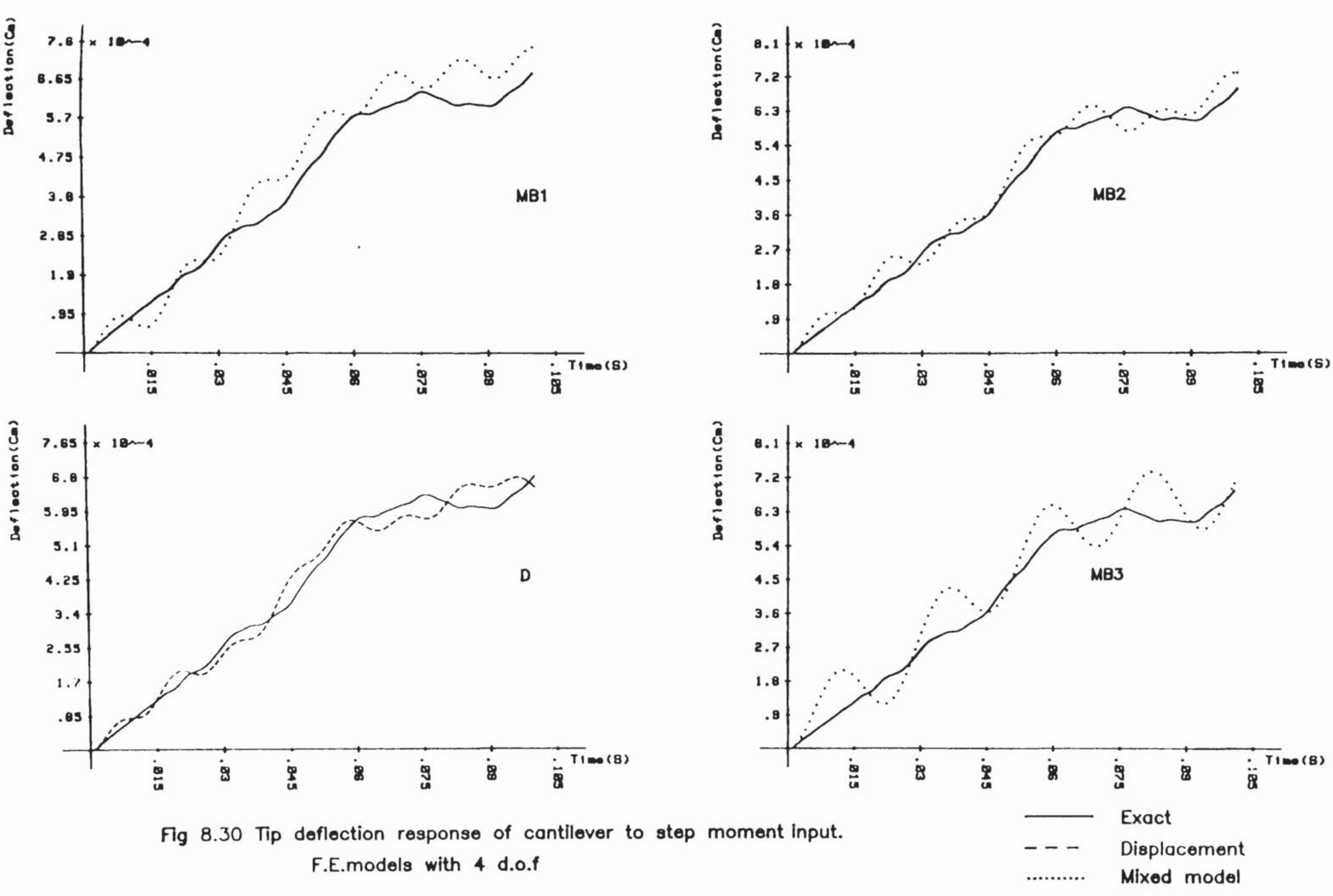


Fig 8.30 Tip deflection response of cantilever to step moment input.
F.E.models with 4 d.o.f

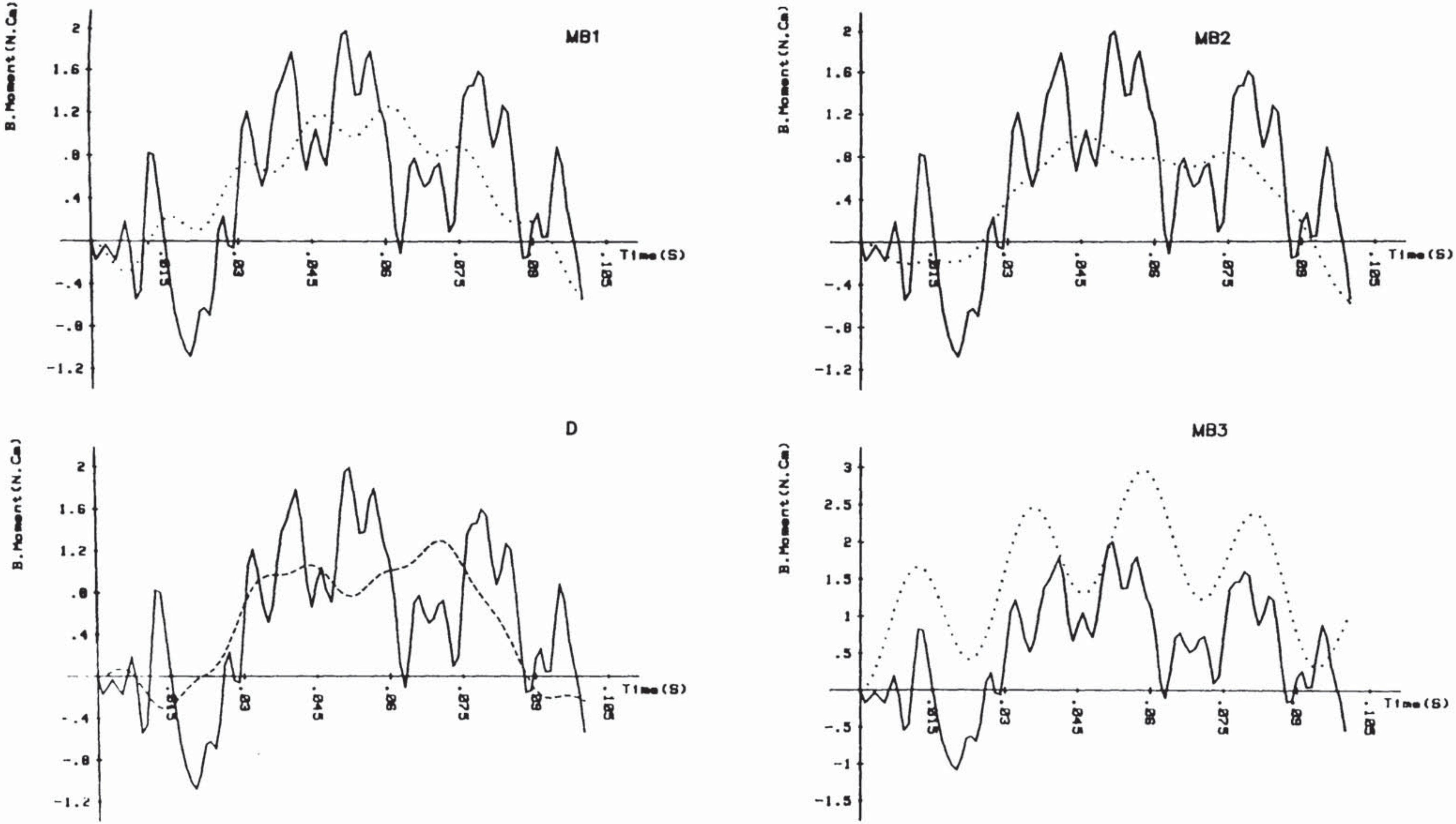


Fig 8.31 Bending moment response of cantilever at x=0.0 to step moment input.
F.E.models with 4 d.o.f

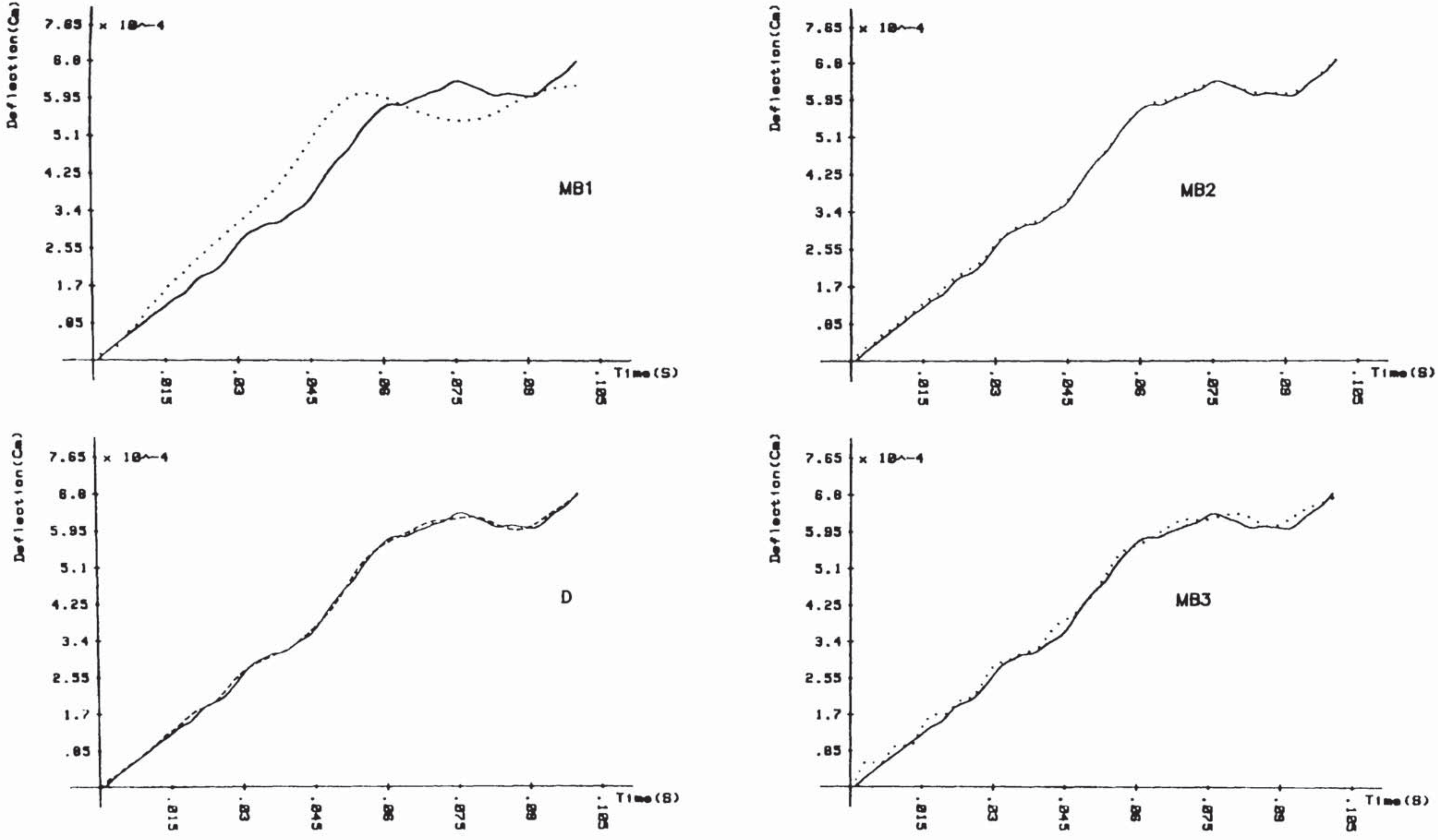


Fig 8.32 Tip deflection response of cantilever to step moment input.
F.E.models with 8 d.o.f

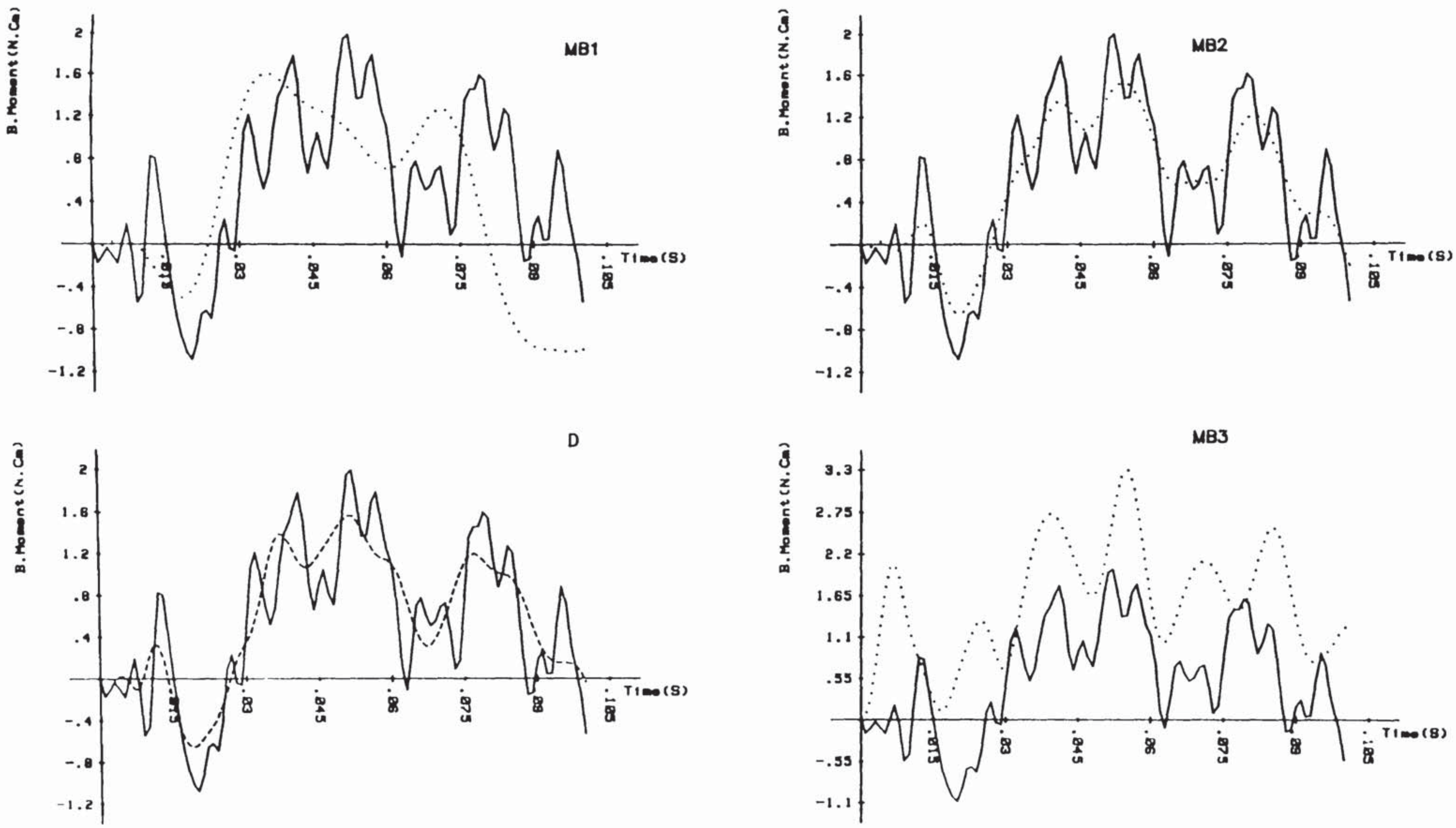


Fig 8.33 Bending moment response of cantilever at x=0.0 to step moment input.
F.E.models with 8 d.o.f

8.4 NUMERICAL EXAMPLES ON FREE VIBRATION OF PLATES

A number of free vibration problems of square plates with various boundary conditions are solved using the mixed quadratic element of Figure (8.1b). The predicted results are compared with the analytical solutions and those obtained from other finite element models (Figs. 8.1a, 8.1c, 8.1d). Frequencies are computed for the following three cases:

(i) Simply supported square plate.

(ii) Square plate simply supported on two opposite edges and clamped on the other two edges.

(iii) Cantilevered square plate.

The constants used in these solutions are:

Plate: $E = 2.07 \times 10^{7}\ ^{N}/_{cm^2}$, $\nu = 0.3$, $\rho = 7.8 \times 10^{-3}\ ^{Kg}/_{cm^3}$

$h = 1$ cm, $a = b = 120$ cm.

8.4.1 Simply-supported plate

The natural frequencies of the simply supported square plate shown in Figure (8.34) were predicted by the mixed quadratic element QR8, using (2 x 2), (3 x 3) and (6 x 6) finite element meshes. The results for the (6 x 6) mesh are obtained by solving a (3 x 3) mesh representation of one quarter of the plate, taking advantage of the symmetry of the problem. In this case only the symmetric modes are obtained.

Table (8.4) contains computed frequencies for various meshes. The accuracy of the fundamental frequency predicted by (2 x 2), (3 x 3) and (6 x 6) meshes are within 1.47, 0.15 and .005% of the analytical value. It should be noted that for the (2 x 2), (3 x 3) and (6 x 6) meshes, the final eigenvalue problem has 5, 16 and 85 degrees of freedom respectively. Figures (8.35) and (8.36) compare the accuracy of the developed mixed element, QR8, with the linear, QR4, element when predicting the first and second natural frequencies of the SS plate. It is seen that the mixed quadratic element is more efficient than the corresponding linear one. Also Figures (8.37) and (8.38) show the convergence curves for different types of elements when predicting the first and fifth natural frequencies. It is observed that the mixed 4 node and 8 node elements provide better results than the corresponding displacement type elements.

8.4.2 Clamped-simply supported square plate

The natural frequencies for a clamped simply supported square plate are obtained by using the mixed quadratic element with various meshes. These results are presented in table (8.5). It shows that a (4 x 6) mesh is capable of predicting the first natural frequency within 0.138%. The coarsest mesh used in this example is (2 x 2) and is capable of predicting this frequency to within 1.54%.

Figures (8.39) to (8.42) compare the performance of mixed quadratic element with the linear one in predicting the lowest four natural frequencies. The comparison with other types of elements are presented in Figures (8.43) and (8.44), corresponding to the second and third natural frequencies of the CSCS plate. It is seen that QR8 element has a much better performance than the 4-

noded QR4, and QD4 elements and is comparable with the 8-noded QD8 element.

8.4.3 Cantilevered square plate

Natural frequencies for a cantilever square plate are computed by using different types of elements presented in section 8.1. No exact solution exists for this problem and therefore the results are compared with the experimental (69) and other types of numerical solutions. These are shown in Table (8.6).

It can be seen that the mixed models using element QR8, have computed the first 5 natural frequencies with good accuracy, but the convergence to the experimental values is not necessarily monotonic. For a (2 x 2) mesh of QR8 element which leads to the final eigen problem of 16 degrees of freedom, the discrepancies of the values of the first five frequencies with reference to the Ritz solution (10) are 0.621, 5.86, - 2.9, 0.962 and 5.924%.

The equivalent discrepancies of a (2 x 2) mesh of displacement element QD8 with 48 degrees of freedom are 1.795, - 1.717, 1.068, - .54 and -1.7%. IN this case, the mixed model discrepancies are larger than the displacement models. It should however be noted that these are obtained using a much smaller eigenvalue problem than the displacement problem.

In application to eigenvalue problems, mixed models possess an important advantage over the conventional displacement models. This is because the reduction of degrees of freedom from total (displacements plus stresses) to the final having either displacements or stresses alone, is an exact operation. In the displacement models, the eigenvalue problem can be reduced by means of the so-called "eigenvalue economizer" method. In this operation, however, the accuracy of the computed eigenvalues decreases.

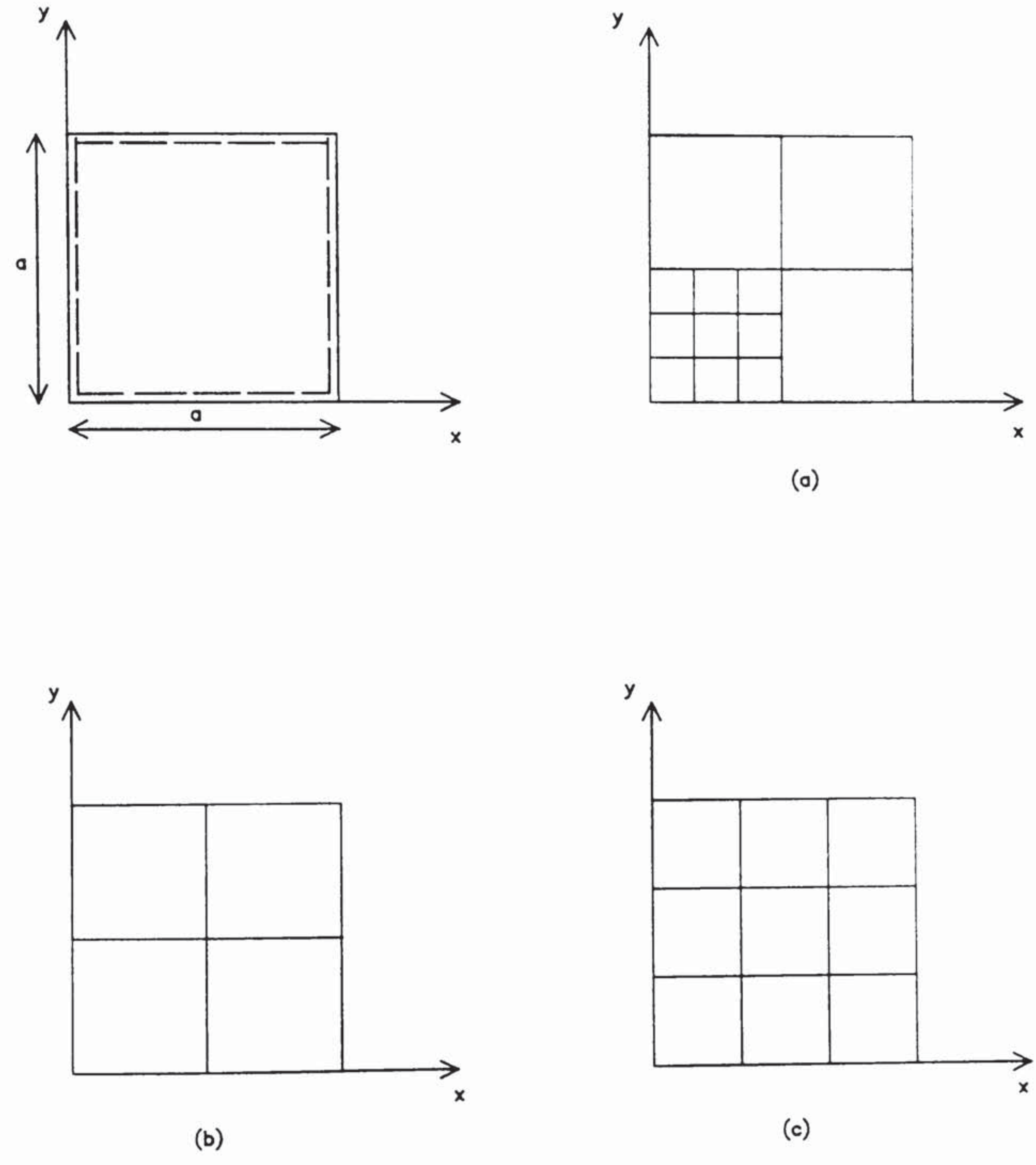


Fig 8.34 Finite element idealisations for a simply supported square plate.

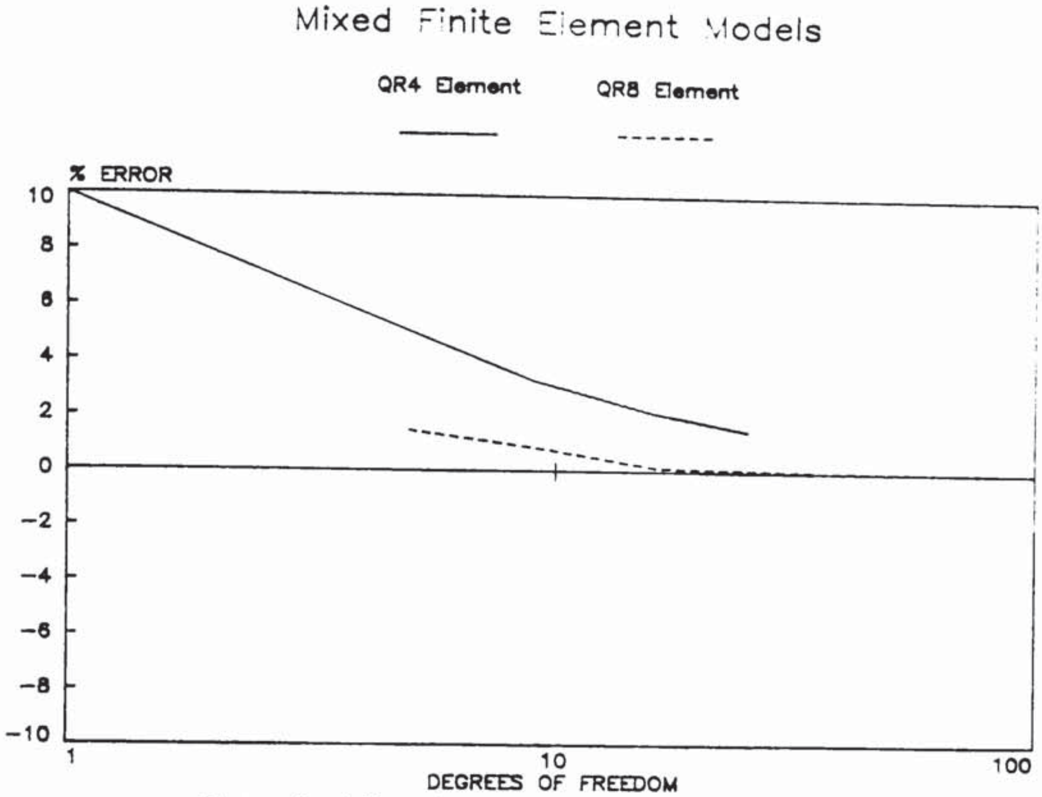


Fig 8·35 Prediction of lowest natural frequency of SSSS plate

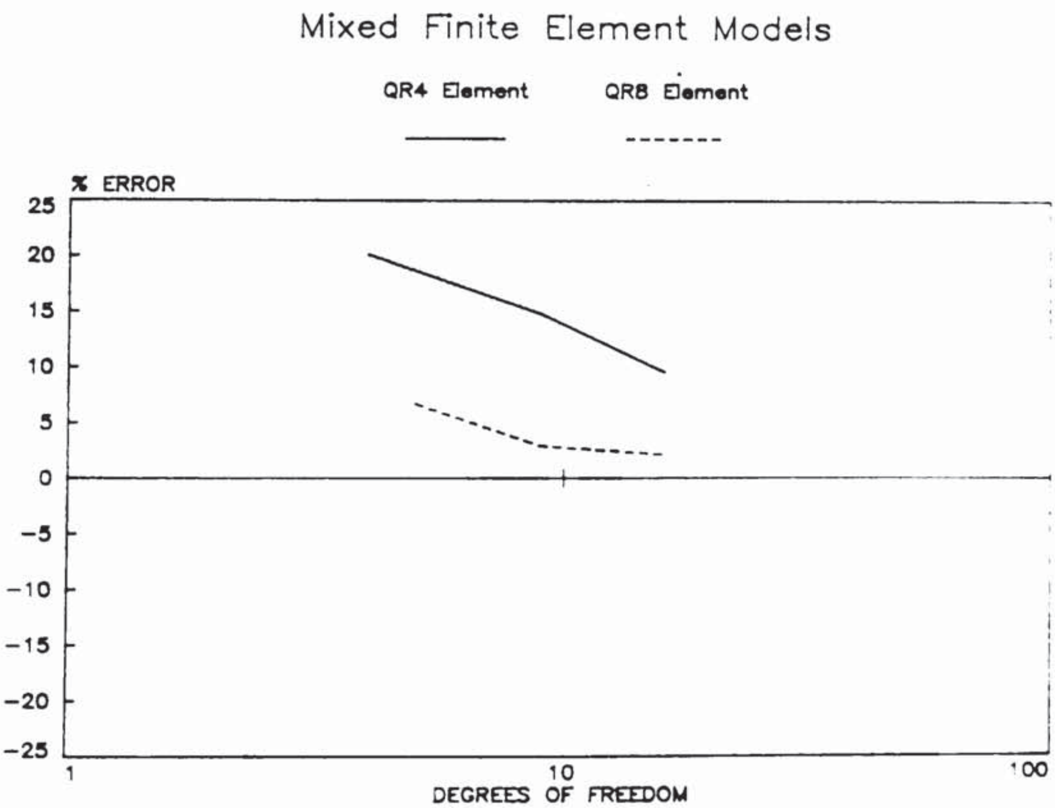


Fig 8·36 Prediction of second natural frequency of SSSS plate

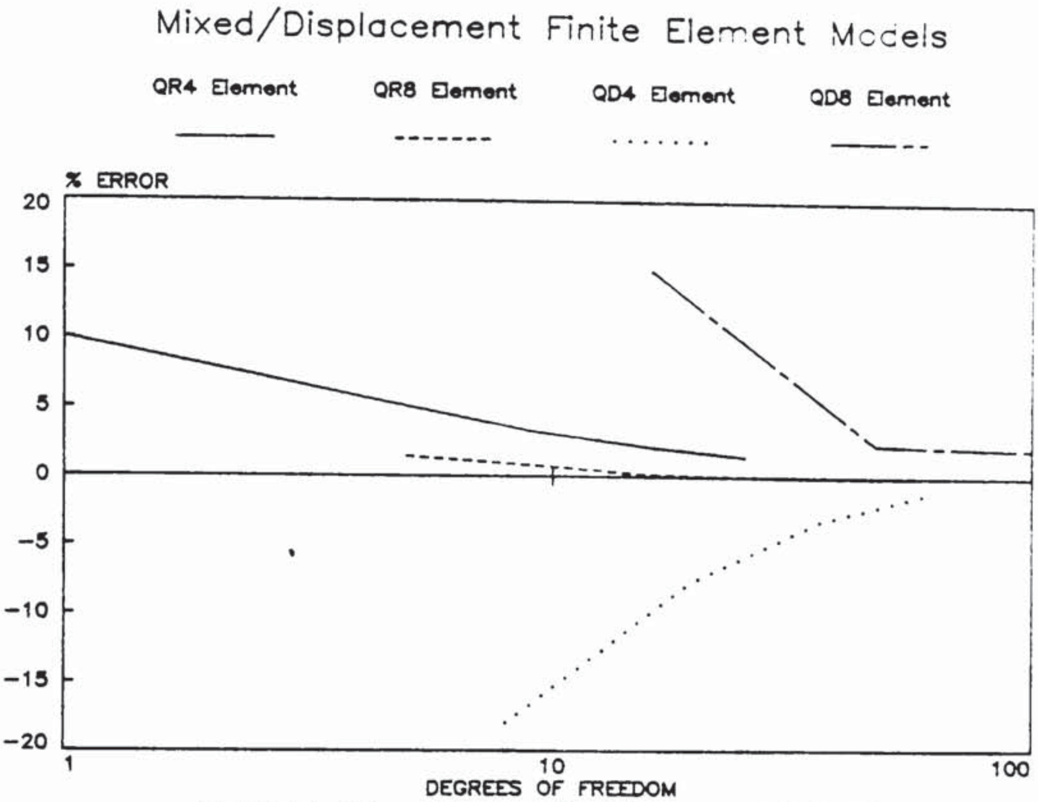


Fig 8.37 Prediction of lowest natural frequency of SSSS plate

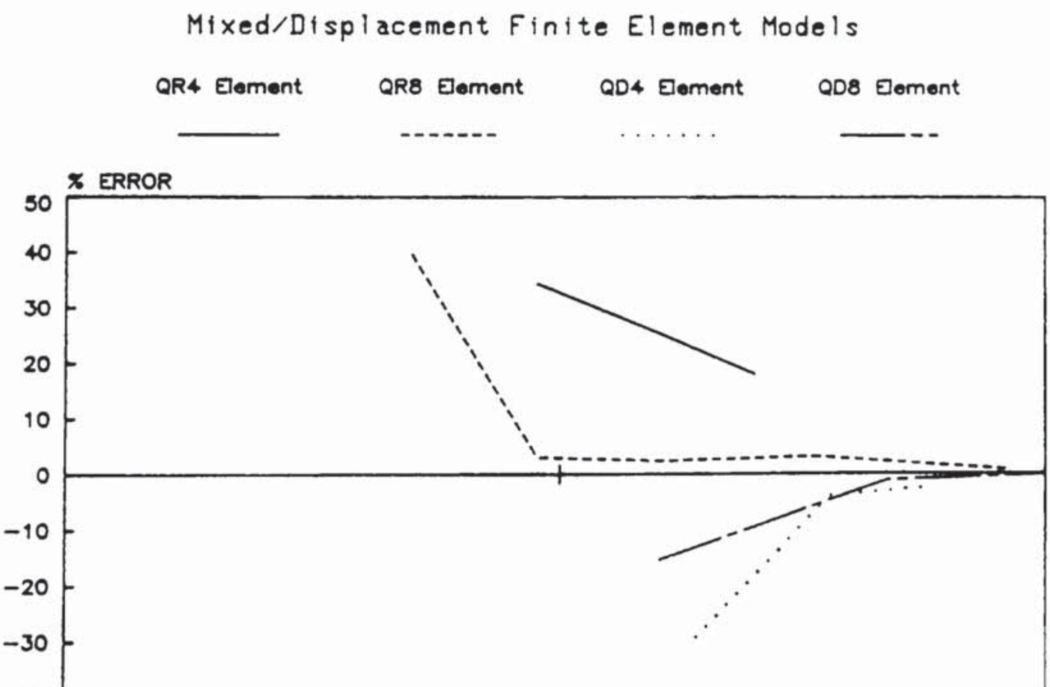


Fig 8.38 Prediction of fifth natural frequency of SSSS plate.m=3,n=1

Mixed Finite Element Models

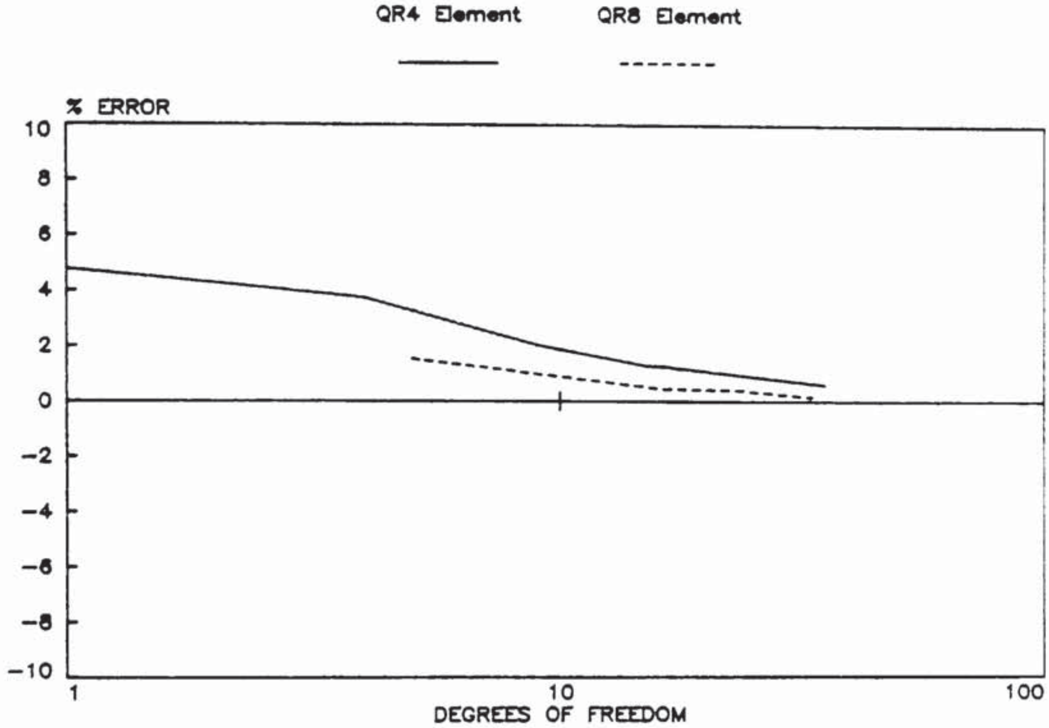


Fig 8·39 Prediction of lowest natural frequency of CSCS plate

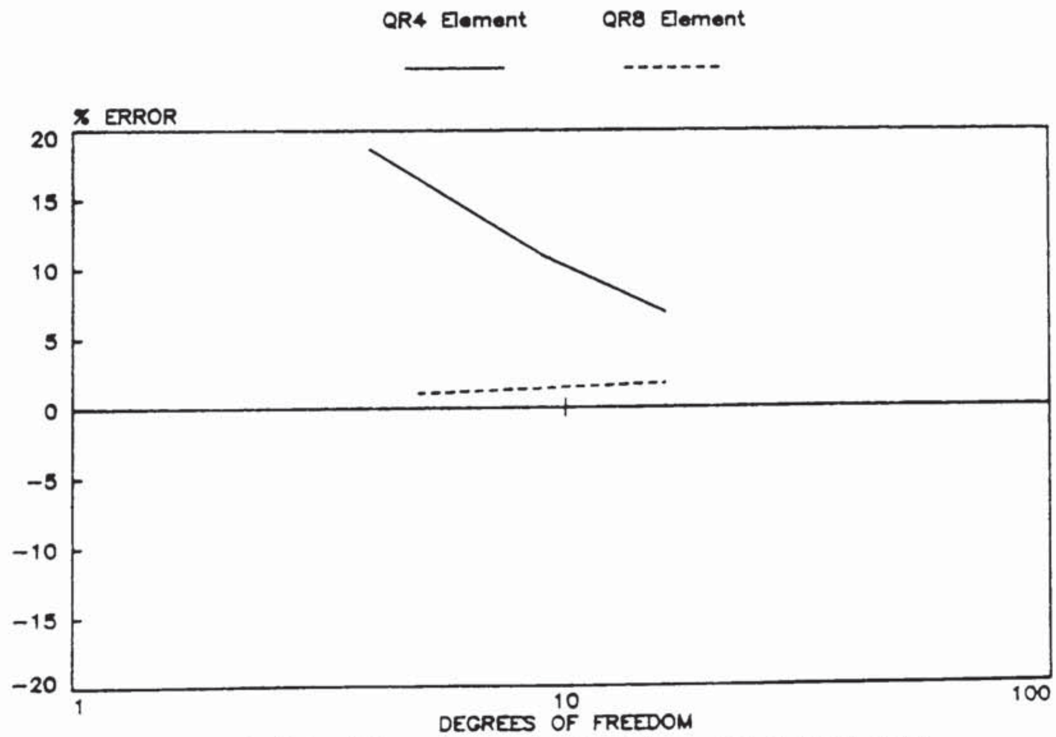


Fig 8·40 Prediction of second natural frequency of CSCS plate

Mixed Finite Element Models

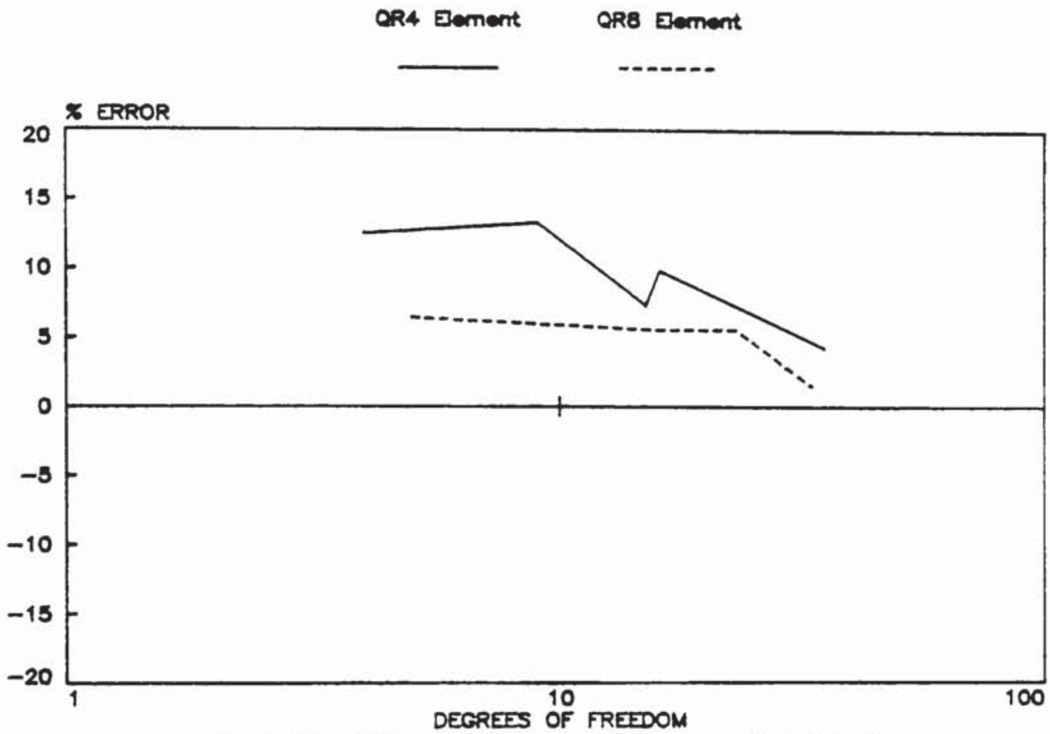


Fig 8·41 Prediction of third natural frequency of CSCS plate

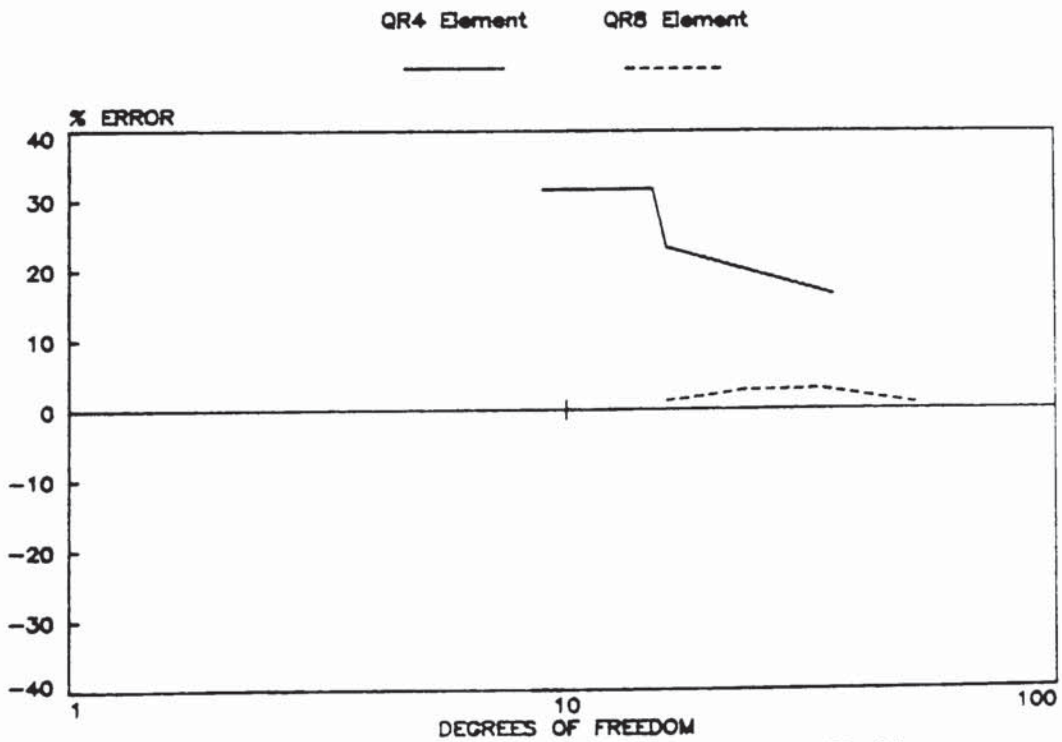


Fig 8·42 Prediction of fourth natural frequency of CSCS plate

Mixed/Displacement Finite Element Models

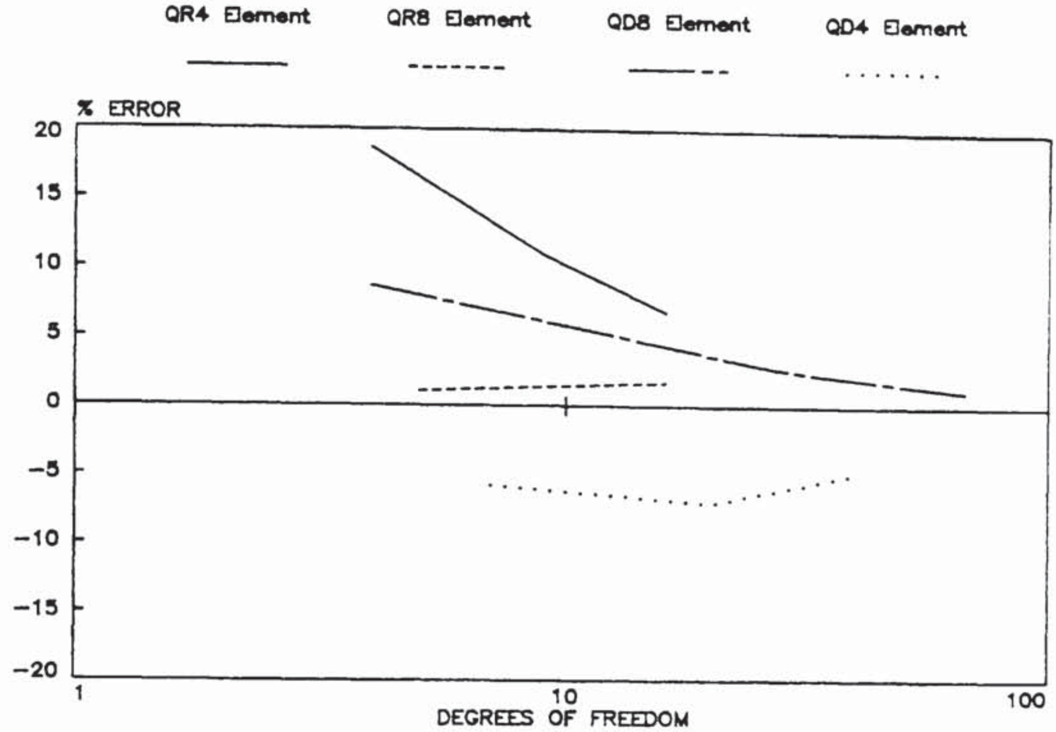


Fig 8.43 Prediction of second natural frequency of CSCS plate

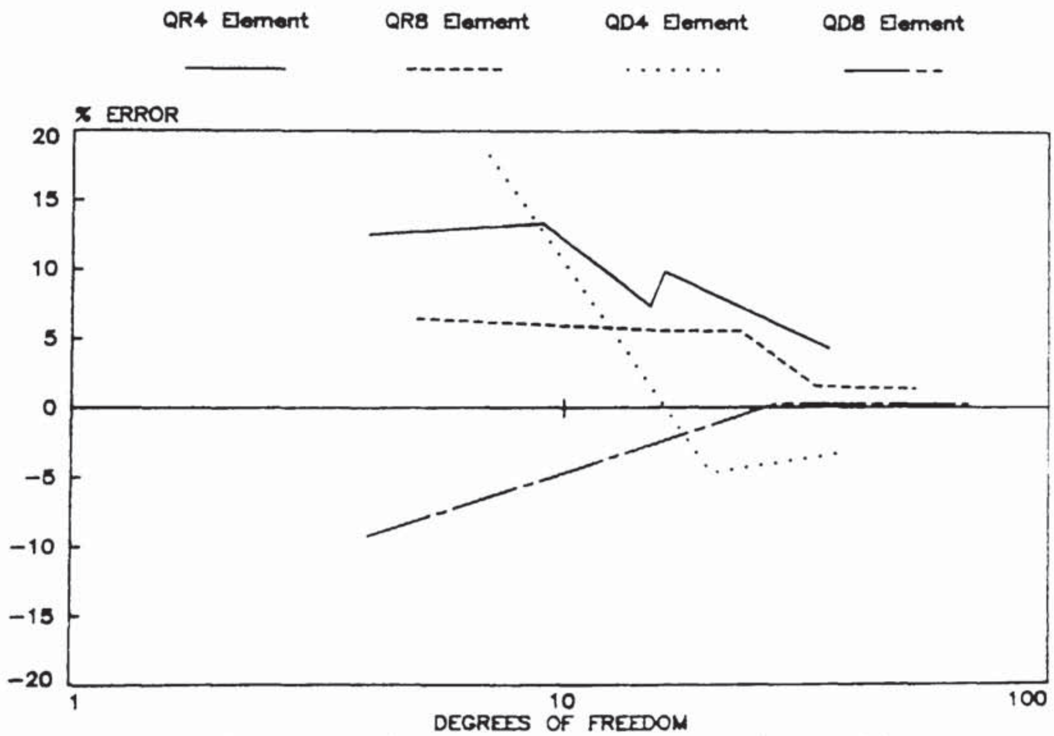


Fig 8.44 Prediction of third natural frequency of CSCS plate

Table 8.4 Eigenvalues of a simply supported plate.

Square Plate	$\lambda = \omega\ a^2\sqrt{\frac{\rho h}{D}}$			
Number of half waves in x & y directions	Exact Ref(10)	Mesh(QR8)		
		2x2	3x3	6x6
m=1,n=1	19.74	20.03	19.77	19.741
m=2,n=1	49.35	52.63	50.393	
m=2,n=2	78.96	135.53	83.122	
m=3,n=1	98.70	137.74	101.06	99.36

Table 8.5 Eigenvalues of a clamped–simply supported plate.

Square Plate	$\lambda = \omega a^2\sqrt{\frac{\rho h}{D}}$			
Mode Number	Exact Ref(10)	Mesh(QR8)		
		2x2	4x4	4x6
1	28.95	29.397	29.07	28.99
2	54.74	55.333		
3	69.32	73.803	73.163	70.36
4	94.59	137.76		
5	102.2	169.372	105.966	102.877

Table 8.6 Vibration eigenvalues of square cantilever plate

Square Plate	$\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}}$					
Number of degrees of freedom	Vibration Mode					
	Source	1	2	3	4	5
6	QR4	3.364	7.067	22.047	24.853	25.947
12		3.426	7.575	22.607	27.37	28.254
30		3.454	7.986	21.856	27.152	29.681
5	QR8	3.545	6.993	20.53	26.055	26.388
16		3.470	8.025	22.049	27.068	29.268
33		3.467	8.205	21.36	26.8	29.87
6	QD4	3.329	9.256	30.632	35.934	47.452
12		3.296	8.865	17.137	31.267	31.939
18		3.458	8.849	21.796	26.686	30.951
36		3.468	8.825	21.709	27.185	31.5
15	QD8	3.442	8.782	21.60	28.192	31.397
48		3.458	8.785	21.28	27.47	31.795
99		3.429	8.671	21.20	27.48	31.64
Experi-mental	Ref(69)	3.37	8.26	20.55	27.15	29.75
Energy solution	Ref(70)	3.494	8.547	21.44	27.46	31.17
Ritz Method	Ref(10)	3.4917	8.5246	21.429	27.331	31.111

8.5 NUMERICAL EXAMPLES ON FORCED VIBRATION OF PLATES

In this section some numerical tests for the solution of problems concerned with the forced vibration of plates are presented. The tests are aimed at illustrating the performance of mixed quadrilateral elements (QR4, QR8) in the solution of dynamic transient problems. The results are compared with the solutions from analytical and from displacement type formulations. The convergence and accuracy of the results are determined as the element sub-division of the plate is refined. In these examples, the solution of the dynamic equilibrium equations are obtained using the unconditionally stable direct integration of Wilson θ with a time step size of $\Delta t = .001$ sec.

Some numerical tests are given at the end of this section to show the effect of time-step size on the stability and accuracy of the solution. It is also possible to use the mode superposition approach to solve the equilibrium equations. Two further examples are given to show the effect of number of modes, included in a mode superposition analysis, on the accuracy of the solution.

8.5.1 Simply-supported square plate under uniform loading, varying sinusoidally with time

This example is chosen to show the accuracy and convergence rate of the present finite element models by comparing the results with the exact solution given in Ref. (32). The finite element meshes used for this example are shown in Figure (8.45) together with the loading condition. Using symmetry, only a quarter of the plate is analysed. From equation (3.81), it can be easily checked that under these conditions, assymetric modes can not be excited and therefore need not be included in the response calculations. The convergence of the centre deflection of the plate with mesh refinement is presented in Figures (8.46) and (8.47). The moment response M_x is plotted in Figures (8.48) and (8.49) for the (1 x 1) and (2 x 2) meshes respectively.

The results show that they converge rapidly towards the correct solution as the mesh is refined. It is also observed that mixed models have predicted more accurate results than the displacement models. This is despite the fact that mixed models involve fewer number of degrees of freedom than the displacement models.

Figure (8.50) shows the bending moment response M_x obtained by using the 8-noded quadrilateral elements of QR8 and QD8. The finite element idealisation using the QD8 element is (2 x 2) and has 44 degrees of freedom whereas the same mesh with QR8 element has only 12 degrees of freedom. It is seen that the results predicted with use of QR8 element are more accurate than those from QD8 element.

8.5.2 Simply supported square plate under point load, step force input

The purpose of this example is once again to show the accuracy and convergence of the predicted results when the structure is under the severe condition of point loading. The finite element grids used in this test together with the forcing function are shown in Figure (8.51). Due to symmetry only ¼ of the plate is analyzed.

The lateral deflection under the point of application of the load and bending moment M_x are plotted in Figures (8.52) to (8.57) together with the exact results.

From these figures, it is obvious that mixed models based on elements QR4 and QR8 have predicted the transient deflection and bending moment with good accuracy and that the solutions improve as the mesh is refined. Figures (8.54) and (8.57) show the results for deflection and bending moment respectively, obtained by using the displacement type elements of QD4 and QD8. Considering the fact that the number of degrees of freedom used in these models is almost three times that of corresponding mixed models, we can conclude that mixed type elements can be much more efficient than the corresponding displacement elements. For example, the results obtained from a (2 x 2) mesh of QR8 elements with 12 degrees of freedom are comparable with those obtained using the (2 x 2) mesh of QD8 elements having 44 degrees of freedom.

8.5.3 Clamped square plate under point load, step force input

In this example the performance of the mixed linear and quadratic elements (QR4 and QR8) are assessed by comparing with the results obtained from a (2 x 2) finite element idealisation based on the 8-noded displacement element with 28 degrees of freedom.ǂ The plate structure and the forcing function are shown in Figure (8.51). In figures (8.58) and (8.59) the central deflection and bending moment, M_x obtained from various meshes of QD4 element are presented. It is seen that results converge to those predicted from QD8 element, as the element sub-division increases. The same test has been carried out with finite element models using the mixed element QR4 and QR8. Figures (8.60) and (8.61) show the convergence of the mid-point deflection and bending moment for models using QR4 element.

ǂ The accuracy of the solution obtained using this element has been demonstrated in sections (8.5.1) and (8.5.2) for the solution of simply supported plate.

The corresponding solutions based on QR8 element are presented in Figures (8.62) and (8.63). It is seen that mixed models can favourably predict the plate response and the solutions are comparable with those from displacement models.

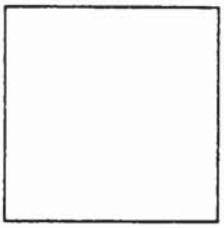
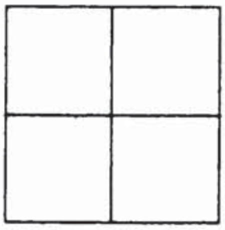
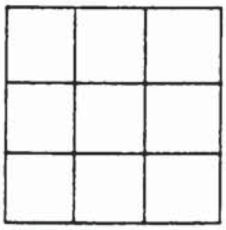
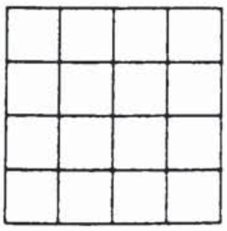
In the numerical tests reported in this section, several convergence plots were obtained indicating that in general, the appropriate order of convergence is obtained with mesh refinement. One important point should however be noted which is related to the computational time for the response calculations. Using the displacement type formulation, the bending moment response at a specific node is to be calculated at each increment of time through differentiating the pre-determined nodal displacements.

This procedure is repeated for all the elements sharing the specific node and the bending moment is determined by averaging the values from each element. This is a time-consuming process and consequently requires much more computational time than in the mixed models where nodal bending moments are calculated through a simple matrix transformation procedure. In table (8.7) the computational time spent at the response stage for various finite element models is indicated.

8.5.4 The effect of time step size, Δt on the numerical stability and accuracy of the solution

In the numerical tests presented in sections 8.5.1 to 8.5.3, integration of the equations of motion of the finite element assemblage was carried out using the Wilson θ method which is an unconditionally stable integration scheme. To test the stability and

Table 8.7 Computer execution time at the response analysis process by Wilson theta method. *

Finite Element Model	Element Type	Number Of Degrees Of Freedom	Time for response calculations(sec)
	QD4	1	240
	QR4	1	5
	QD8	5	600
	QR8	3	18
	QD4	8	300
	QR4	4	20
	QD8	28	840
	QR8	12	60
	QD4	21	420
	QR4	9	30
	QD4	40	660
	QR4	16	50

* Processed with HP 9845B desk top with enhanced processor.

accuracy of the solution as the time step size increases, two cases are studied. Figure (8.64) shows a square simply supported plate which is discretized by a (2 x 2) mesh of QR8 elements. The plate is subjected to either one of the two types of loading shown in Figure (8.64). The time step sizes used in the solutions are $\Delta t = .0001$, $\Delta t = .001$, $\Delta t = .005$ and $\Delta t = .01$.

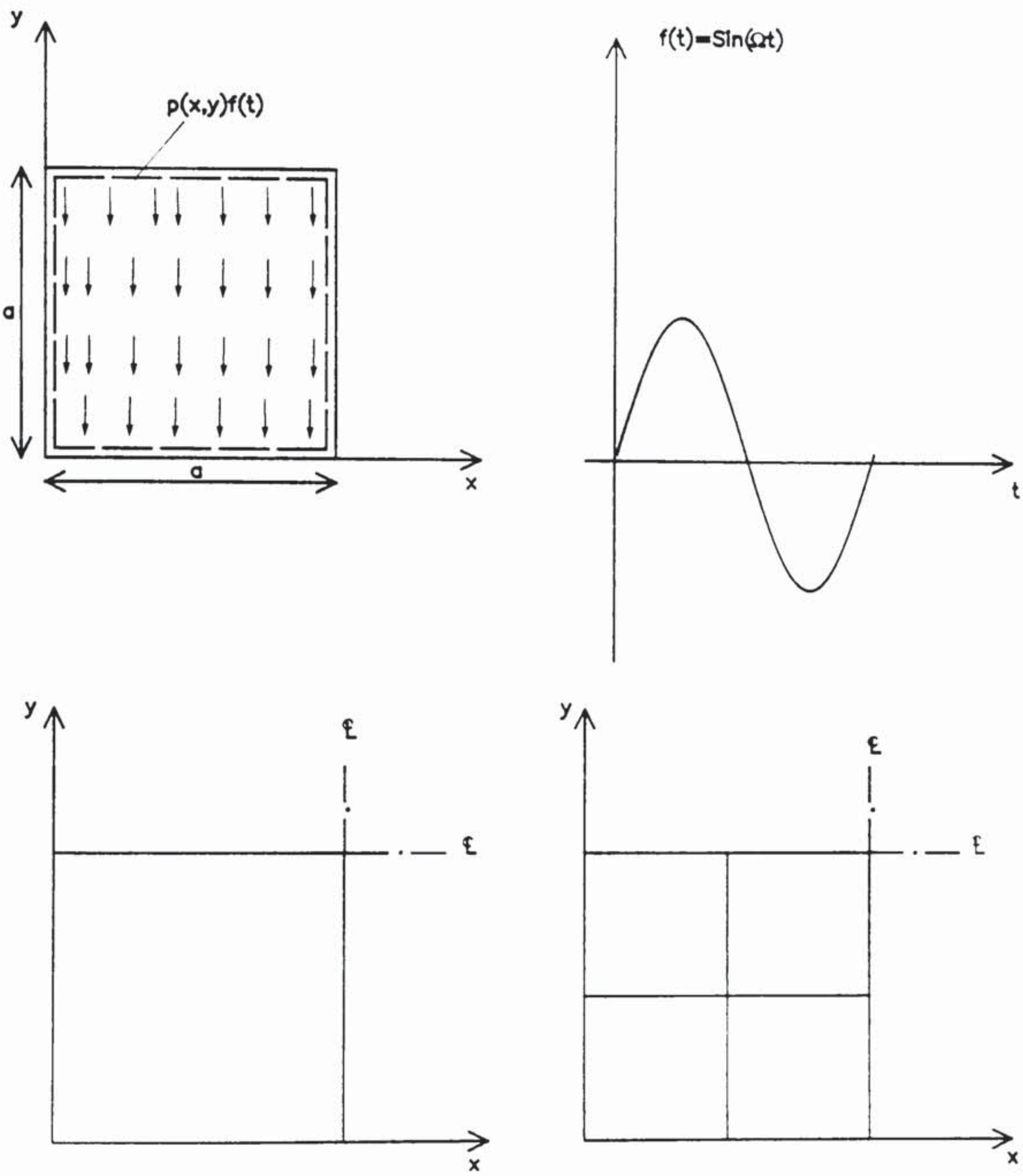
Figures (8.65) to (8.68) show the deflection and bending moment responses of the SS plate under the two conditions of loading. It is interesting to notice that the accuracy of the solution is significantly reduced only for the largest time step size, that is for $\Delta t = .01$. For the other three time-step sizes, the solutions remain bounded and the accuracy of the results is acceptable. In particular, Figure (8.65) shows that the displacement response predicted by a time step size of $\Delta t = .001$ is more accurate than the one from $\Delta t = .0001$. This can be attributed to the fact that with $\Delta t = .0001$, the response will be affected by the contributions from higher, inaccurate modes of the finite element assemblage.

8.5.5 <u>The effect of number of modes on the accuracy of the solution from mode-superposition method</u>

The mode superposition procedure can in some practical problems be more efficient than a direct step by step integration method. To demonstrate this, the simply supported plate of the previous example under the impulsive load ($f(t) = 1 \quad .01 \leq t \leq .015$) is analyzed. The finite element model being used is once again a (2 x 2) mesh of QR8 elements.

Figures (8.69) to (8.71) show the deflection and bending moment responses at the middle of the plate for different number of modes. The total number of modes present in the finite element assemblage is 12.

It is observed that the displacement obtained by using only 1 mode (Fig. 8.69a) is calculated with reasonable accuracy whereas the bending moment is not as good (Fig. 8.70a). On the other hand, the analysis with 3 modes has predicted excellent results for both displacement and bending moment (Figs. (8.69b), (8.70b)). Figures (8.71a) and (8.71b) show the displacement and bending moment responses, respectively, obtained by using the total number of modes in the analysis. It is observed that the solution obtained by using 3-modes compares favourably with the 12-mode solution and no particular accuracy has been gained by increasing the number of modes in the analysis.



$\upsilon = .3$
$h = 1Cm$
$a = 50Cm$
$p(x,y) = 1N/Cm^2$

$E = 2.07 \times 10^7 \ N/Cm^2$
$\rho = 7.8 \times 10^{-3} \ Kg/Cm^3$
$\Omega = 50Hz$

Fig 8.45 Finite element meshes used for the analysis of a thin simply supported plate under steady state loading.

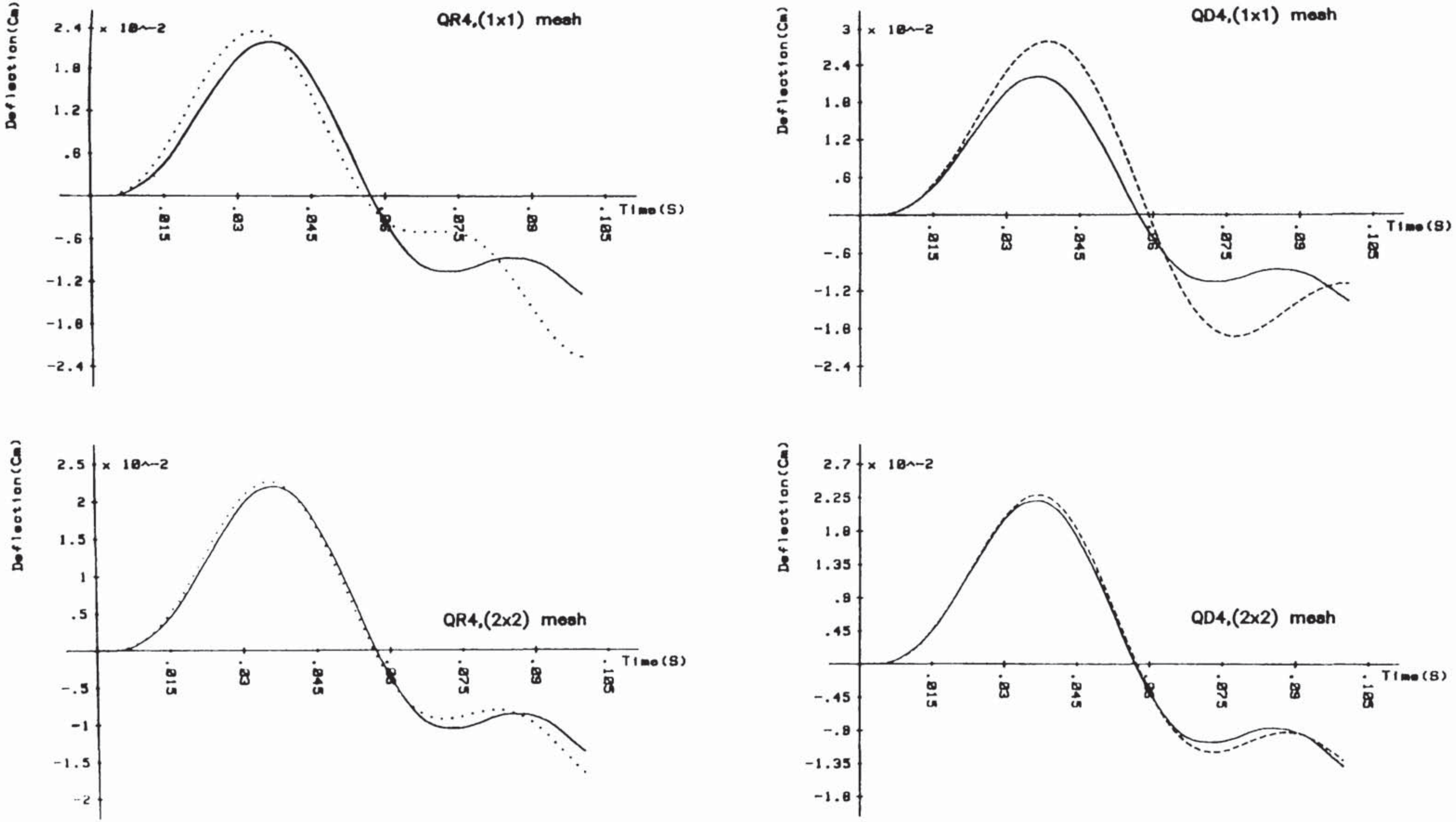


Fig 8.46 Deflection W,vs time;t at the centre of a simply supported plate under uniform loading varying sinusoidally with time.

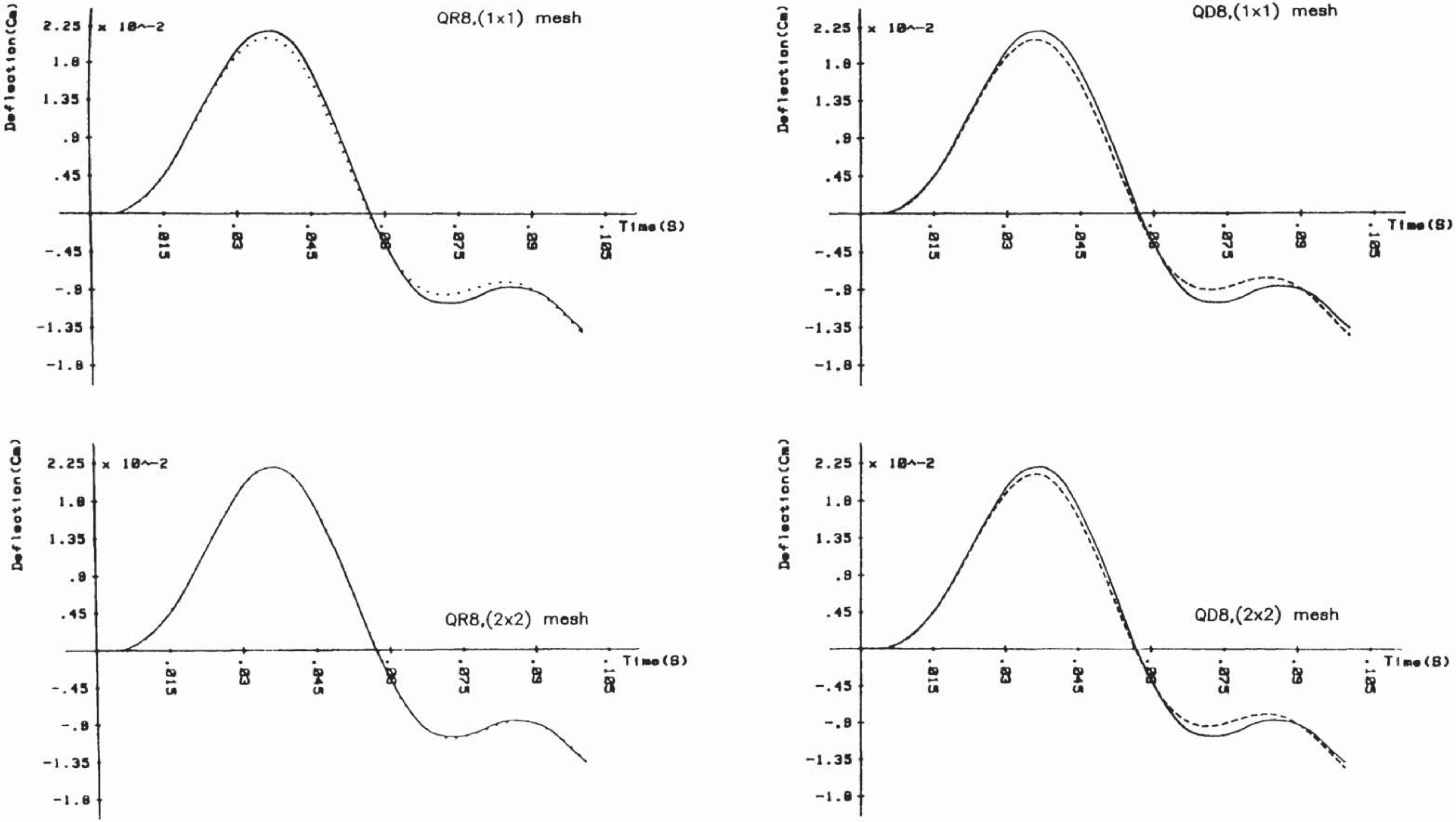


Fig 8.47 **Deflection** W,vs time;t at the centre of a simply supported plate under uniform loading varying sinusoidally with time.

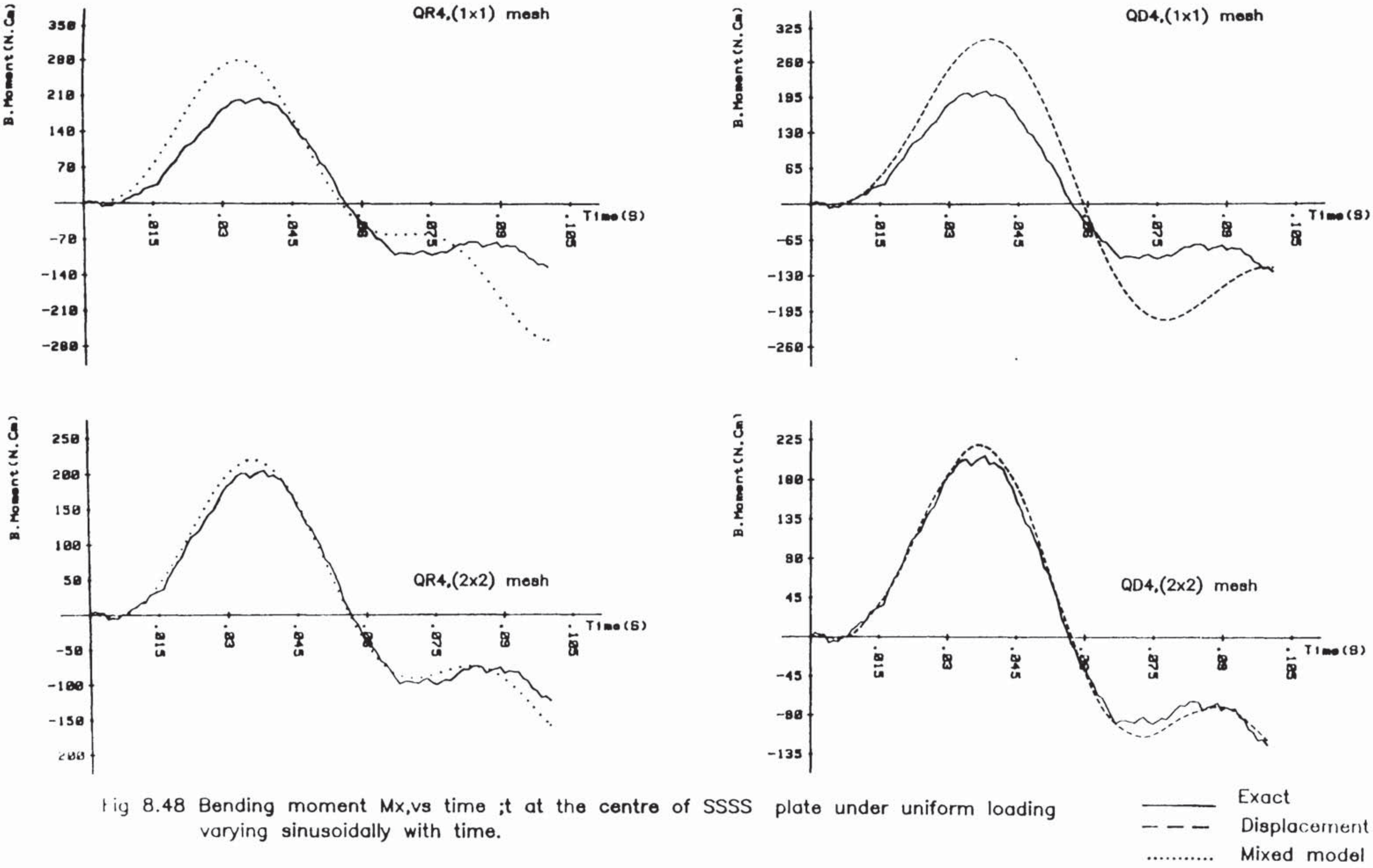


Fig 8.48 Bending moment Mx,vs time ;t at the centre of SSSS plate under uniform loading varying sinusoidally with time.

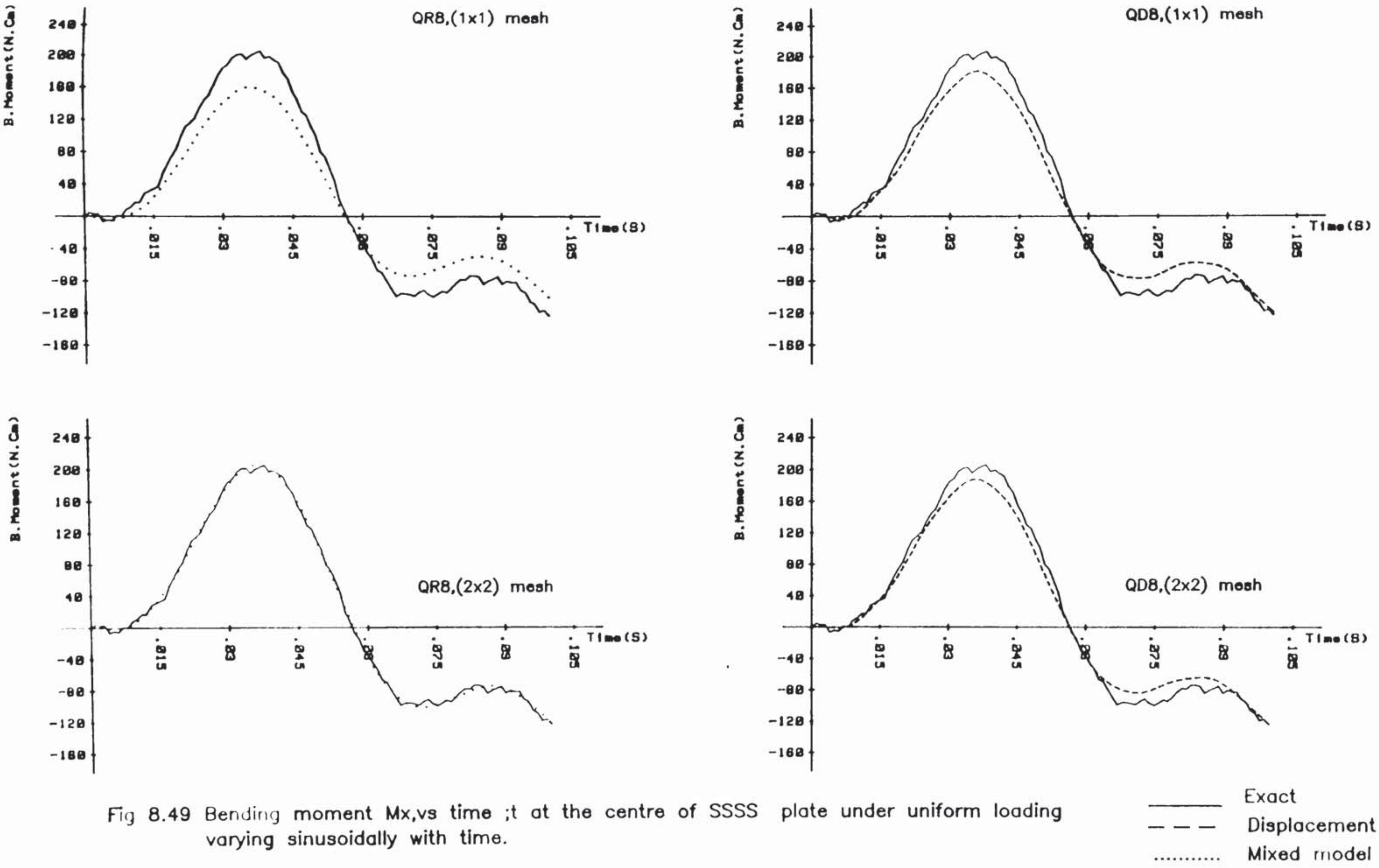


Fig 8.49 Bending moment Mx,vs time ;t at the centre of SSSS plate under uniform loading varying sinusoidally with time.

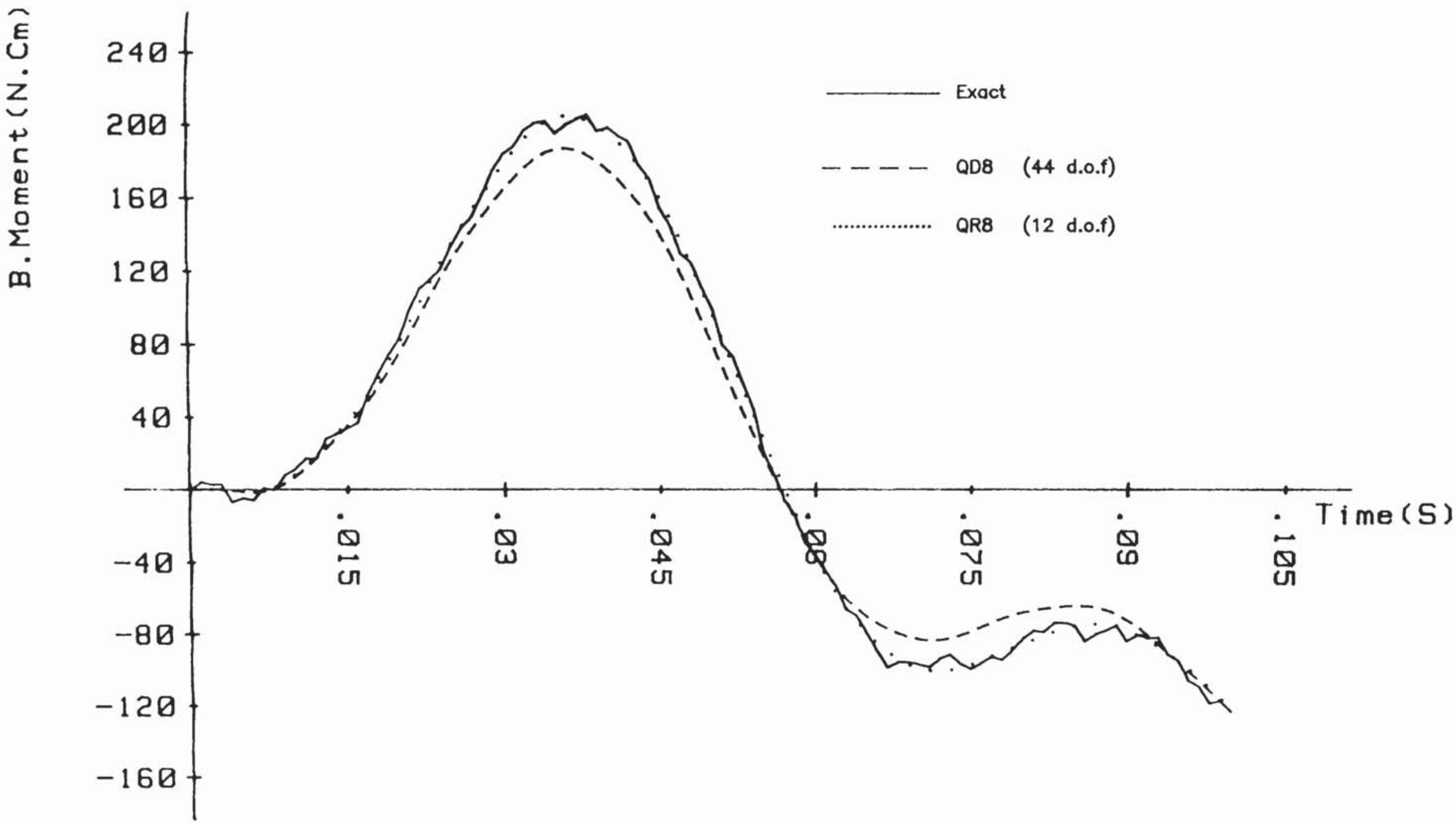


Fig 8.50 Bending moment Mx,vs time;t at the centre of SSSS plate.
Uniform loading,varying sinusoidally with time.

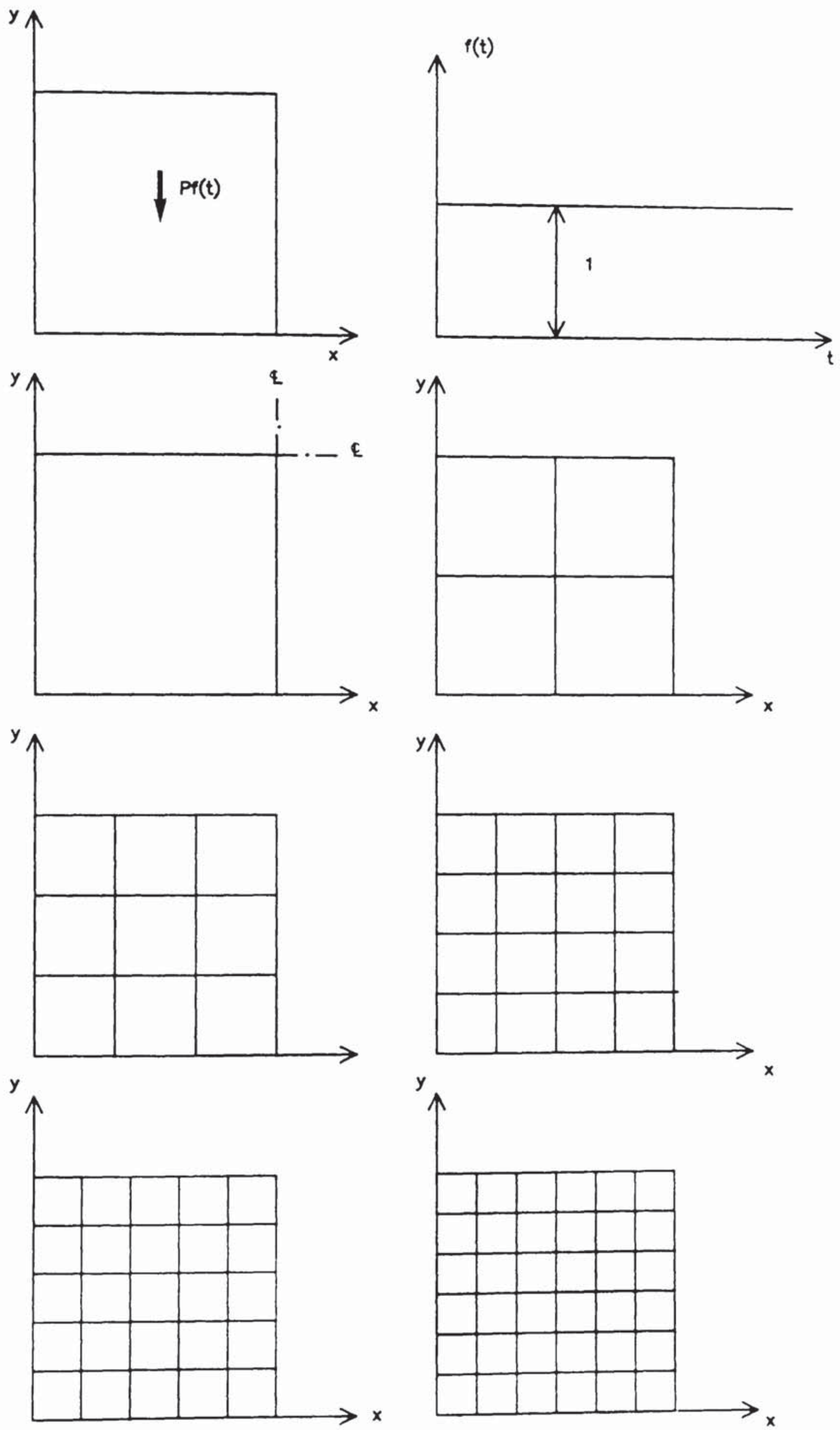


Fig 8.51 Finite element meshes used for the analysis of a square plate under transient point loading.
a=120Cm,h=1Cm,P=100N

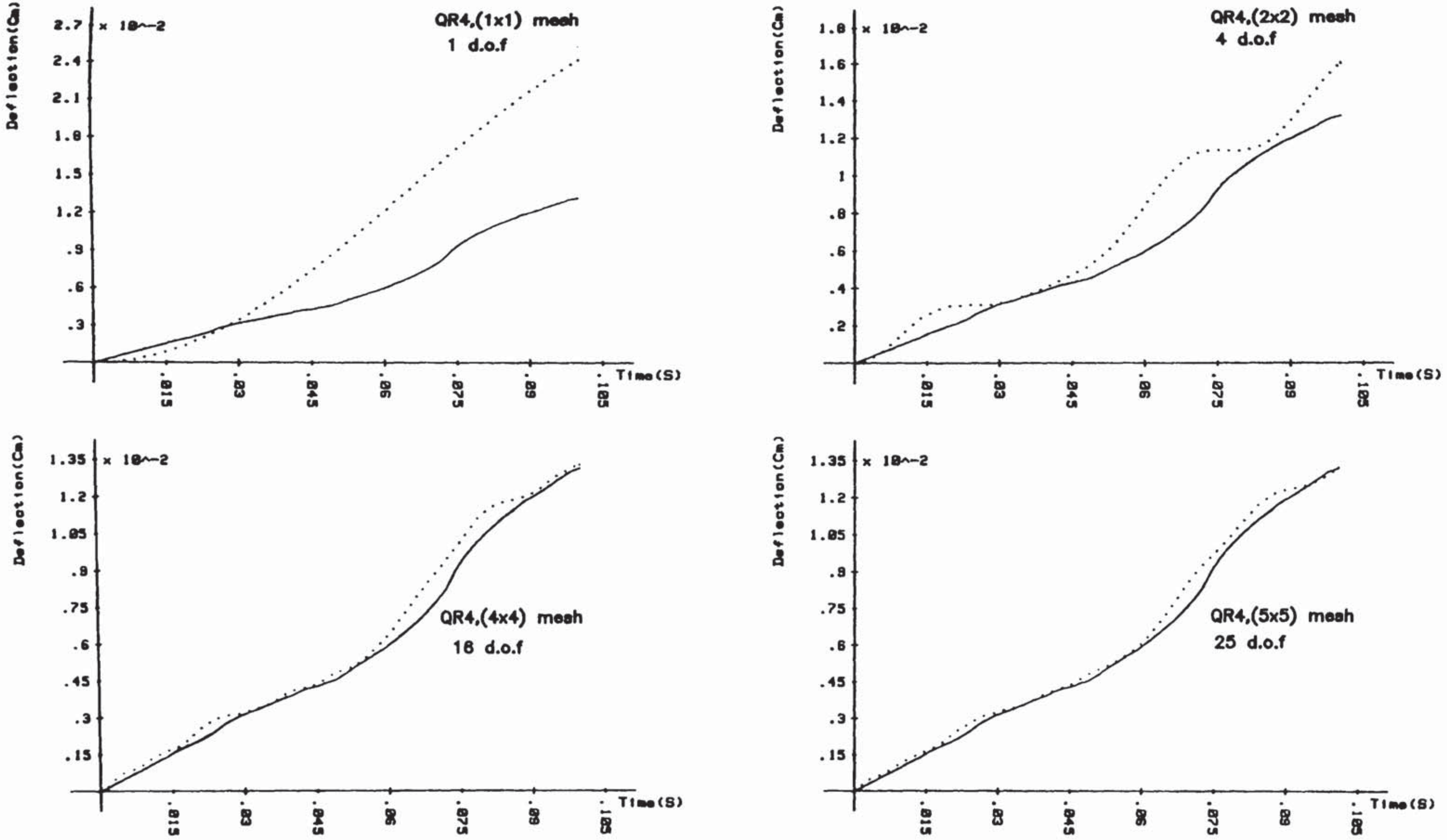


Fig 8.52 Deflection W,vs time ; t at the centre of SSSS plate under transient point load.(QR4 element)

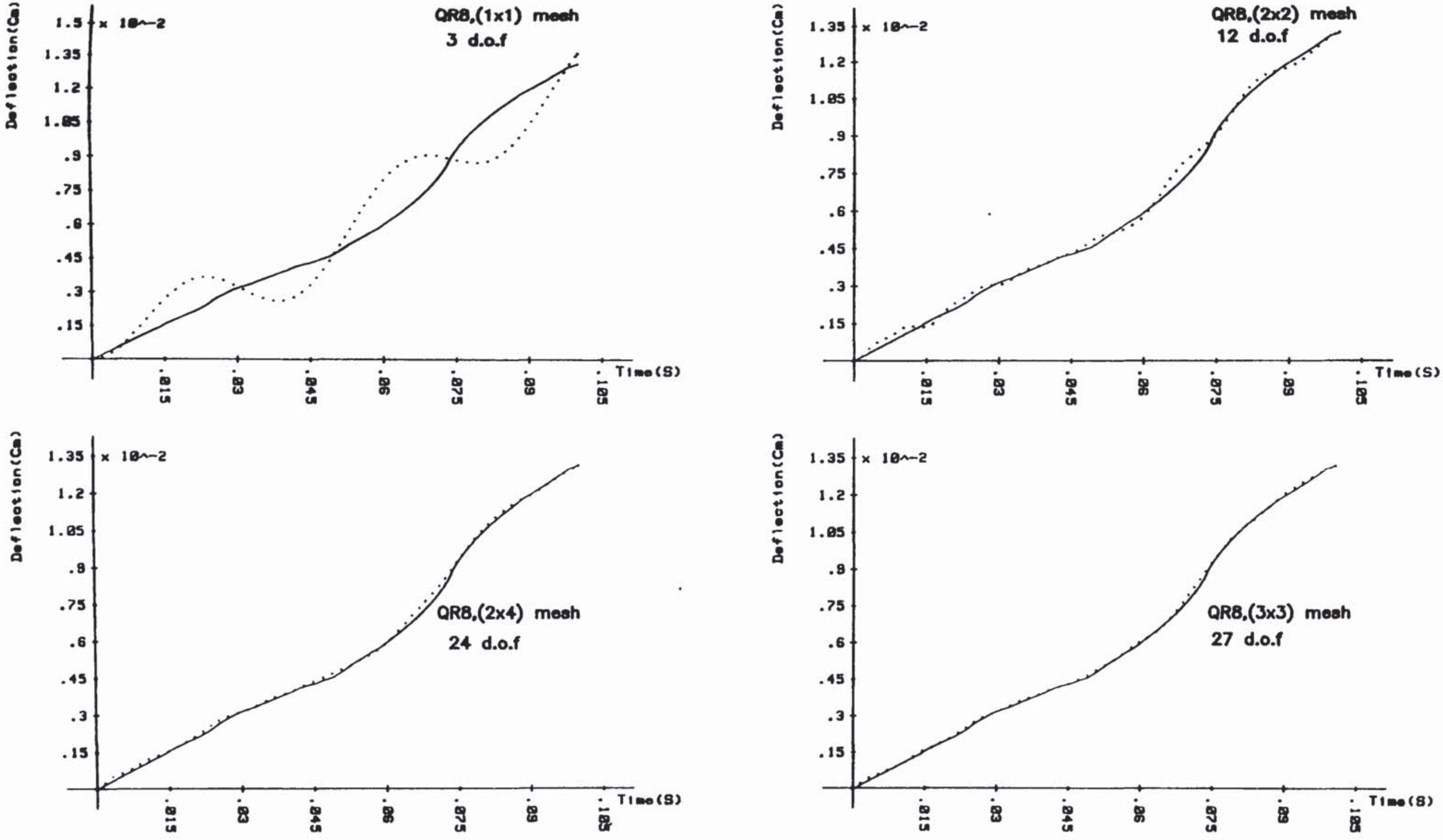


Fig 8.53 Deflection W,vs time ; t at the centre of SSSS plate under transient point load.(QR8 element)

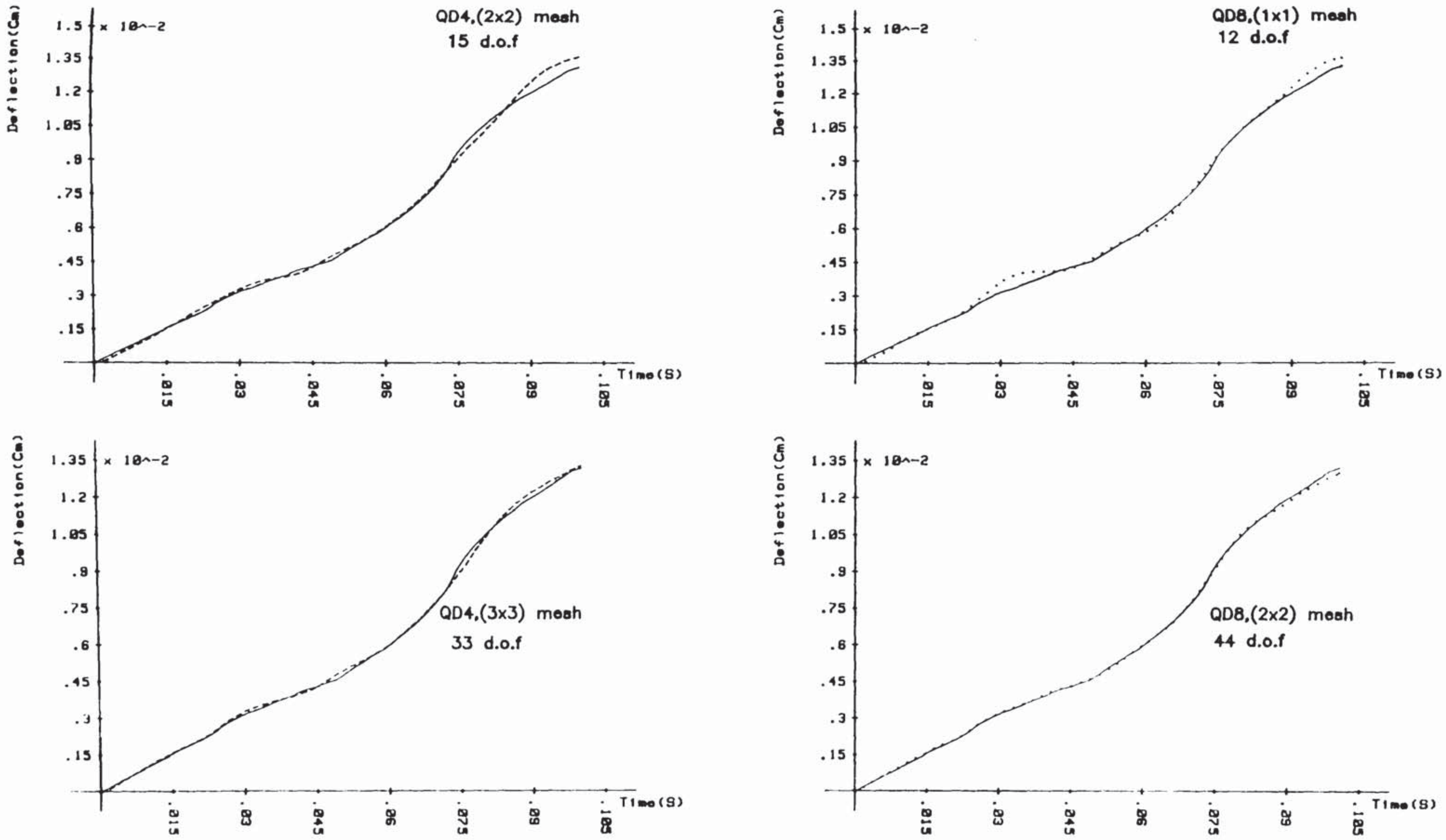


Fig 8.54 Deflection W,vs time ; t at the centre of SSSS plate under transient point load.(QD8 & QD4 elements)

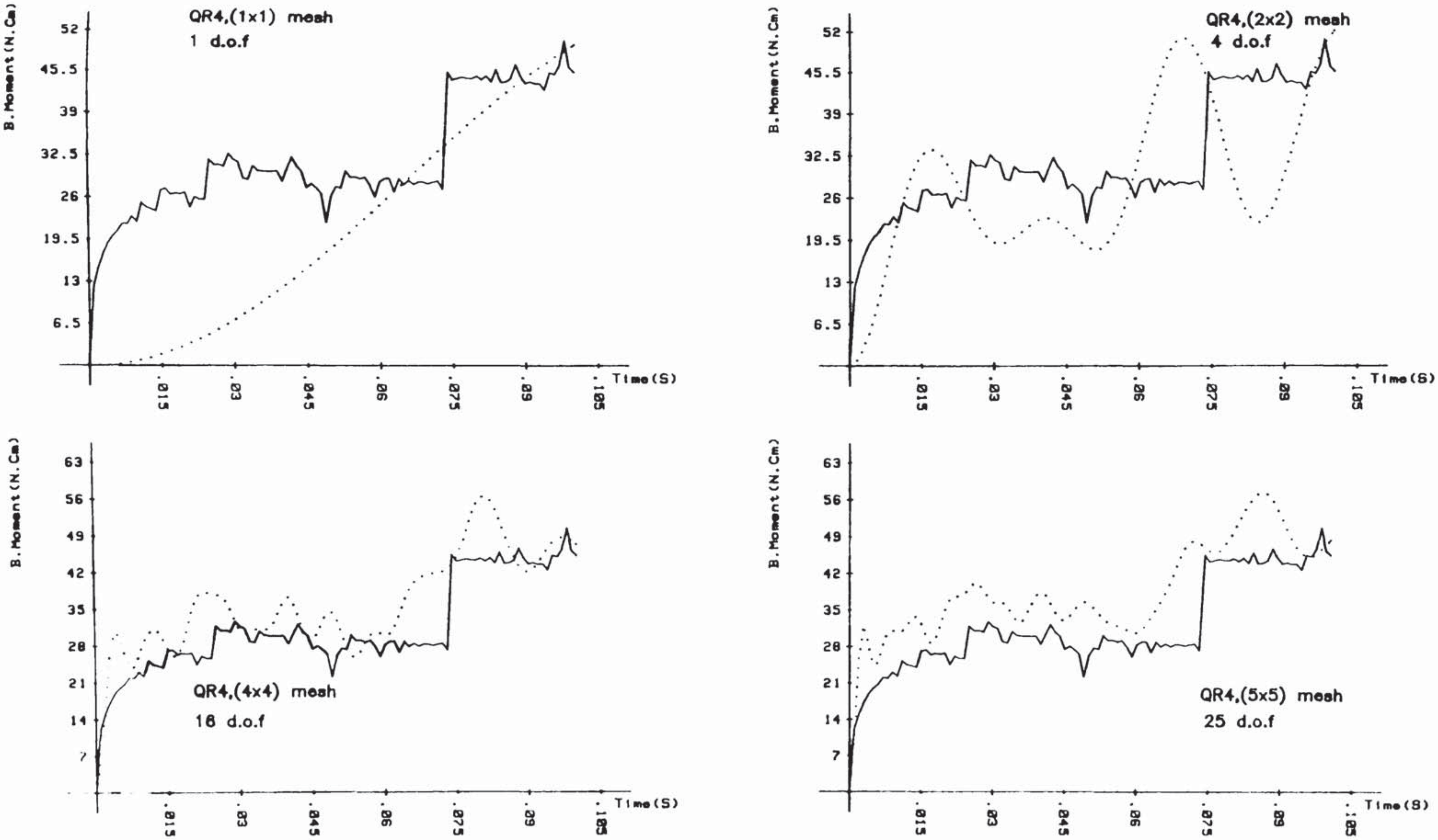


Fig 8.55 Bending moment,Mx vs time ; t at the centre of SSSS plate under transient point load.(QR4 element)

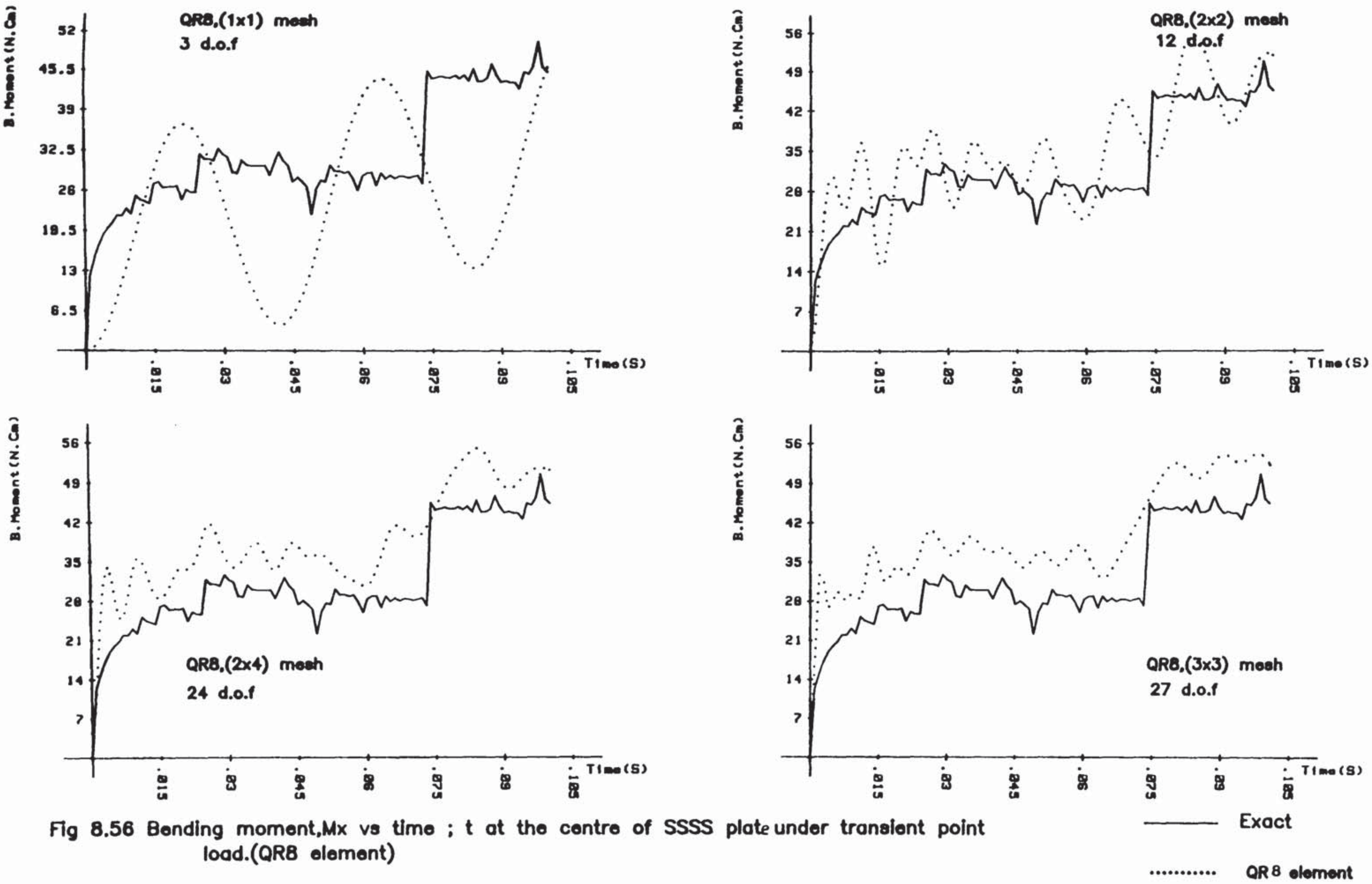


Fig 8.56 Bending moment,Mx vs time ; t at the centre of SSSS plate under transient point load.(QR8 element)

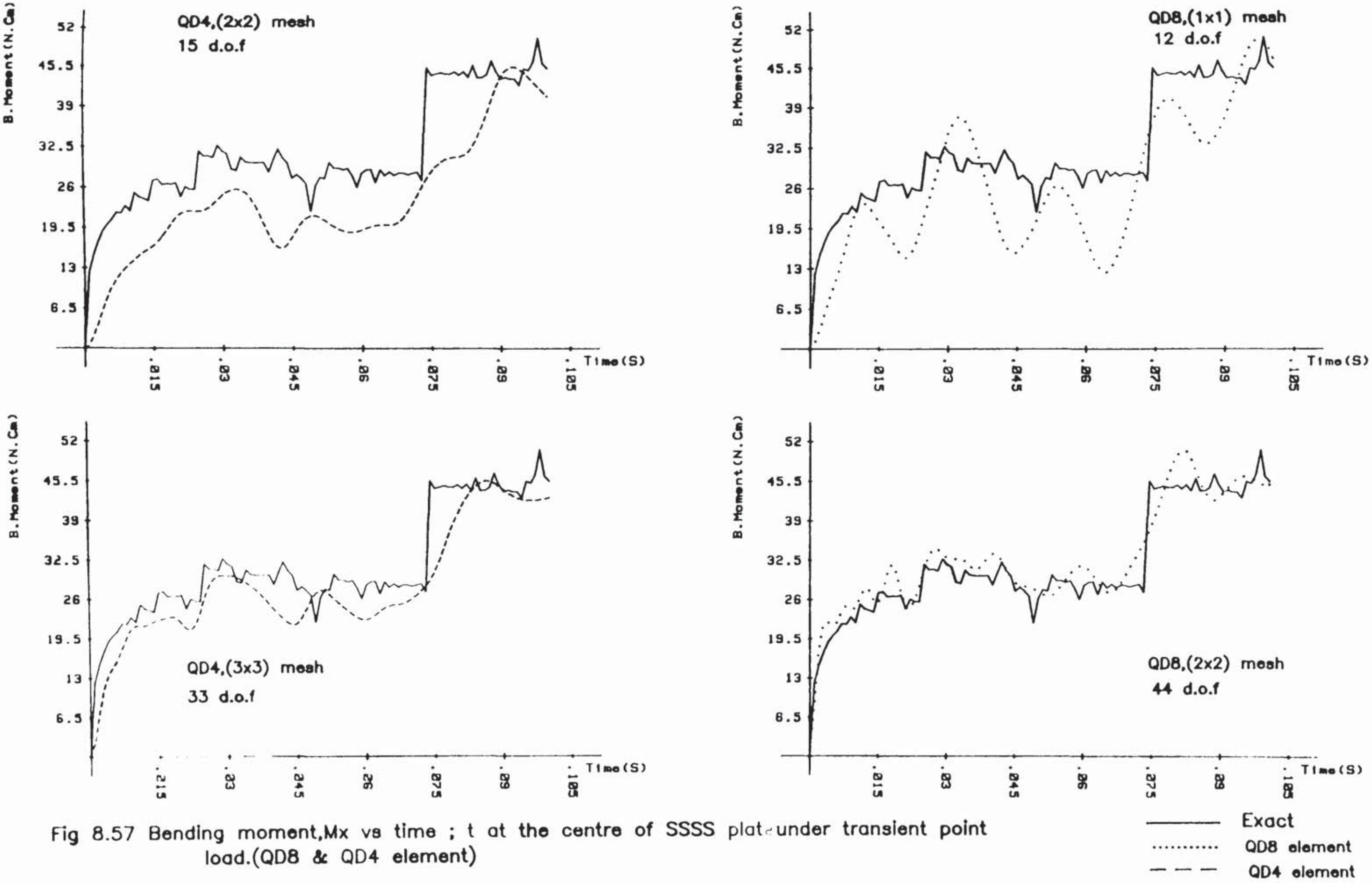


Fig 8.57 Bending moment,Mx vs time ; t at the centre of SSSS plate under transient point load.(QD8 & QD4 element)

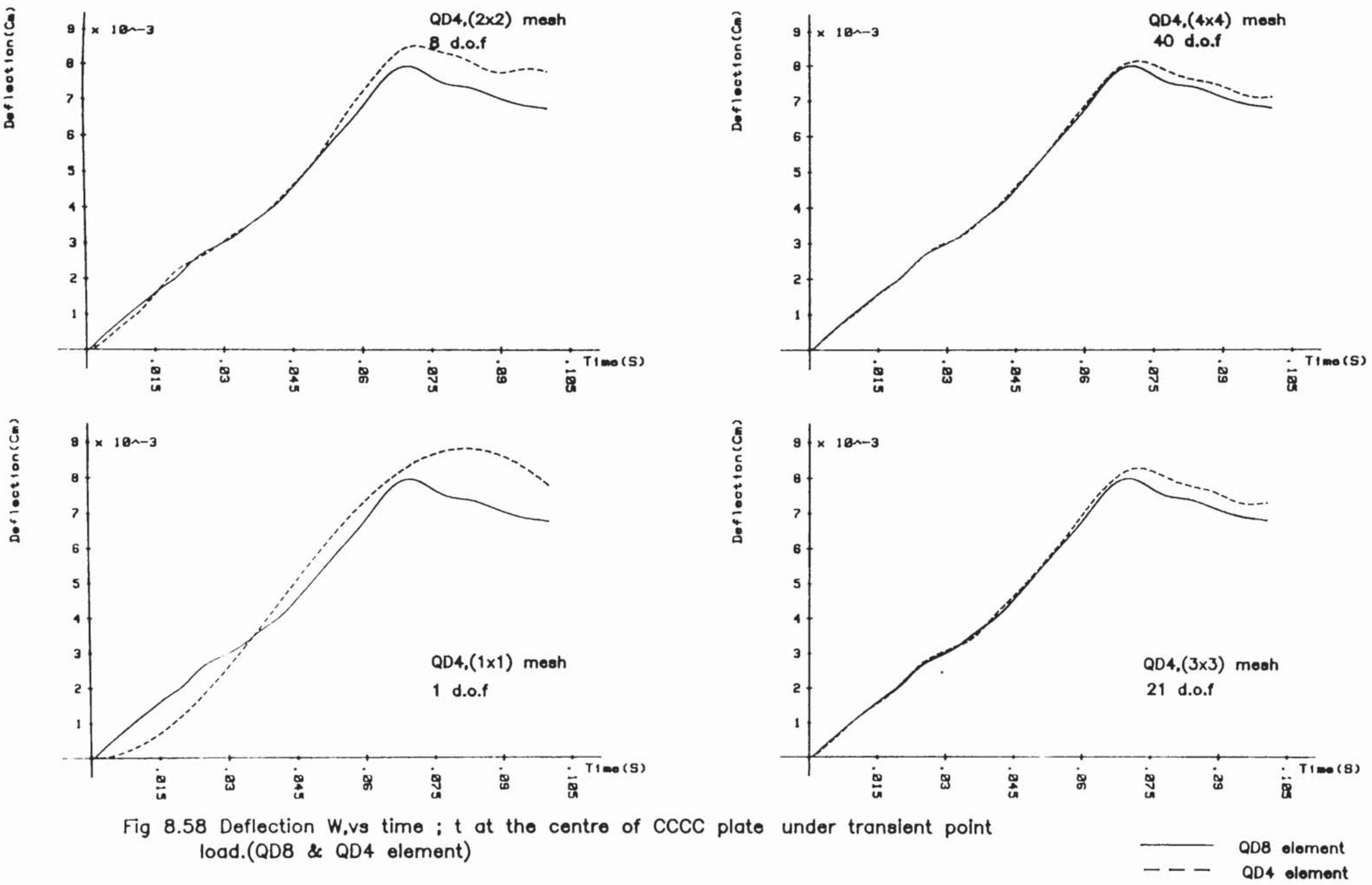


Fig 8.58 Deflection W,vs time ; t at the centre of CCCC plate under transient point load.(QD8 & QD4 element)

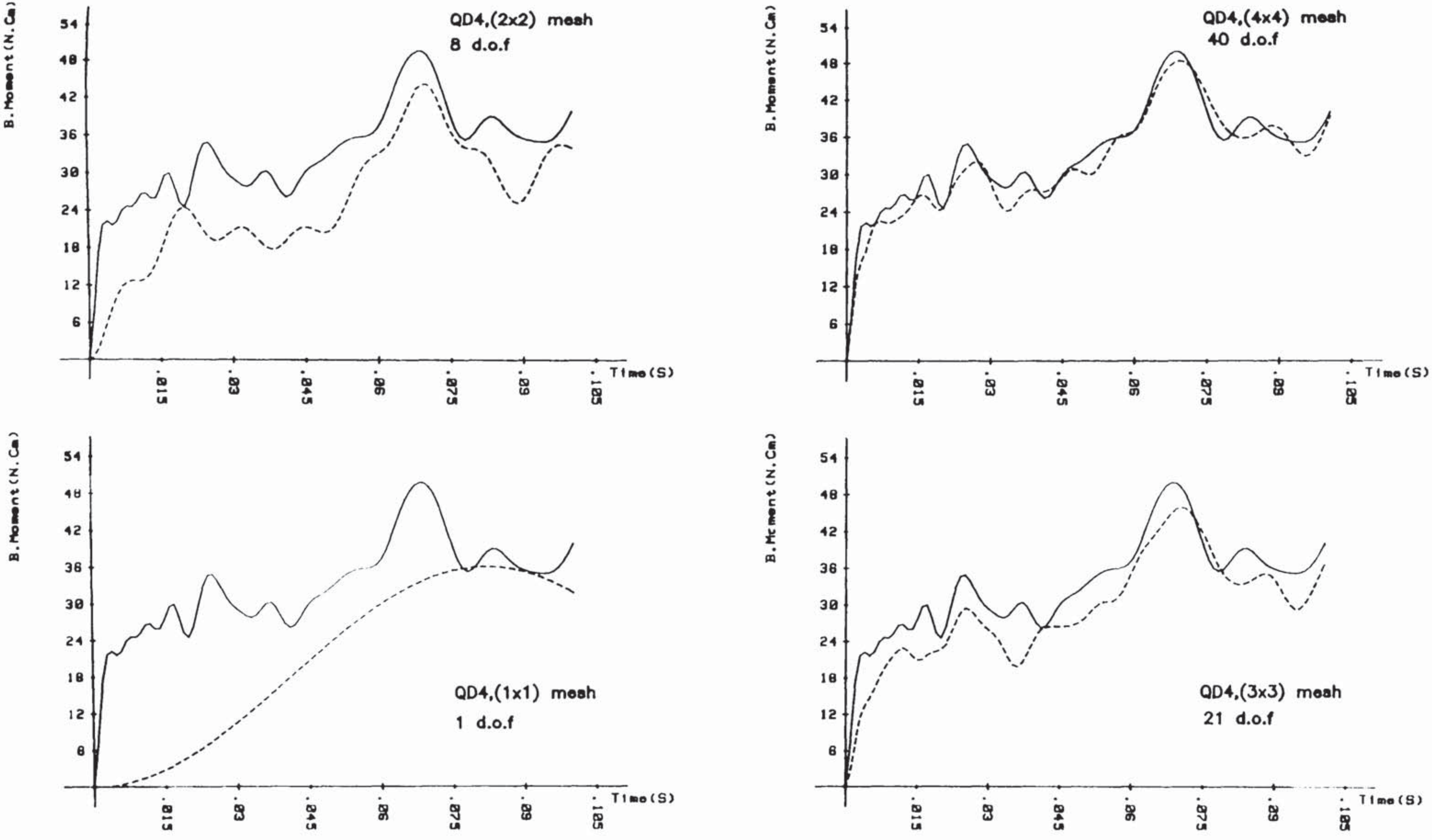


Fig 8.59 Bending moment,Mx vs time ; t at the centre of CCCC plate under transient point load.

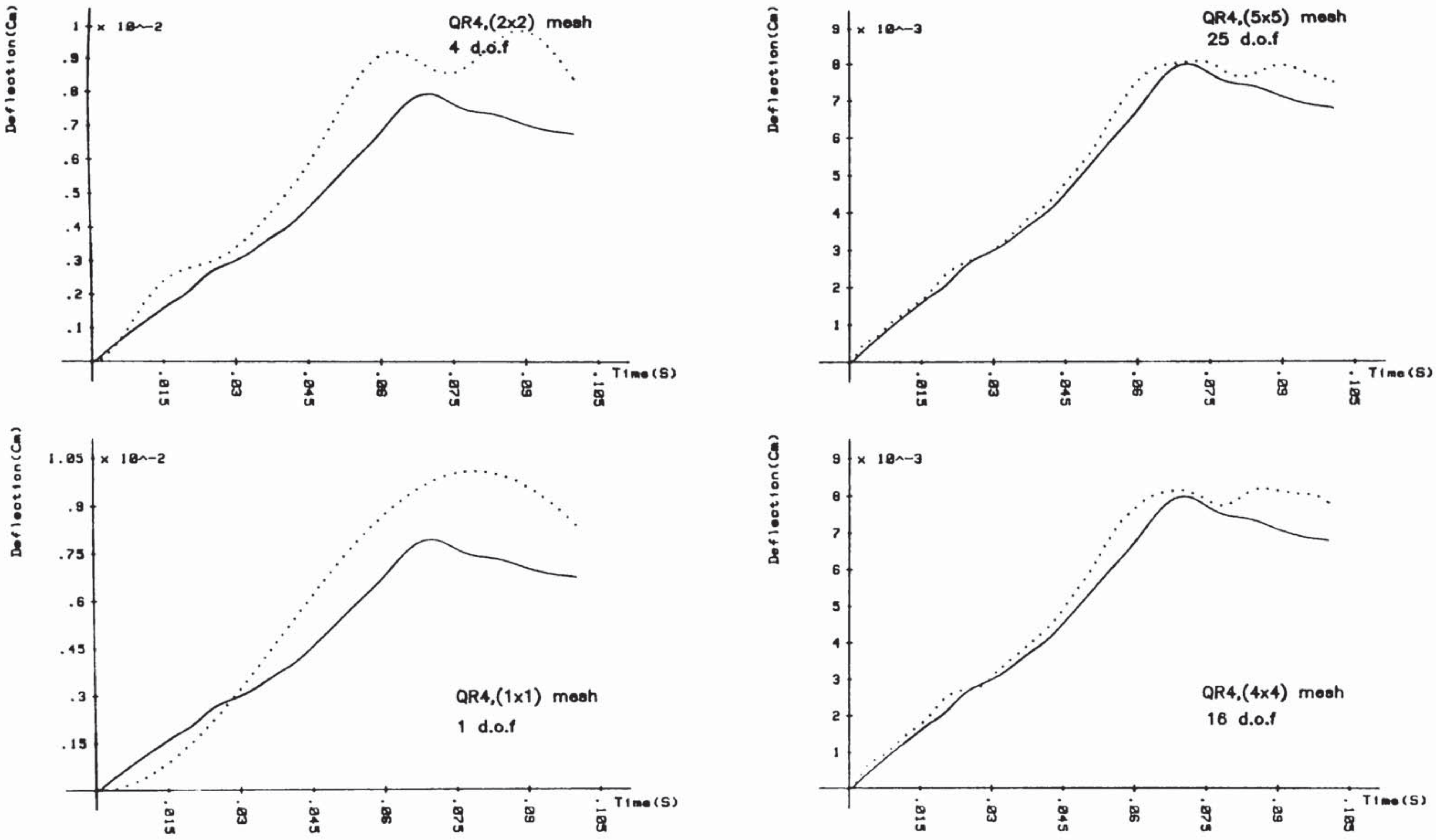


Fig 8.60 Deflection W,vs time ; t at the centre of CCCC plate under transient point load

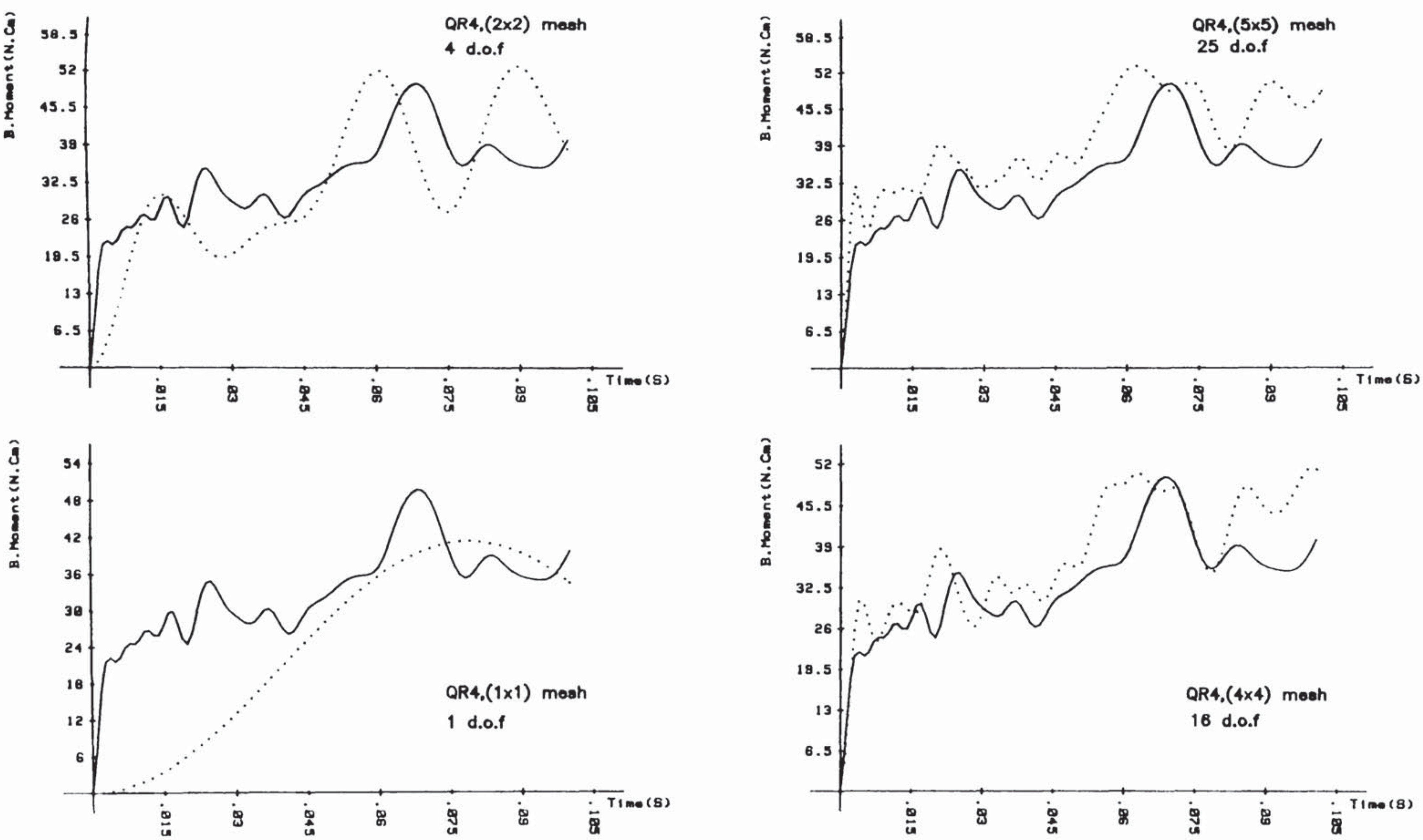
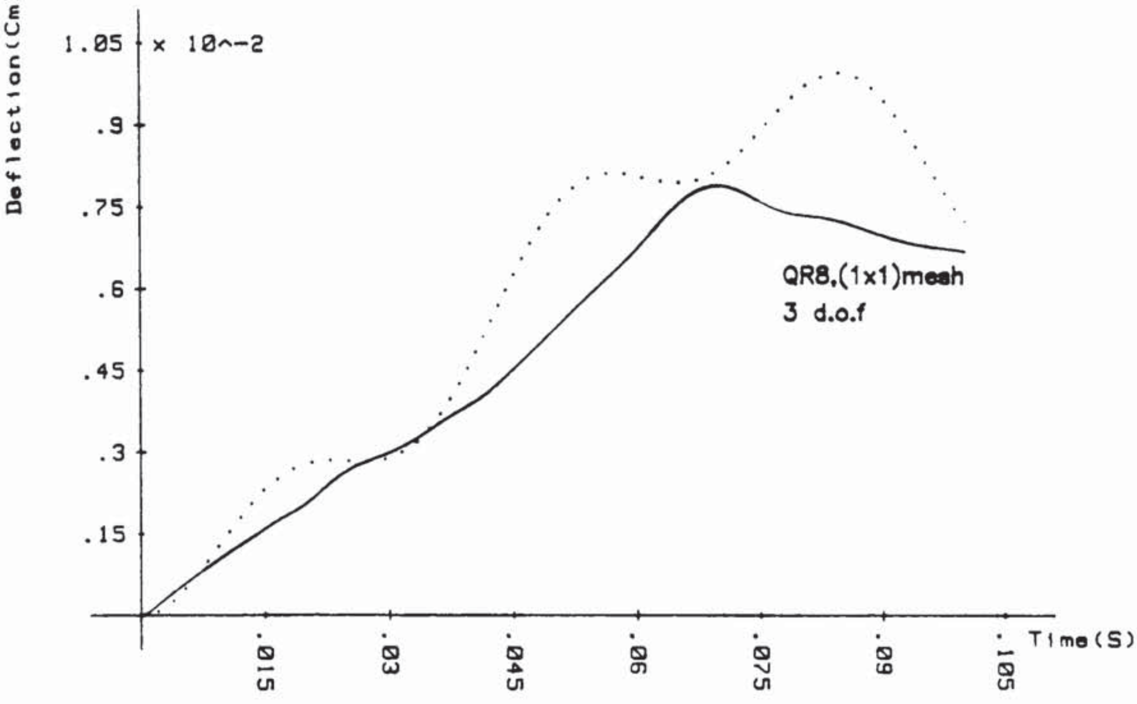


Fig 8.61 Bending moment,Mx vs time ; t at the centre of CCCC plate under transient point loading.



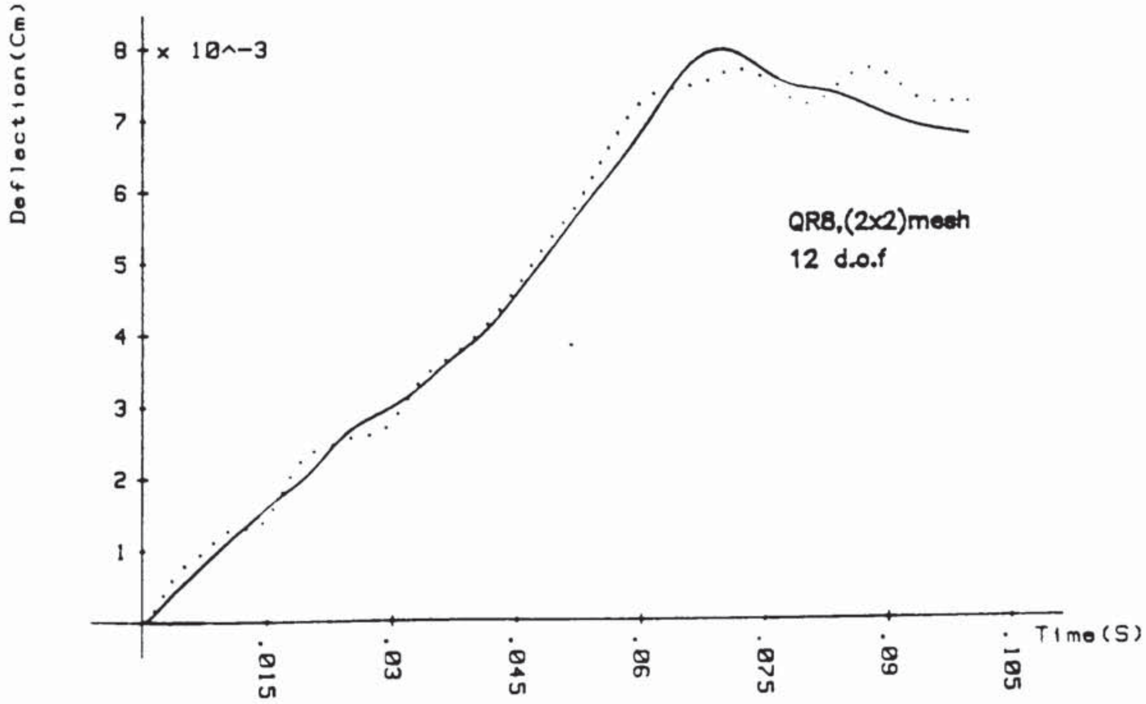
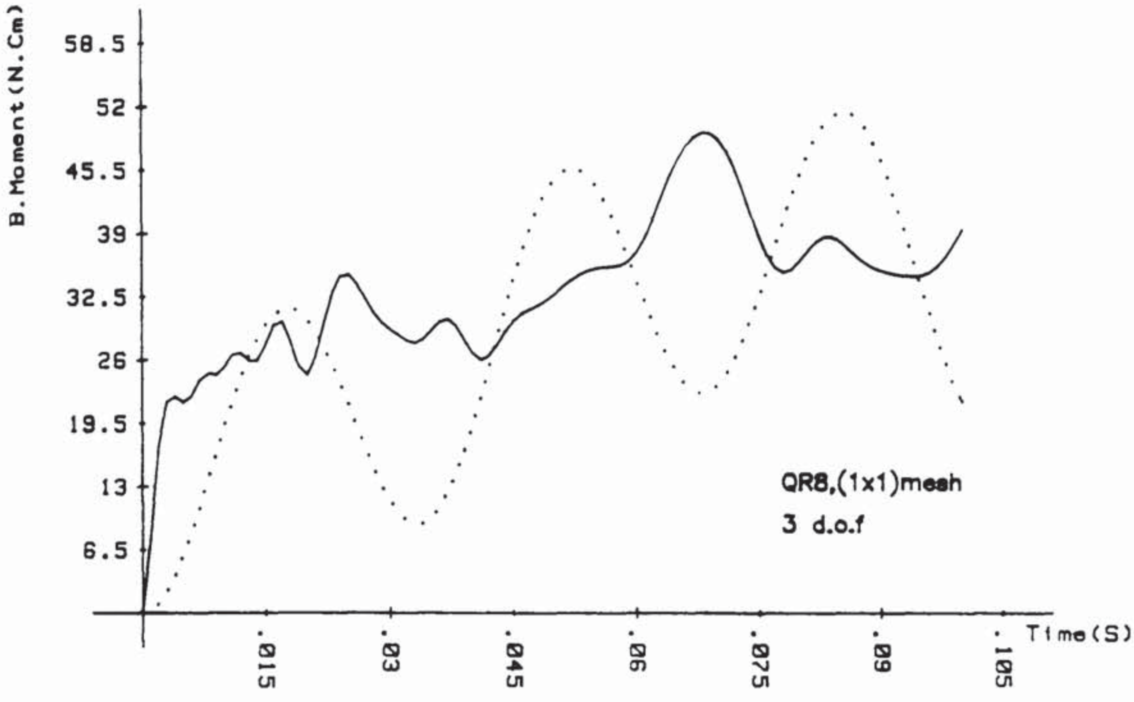


Fig 8.62 Deflection W vs time;t at the centre of clamped plate under transient point load.

——— QD8 element

.... QR8 element



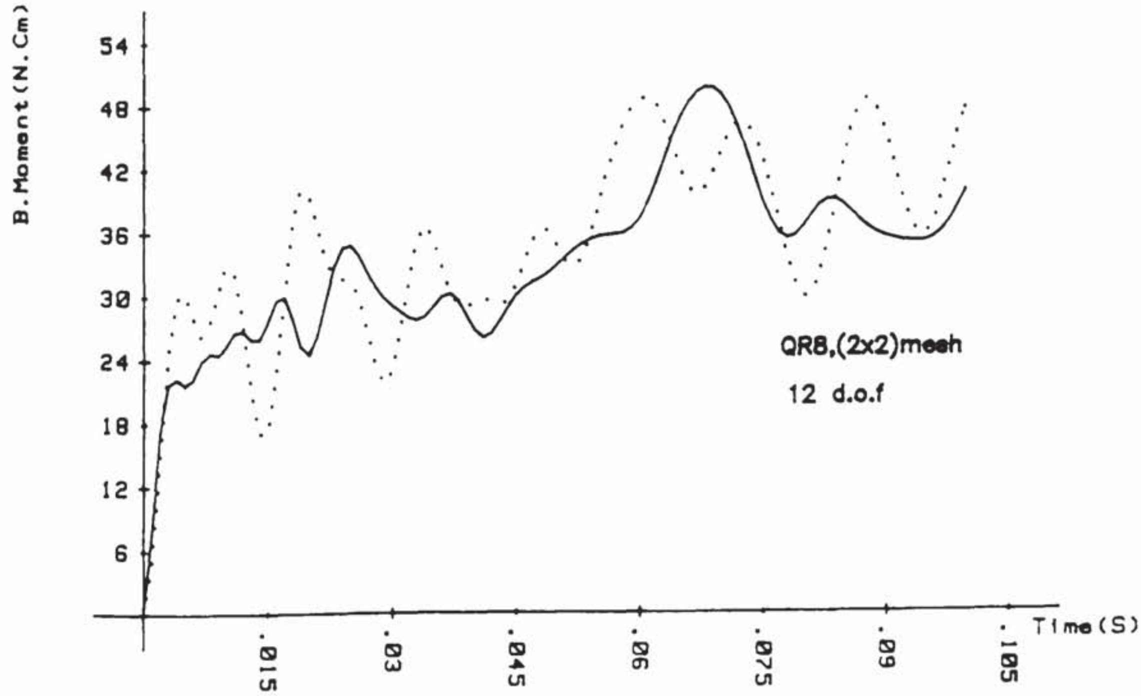


Fig 8.63 Bending moment Mx vs time;t at the centre of clamped plate under transient point load. ———— QD8 element
.... QR8 element

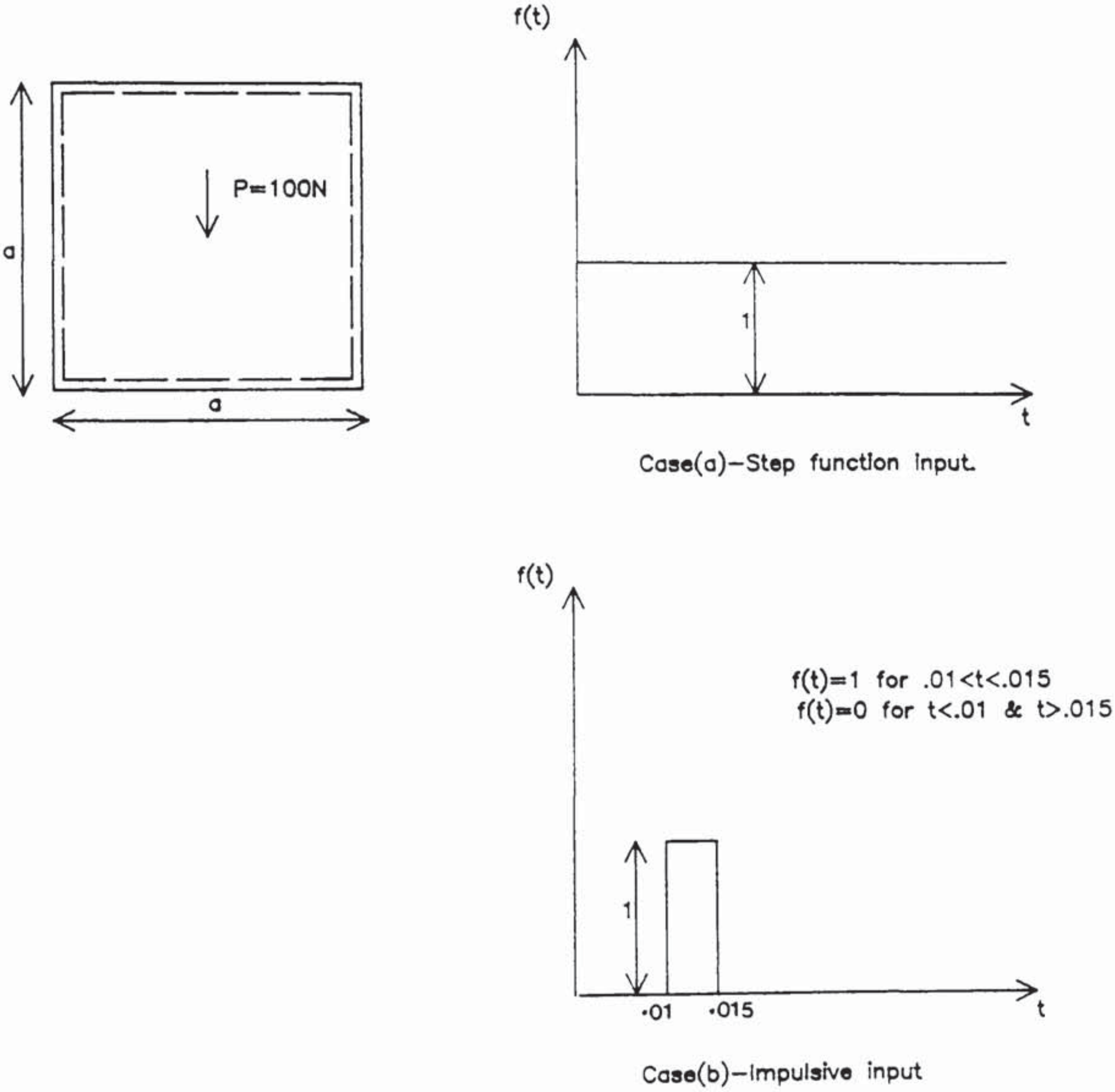


Fig 8.64 Simply supported square plate subjected to:
(a)Step function input,(b)Impulsive input.

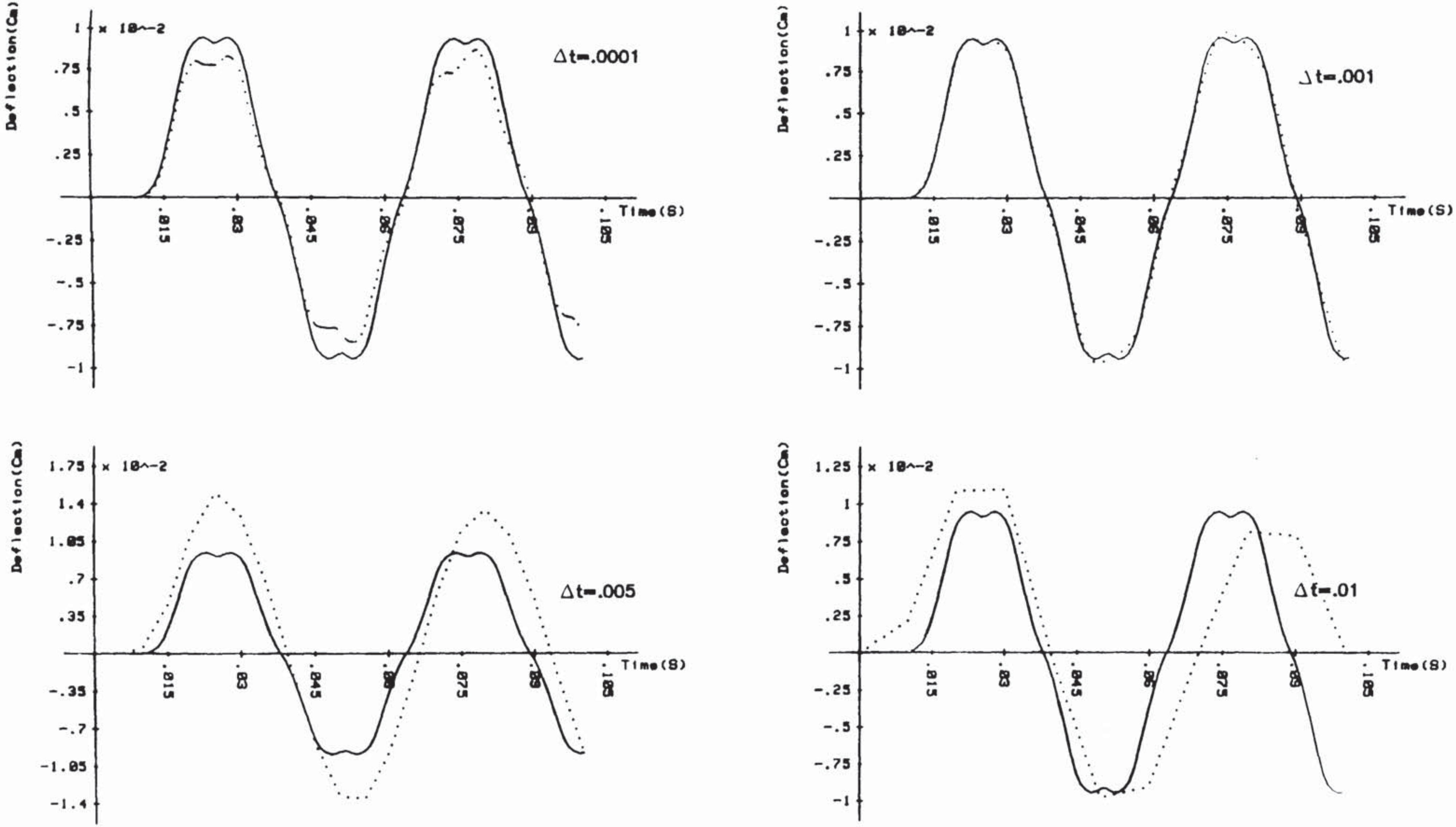


Fig 8.65 Displacement response predicted with increasing time step size Δ t;Wilson θ method,θ=1.4(case a)

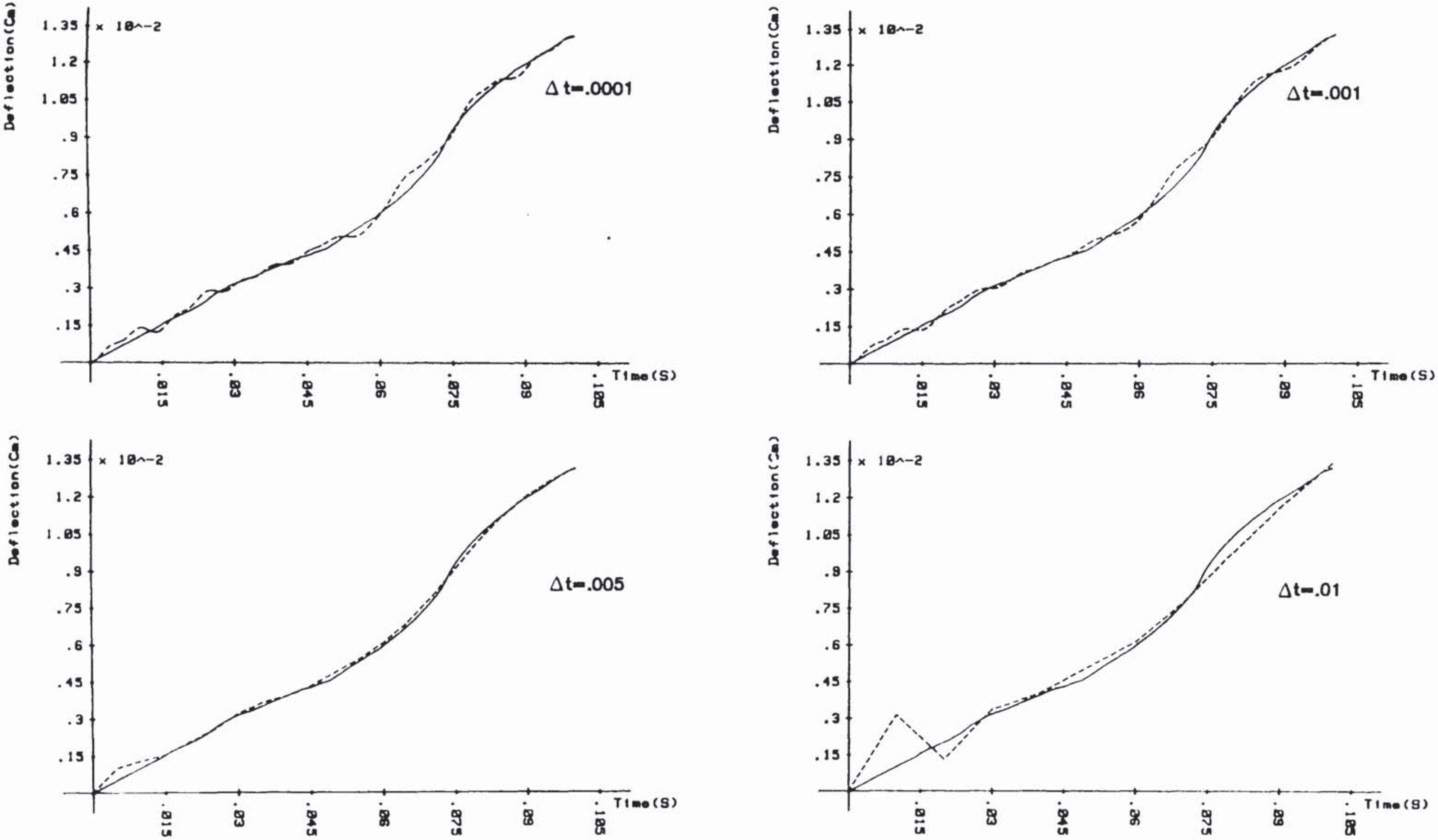


Fig 8.66 Displacement response predicted with increasing time step size Δ t;Wilson θ method,θ=1.4(case b)

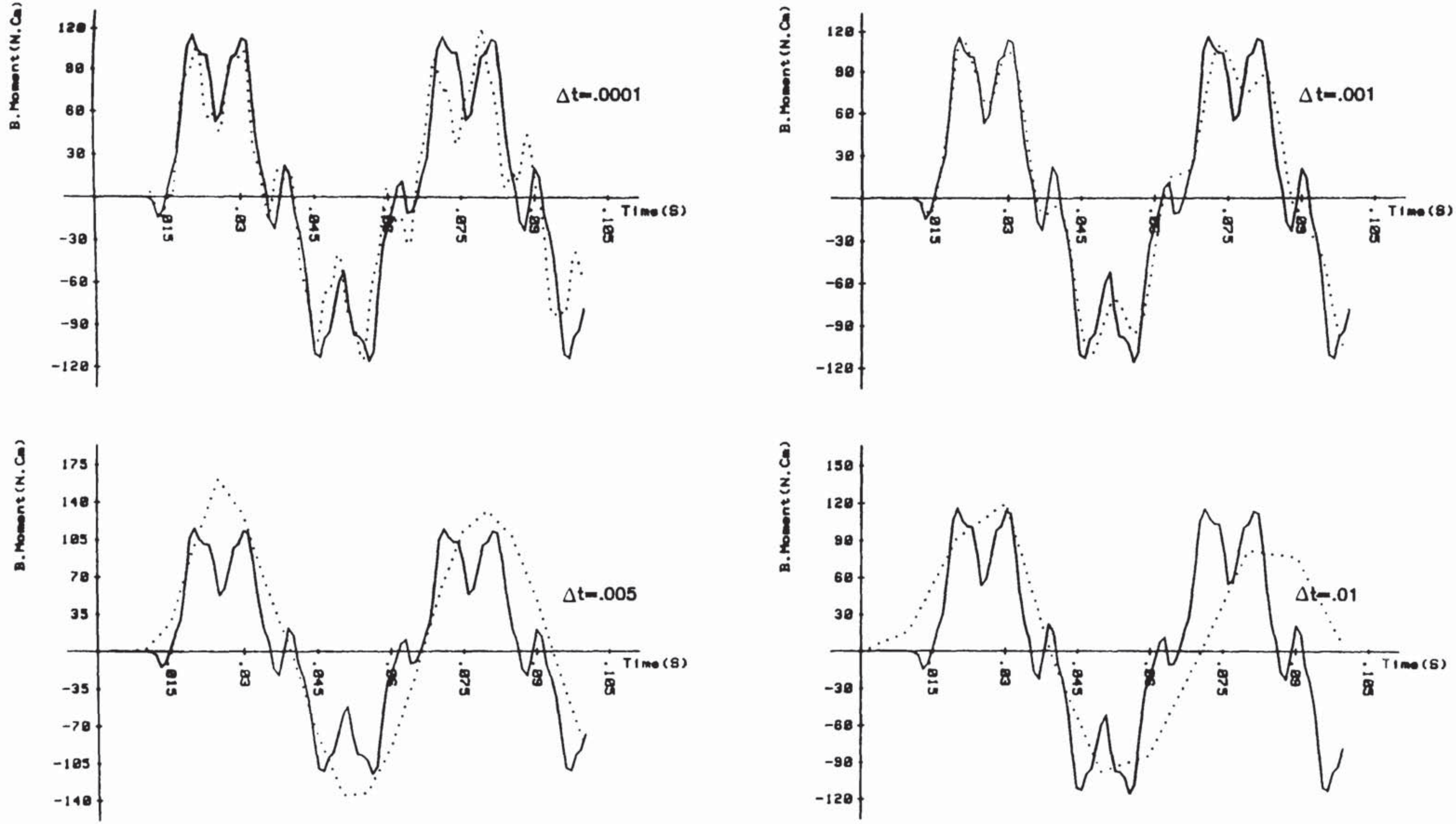


Fig 8.67 Bending moment response predicted with increasing time step size Δ t;Wilson θ method,θ=1.4(case a)

——— Exact
.... F.E model

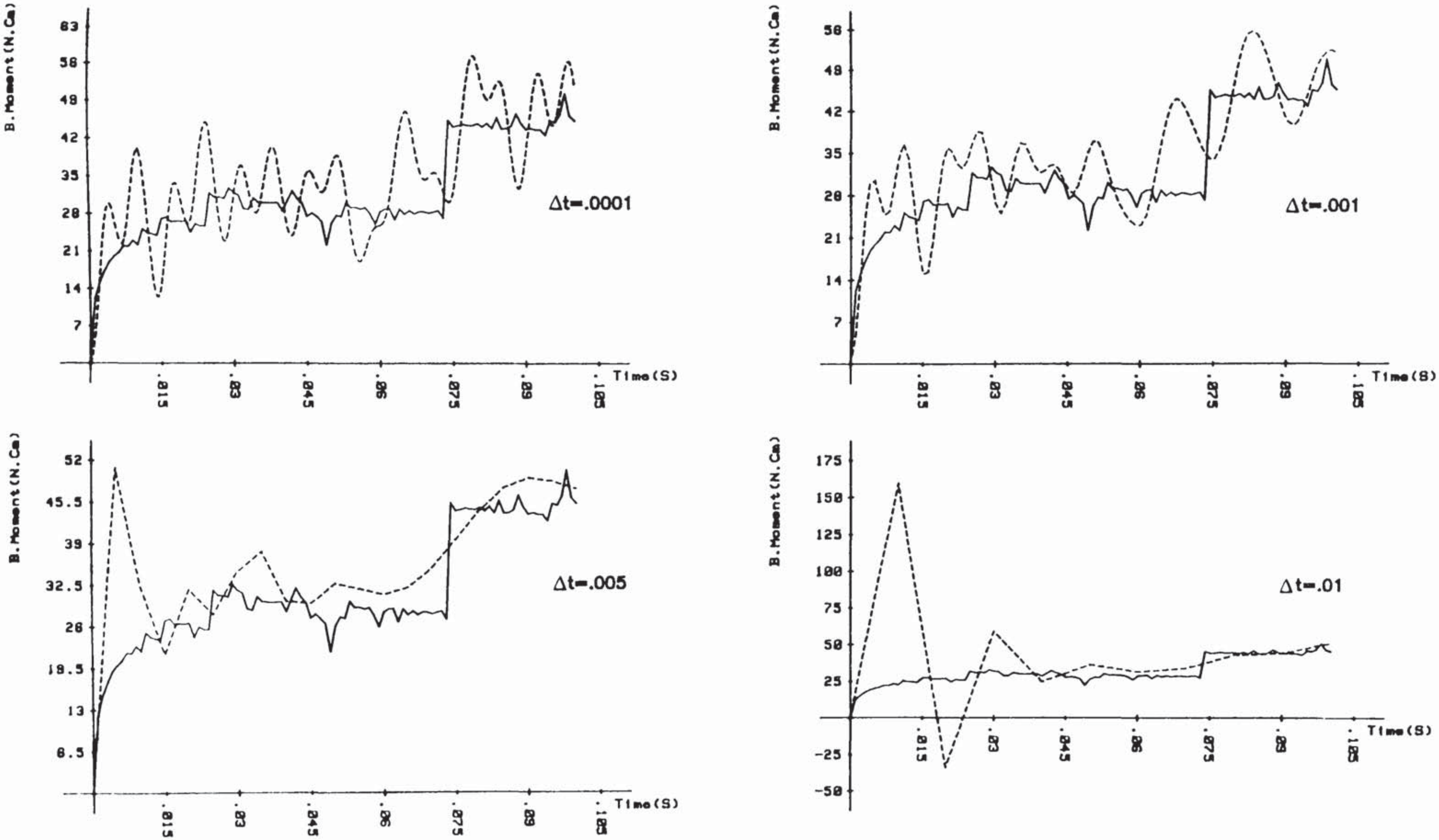
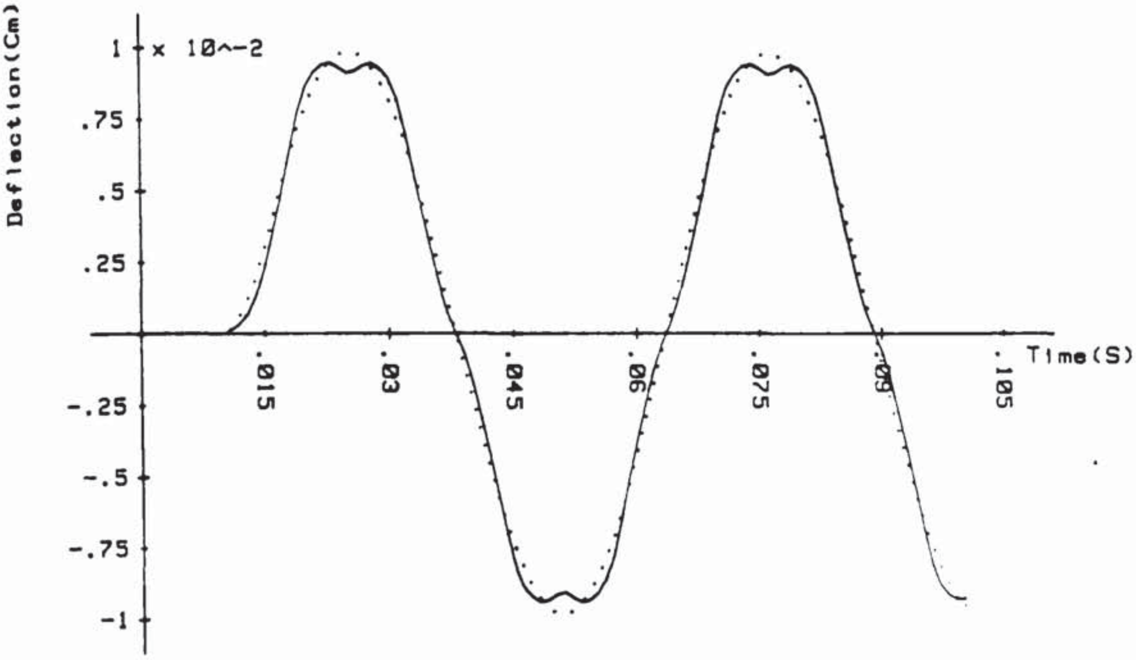
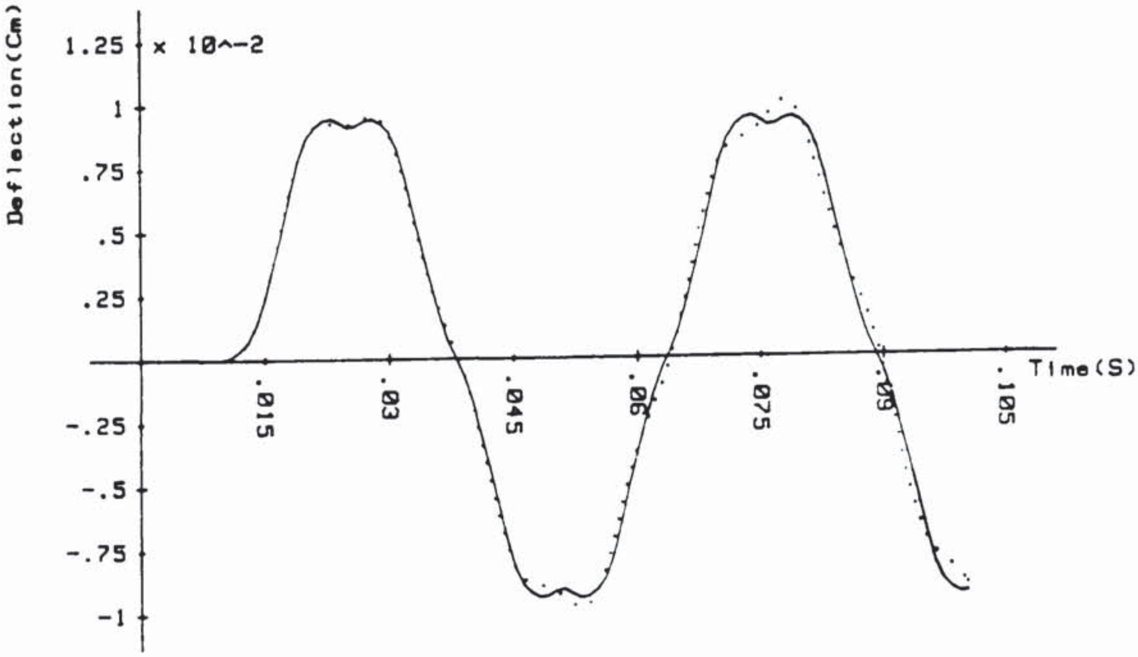


Fig 8.68 Bending moment response predicted with increasing time step size Δt;Wilson θ method,θ=1.4(case b)

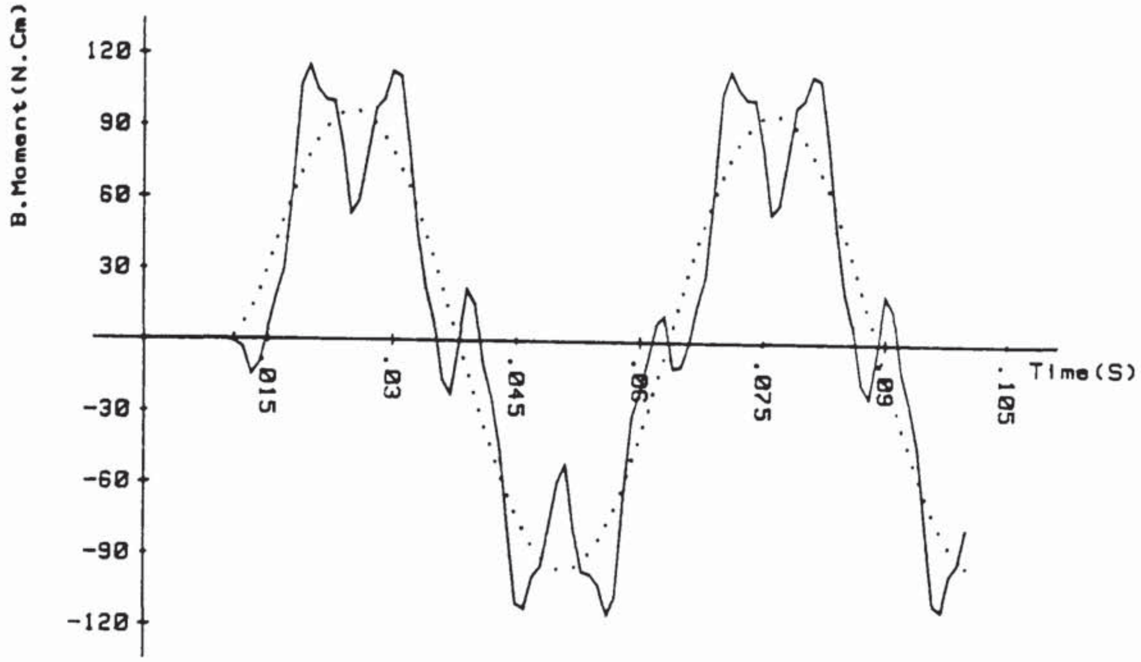


(a)

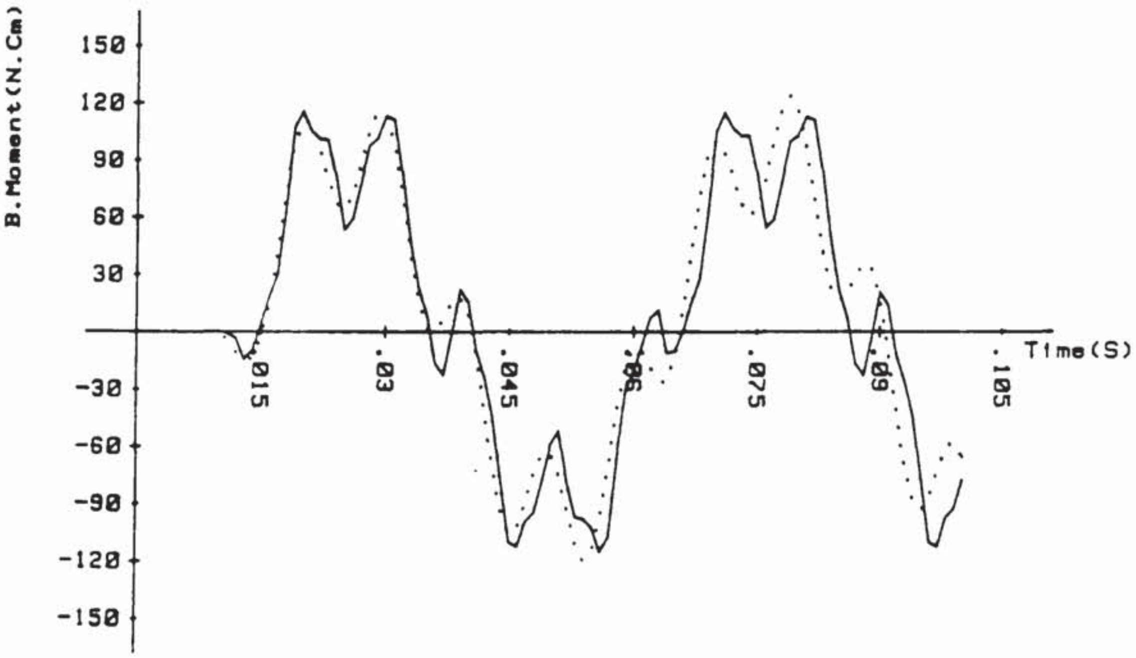


(b)

Fig 8.69 Displacement response by mode superposition method.
(a–1mode,b–3modes used) ——— Exact
.... F.E Model

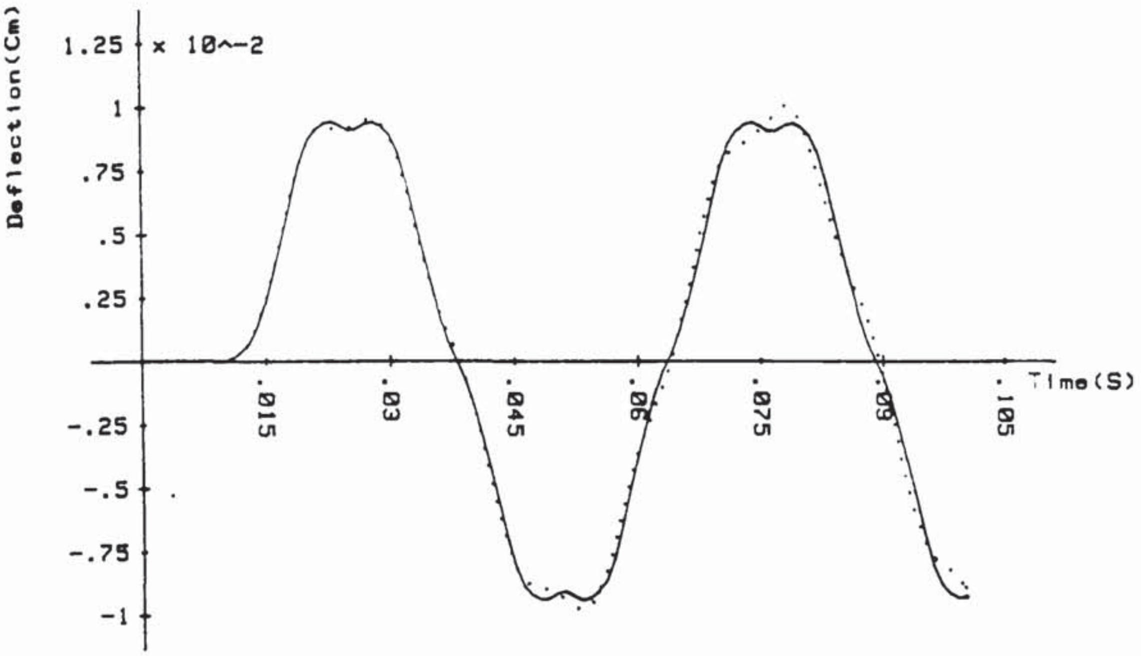


(a)

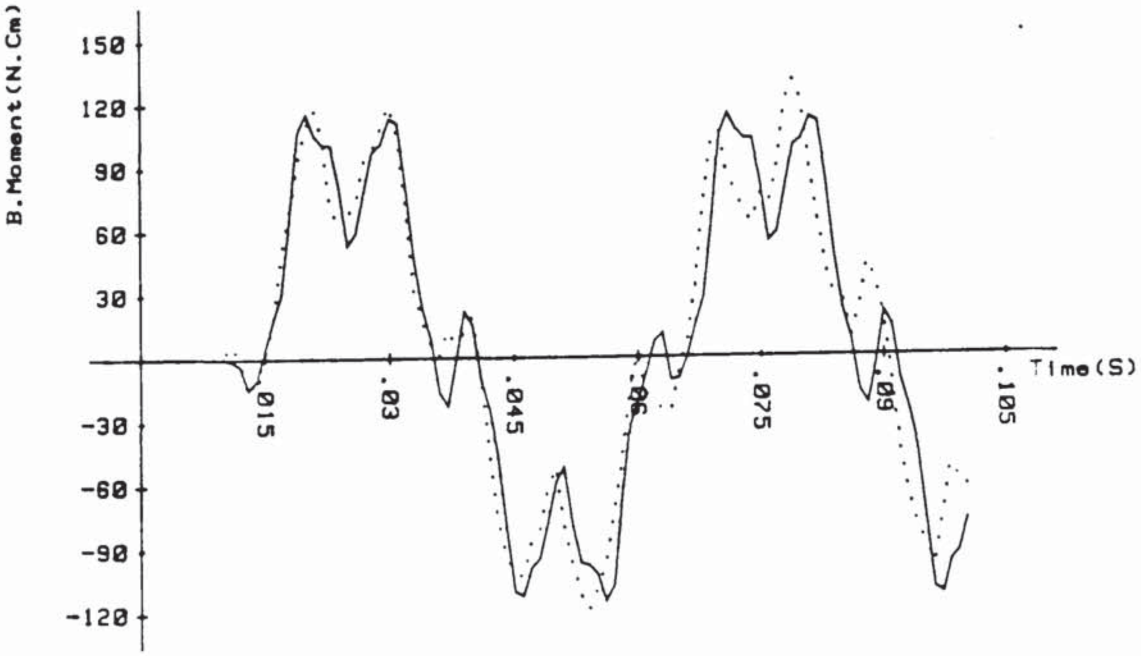


(b)

Fig 8.70 Bending moment response by mode superposition method.
(a—1mode,b—3modes used) ———— Exact
.... F.E Model



(a)



(b)

Fig 8.71 Forced response analysis of a SSSS plate by mode superposition method using 12 modes.
(a)Mid point displacement response
(b)Mid point bending moment response

———— Exact
.... F.E Model

CHAPTER 9

DISCUSSION AND CONCLUSIONS

9. DISCUSSION AND CONCLUSION

The application of Reissner's variational principle in the finite element analysis of structures has seen rapid progress during the past decade. This principle can be derived by the generalization of the minimum potential energy principle and is characterized by the use of both displacements and stresses as field variables. The so-called "mixed finite element" models by Reissner's principle have the following advantages:

(i) The possibility of relaxing the continuity conditions along the interelement boundaries. Thus allowing the use of simple and low order shape functions for displacements and stresses. This property has been shown to be of particular advantage in the analysis of plate and shell type structures.

(ii) Stresses, which are often of primary importance and interest, are calculated directly. Thus, the accuracy of the solution is comparable with that for displacements.

The earliest mixed finite element models were introduced by Herrmann (3, 61) in the static analysis of plate structures. He used a modified version of Reissner's principle which only imposes C0 continuity on the field of displacement and stresses. As a result of the successful application of Herrmann's mixed models, a number of mixed finite element models for plate problems have appeared to date. Only a few of these investigations have dealt with dynamic (free vibration) problems, however.

The main objective of this project was to study the performance in discretizing the bending moment M and displacement W. The elements are, in general, capable of predicting the natural frequencies with good accuracy and the results are in good agreement with the displacement type of solution. However, the solutions from two elements of C0 continuous class, namely MB7 with constant-parabolic shape functions and MB8 with parabolic-constant shape functions were wrong. The misbehaviour of those elements was attributed to the existence of untrue rigid body modes. These elements were excluded from the forced vibration analysis. The accuracy and convergency properties of mixed elements were also studied in the solution of beam response problems. It was observed that with a very few number of degrees of freedom, the elements are capable of predicting the transient response (deflection and moments) with very good accuracy. And in most cases, it was seen that the accuracy of the moments predicted by mixed elements is superior to that of displacement models, for the same number of degrees of freedom.

Determination of suitable shape functions of C1 continuity, in the formulation of plate elements is much more complex. If complete slope continuity is required on the interfaces between various elements, the mathematical and computational difficulties rise disproportionately fast. For this reason, the modified Reissner's principle introduced by Herrmann was employed in the development of mixed plate elements. The first element is a linear quadrilateral element with 16 degrees of freedom. This element has been tested in the solution of static, free vibration and buckling plate problems by Mota Soares (7), and good results were reported. The second element, developed in this work, is a parabolic quadrilateral element with 8 nodes, and 32 degrees of freedom. This element is suitable for representing plates of arbitrary shape. In addition to these, the two non-conforming displacement type elements of Ref. (9) were of the mixed finite element models based on Reissner's variational principle in the solution of dynamic structural problems. These problems include both free and forced vibration analysis. In design work, a knowledge of the system natural frequencies and mode shapes, obtained from the free vibration analysis, is helpful in avoiding the peak response which occur in the vicinity of the natural frequencies. In the forced vibration analysis, the effects of dynamic loads on the behaviour of the structure are investigated. In large and complex structures, these effects can become dominant.

The work presented in this thesis deals with the free and forced vibration of beam and plate type structures. As a prerequisite for this work, it was necessary to derive a dynamic version of Reissner's principle. In Chapter 2, it was shown that this principle can be obtained from the minimum potential energy principle by introducing the strain-displacement equations as conditions of constraint and the corresponding Lagrange multipliers, which are the stresses, as additional variables, and then by eliminating the strains as variables using the stress-strain relations. The extension to dynamic problem, which also includes velocity dependent damping forces, was performed in a similar manner using Hamilton's generalized principle. A convenient version of Reissner's principle, for application to beam and plate problems can be derived by single integration by parts of the terms with second order derivatives of displacement. This version of Reissner's principle allows the use of C0 continuous shape functions and was derived in section 3.3 for beam and in section 4.3.2 for plate analysis. The inclusion of C_1 continuous shape functions in the beam element formulation does not raise any difficulty. Thus both versions of Reissner's principle were employed in the derivation of beam elements characteristics. Eight elements were developed, with various sets of shape functions used extended to the forced vibration case. The input data preparation was performed by means of an automatic mesh generation program.

In applications to free vibration plate problems, it was seen that the mixed models are capable of predicting the lower natural frequencies with good accuracy. The results from the linear mixed element are reasonable and the parabolic element results are significantly better than the linear ones. An advantage of the mixed models in the solution of eigenvalue problems is that the size of the eigen problem can be considerably reduced without affecting the solution accuracy. In this work, the eigenvalue problem was reduced to one having only nodal deflections asthe unknowns.

In the forced vibration problems, the mixed equations were formulated in terms of the nodal deflections. Having determined the transient displacements, the bending or twisting moments could then be calculated by a simple matrix transformation procedure. Some numerical tests were performed and the results were compared with the available analytical and the non-conforming displacement type solutions. In all these applications relatively coarse meshes of mixed elements were capable of predicting the transient displacements and moments with comparable (in some cases with better) accuracy than the displacement models. In particular, it was observed that the computational time spent in the process of calculating the dynamic moments is considerably less than that in the displacement type formulation.

9.1 Further improvements

In this work, mixed isoparametric quadrilateral elements were used in the solution of dynamic plate problems. The discretization of plate structures was performed by an auto-mesh generation program. This program is also capable of generating meshes of 6-node triangular elements. The suite of programs can, with little modification, be made to accomodate this type of element. An advantage of the triangular element is that it is more versatile in representing the general shape of the boundary and contains fewer number of degrees of freedom than the 8-node quadrilateral element. (Moments and displacement may be assumed to vary parabolically within the element).

The modifications of the existing programs for the solution of static plate bending problems can be an objective of a further extention of the work of this thesis. For the solution of static problems, the mixed equations can be rearranged to yield a single matrix equation with nodal deflections and nodal bending moments as unknowns to be determined in a single operation. The corresponding mixed matrix is sparsely populated and the non-zero elements are located near the leading diagonal in a band form. In this way, it is only necessary to store the complete band form of the mixed matrix. This has the advantage of reducing the computer storage requirements. However, the overall mixed matrix is non-positive definite and the Gauss elimination method with row interchanges must be used in the solution of static equations.

REFERENCES

1. TURNER M.J., CLOUGH R.W., MARTIN H.C., and TOPP L.C. "Stiffness and deflection analysis of complex structures", *J. Aeronaut. Sci.* Vol. 23, No. 9, (1956)

2. REISSNER E. "On a variational theorem in elasticity", *J. Math Phys.* 29, p.90, (1950)

3. HERRMANN L.R., "Finite element bending analysis of plates", *J. Eng. Mechanics. Div.,* ASCE, 94, No. EM5, pp 13-25 (1968)

4. VISSER W., "A refined mixed type plate bending element", *AIAA Journal* 7, pp 1801-1803, (1969)

5. COOK R.D., "Eigenvalue problems with mixed plate elements", *AIAA Journal,* Vol. 7, No. 4, pp 982-983, (1969)

6. KIKUCHI F., ANDO Y., "Rectangular finite element for plate bending analysis based on Hellinger-Reissner's variational principle", *J. Nucl. Sci. Techn.* 9, pp 28-35, (1972)

7. MOTA SOARES, C.M., *"A study of mixed formulation for the finite element analysis of plates",* PhD thesis, University of Aston in Birmingham, (1976)

8. TSAY C.S., and REDDY J.N., "Free vibration of thin rectangular plates by a mixed finite element; ASME, paper N77, (1977)

9. HENSHELL R.D., WALTERS D, and WARBURTON, G.B. "A new family of curvilinear plate bending elements for vibration and stability", *Journal of Sound and Vibration,* 20(3), pp 381-397, (1972)

10. LEISSA A.W., "The free vibration of rectangular plates", *Journal of Sound & Vibration*, 31(3), pp 257-293, (1973)

11. LEIPHOLZ H.H.E., "Six lectures on variational principles in structural engineering", Solid Mechanics Division, University of Waterloo, Waterloo, Canada, (1978)

12. LOVE A.E.H., *"A treatise on the mathematical theory of elasticity,"* Cambridge University Press, 4th edition,(1927)

13. TIMOSHENKO S, AND GODIER J.N., *"Theory of elasticity"*, McGraw-Hill, (1951)

14. WASHIZU K, *"Variational methods in elasticity and plasticity"*, Pergamon Press, Oxford, (1968)

15. LANGHAAR H.L., *"Energy methods in applied mechanics"* John Wiley & Sons, Inc. New York, N.Y., (1962)

16. DYM C.L., and SHAMES I.H., *"Solid mechanics: variational approach"*, McGraw-Hill, (1973)

17. KANTOROVICH L.V., *"Approximate methods of high analysis"*, Inter Science Pub. (1964)

18. WILKINSON J.H., *"The algebraic eigenvalue problem"*, Clarendon Press, Oxford,(1965)

19. CLOUGH R.W., "The finite element in plane stress analysis", Proc. 2nd ASCE Conf. on electronic computation, Pittsburgh, Pa. September 1960

20. ZIENKIEWICZ, O.C. *"The finite element method"* 3rd Edition, McGraw-Hill Book Co. (UK) Ltd. (1977)

21. DESAI C.S., and ABEL J.F., *"Introduction to the finite element method"*, Van Nostrand Reinhold Co. New York (1972)

22. BEREBBIA C., TOTTENHAM H., *"Finite element techniques in structural mechanics"*, Southampton University Press, (1971)

23. COOK R.D. *"Concepts and applications of finite element analysis"* John Wiley & Son (1974)

24. PIAN T.H.H., TONG P., "Basis of finite element methods for solid continua", *International Journal for Numerical Methods in Engineering*, Vol. 1, (1969), pp 3-28

25. LEISSA A., "Recent research in plate vibration 1973-1976: classical theory," *Shock and Vibration Digest*,1977

26. LEISSA A., "Recent research in plate vibration 1973-1976: complicating effects", *Shock and Vibration Digest*,1977

27. BISHOP R.E.D. and JOHNSON D.C., *"Vibration analysis tables"*, Cambridge University Press, Cambridge (1956)

28. WARBURTON G.B., *"The dynamical behaviour of structures"*, 2nd edition, (1976), Pergamon Press, Oxford

29. TIMOSHENKO S.P. and KRIEGER S.W., *"Theory of plates and shells"*, 2nd edition, McGraw-Hill, Tokyo, (1959)

30. TIMOSHENKO S.P., GERE J.M. *"Theory of elastic stability"*, McGraw-Hill, 2nd edition, (1961)

31. LEISSA W.A., "Vibration of plates", Nasa Sp-160, National Aeronautics and Space Administration, Washington, (1969)

32. MEIROVITCH L., *"Analytical methods in vibration"*, McMillan Co. New York (1967)

33. TONG P., "New displacement hybrid finite element models for solid continua", *Int. J. Num. Meth. Engng.* 2, pp 73-83, (1970)

34. de VEUBEKE B.F. "Displacement and equilibrium models in finite element method", in *Stress analysis* (eds. Zienkiewicz and Hollister), Wiley, (1965)

35. SPILKER R.L. and MUNIR R.L., "The hybrid stress model for thin plates", *Int. J. Num. Meth. Engng.* 15, pp 1239-1260, (1980)

36. MIRZA F.A., OLSON M.D. "The mixed finite element method in plane elasticity", *Int. J. Num. Meth. Engng.* 15, p 273-289, (1980)

37. PRAGER W., "Variational principles of linear elastostatics for discontinuous displacements, strains and stresses", *Recent progress in applied mechanics, The Folke Odqvist Volume* B. Brcberg, J. Hult, and F. Niodson (eds) Odqvist & Wiksell, Stockholm, pp 463-474, (1967)

38. NEMAT-NASSER, S., "Application of general variational methods with discontinuous fields to bending, buckling and vibration of beams", *Computer Meth. in Appl. Mech. and Engng*, 2, pp 33-41, (1973)

39. PIAN, T.H.H., and TONG P., "Reissner's principle in finite element formulation", *Mechanics Today*, Vol. 5, pp 377-395, (1980)

40. LAZAN B.J. *Damping of materials and members in structural mechanics*, Pergamon Press, 1968, Oxford

41. BERT, C.W. "Material damping: an introductory review of mathematical models, measures and experimental techniques", *Journal of Sound and Vibration*, 29(2), pp 129-153, (1973)

42. CLOUGH R.W., and PENZIEN J., *Dynamics of structures* McGraw-Hill Book Co., New York, (1975)

43. WILSON E.L. and PENZIEN J., "Evaluation of orthogonal damping matrices," *International Journal for Numerical Methods in Engineering*, Vol. 4, No. 1, pp 5-10, January (1972)

44. BATHE K.J. and WILSON E.L. *Numerical methods in finite element analysis*, Prentice-Hall, Inc. New Jersey (1976)

45. HITCHINGS D and DANCE S.H., "Response of nuclear structural systems to transient and random excitations, using both deterministic and probabilistic methods", *Nuclear Engineering and Design*, 29 (1974), pp 311-337

46. GALLAGHER R.H., "Analysis of plate and shell structures", Proc. of Conf. on Application of Finite Element Method in Civil Eng., Vanderbilt University, Nashville, pp 155-206, (1969)

47. BOGNER F.K., FOX R.L., and SCHMIT L.A., "The generation of interelement, compatible stiffness and mass matrices by the use of interpolation formulas", Proc. (1st) Conf. on Matrix Methods in Struct. Mech., AFFDL TR 66-80, Nov., (1965)

48. BUTLIN G.A. and LECKIE F.A. "A study of finite elements applied to plate flexure", Symposium papers, Numerical Methods for Vibration Problems, Vol. 3, July (1966), University of Southampton

49. MASON V., "Rectangular finite elements for analysis of plate vibration", *Journal of Sound and Vibration*, Vol. 7, (1968), pp 437-448

50. COWPER G.R., KOSKO E., LINDBERG G.M. and OLSON M.D. "Static and dynamic applications of a high-precision triangular plate bending element", *AIAA Journal*, Vol. 7, No. 10, October (1969

51. BAZELEY G.P. et al., "Triangular elements in plate bending - conforming and non-conforming solutions" Matrix methods in structural mechanics, AFFDL - TR - 66-80, (1966), Wright-Patterson Air Force Base, Ohio, pp 547-576

52. CLOUGH R.W. and TOCHER J.L., "Finite element stiffness matrices for analysis of plate bending", Matrix methods in structural mechanics, AFFDL - TR - 66-80, (1966), Wright Patterson Air Force Base, Ohio, pp 515-545

53. MINDLIN, R.D., "Influence of rotatory inertia and shear on flexural motion of isotropic elastic plates", *J. Appl. Mech.* 18, pp31-38, (1951)

54. WEMPNER G.A., ODEN J.T., and KROSS D.A., "Finite element analysis of thin shells", Proceedings of the American Society of Civil Engineering, Vol. 94, EM6, December (1968), pp1273-1294

55. FRIED I., "Shear in C^0 and C^1 plate bending elements", *Int. J. Solids and Structures,* 9, 449-460 (1973)

56. ZIENKIEWICZ O.C., TAYLOR R.L. and TOO J.M., "Reduced integration techniques in general analysis of plates and shells", *Int. J. Num. Meth. Engng.* 3, 275-290 (1971)

57. PAWSEY S.E., and CLOUGH R.W., "Improved numerical integration of thick shell finite elements", *Int. J. Num. Meth. Engng.,* 3, 545-586 (1971)

58. LEE S.W. and PIAN T.H.H., "Improvement of plate and shell finite elements by mixed formulation", *AIAA Journal,* Vol. 16, No. 1, January (1978)

59. MORLEY L.S.D., "A triangular equilibrium element with linearly varying bending moments for plate bending problems", *J. Royal Aero. Soc.* 71, pp 715-721, (1967)

60. FRAEIJS de VEUBEKE B. and SANDER G "An equilibrium model for plate bending", *Int. Jnl. Solids and Structs. (G.B.)*, 4, 447-468 (1968)

61. HERRMANN L.R. "A bending analysis for plates", Proc. (1st) Conf. on Matrix Methods in Struct. Mech., AFFDL - TR 66-80, pp 577-604, October (1965)

62. TAHIANI C., *"Analyse des Voiles Minces dans les Domains Lineaire et Geometriquement Non-Lineaire par la Method des Elements Finis Mixtes",* Theses de Doctorat, Department de Genie Civil, Universite Laval, Aout (1971)

63. BRON J., DHATT G., "Mixed quadrilateral elements for bending", *AIAA Journal,* Vol. 10, October(1972)

64. RICHARDS T.H., *Energy methods in stress analysis,* Ellis Horwood Ltd. (1977)

65. FROBERG C.E., *Introduction to numerical analysis,* 2nd edition, Addison-Wesely Publishers, Comp. (1970)

66. GUNNAR E.N.,*"Computer solution of the eigenvalue problem in vibration analysis",* MSc dissertation, University of Aston in Birmingha, November (1979)

67. WOOD P.C. *"Application of finite element method to problems in fracture mechanics",* PhD thesis, University of Aston in Birmingham, (1979)

68. WILKINSON J.H., MARTIN R.S., PETERS G., "Symmetric decomposition of a positive definite matrix", *Numerische Mathematik*, 7, pp 362-383, (1965)

69. RIPPERPER E.A., DALLY J.W., "Experimental values of natural frequencies for skew and rectangular cantilever plates", Report No. DRL 231, CF - 1354, Defence Research Lab. University of Texas, Austin (1949)

70. BARTON M.V., "Vibration of rectangular and skew plates", *Journal of Appl. Mech.*, 18, 2, (1951)

The undamped free vibration mode shapes and frequencies for an N degree of freedom system are determined by solving the eigenvalue equation (A.1),

$$[K][\hat{U}] = [M][\hat{U}][\omega^2] \qquad (A.1)$$

in which $[\hat{U}]$ is the full (NxN) mode shape matrix and $[\omega^2]$ is an N x N diagonal frequency matrix containing the N squared natural frequencies. The undamped normal modes are then used to uncouple the equation of motion (A.2),

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{R\} \qquad (A.2)$$

Thus, introducing normal-coordinate transformation,

$$\{U\} = [\hat{U}]\{q\} \qquad (A.3)$$

into equation (A.2), we obtain:

$$m_r \ddot{q}_r + c_r \dot{q}_r + k_r q_r = R_r \qquad (A.4)$$

$r = 1,2,\ldots N$

where

$$m_r = \{\hat{U}\}_r^t [M] \{\hat{U}\}_r \qquad (a)$$

$$c_r = \{\hat{U}\}_r^t [C] \{\hat{U}\}_r = 2m_r\omega_r\xi_r \qquad (b)$$

$$k_r = \{\hat{U}\}_r^t [K] \{\hat{U}\}_r = m_r\omega_r^2 \qquad (c)$$

$$R_r = \{\hat{U}\}_r^t \{R\} \qquad (d)$$

(A.5)

and ζ_r is the damping ratio of the rth mode of vibration.

After simple matrix manipulation, the following relation is obtained:

$$[C] = [\hat{U}]^{-t} [c_r][\hat{U}]^{-1} \qquad (A.6)$$

Using the first part of equation (A.5), it can be shown that

$$[\hat{U}]^{-t} = [M][\hat{U}][m_r]^{-1} \quad (a)$$

and $$[\hat{U}]^{-1} = [m_r]^{-1}[\hat{U}]^{t}[M] \quad (b) \qquad (A.7)$$

substituting from (A.7) into (A.6) yields:

$$[C] = [\Phi][\beta][\Phi]^{t} \qquad (A.8)$$

where $[\Phi]$ is the mass normalized mode shape matrix defined by:

$$[\Phi] = [M][\hat{U}] \qquad (A.9)$$

and $[\beta]$ is a diagonal matrix in which the terms are given by

$$\beta_r = \frac{2\omega_r\zeta_r}{m_r} \qquad (A.10)$$

Equation (A.8) can be written in an alternate form as a summation of modal damping matrices c_r i.e.

$$[C] = \sum_{r=1}^{N} [c_r]$$

where $[c_r]$ produces damping in mode r only and may be calculated directly from the mass normalized shape vector $\{\phi\}_r$ thus:

$$c_r = \beta_r \{\phi\}_r \{\phi\}_r^t \tag{A.11}$$

The damping matrix $[\,C\,]$ is particularly useful in the evaluation of the dynamic response of structures when the direct step-by-step integration method is preferred to the normal mode superposition method.

APPENDIX B - INPUT DATA FOR MESH GENERATION PROGRAM

B.1 Introduction

The mesh generation scheme is based on using "isoparametric" curvilinear mapping of quadrilaterals, which allows a unique coordinate mapping of curvilinear and cartesian coordinates (20).

In this program, a structure is divided into a "chequer board" pattern of quadrilateral zones. Each of these may define a material with a single property - and if such property is specified as zero - a void is achieved, allowing multiply connected zones to be mapped.

In this section, the input data required by the mesh generation program are described. Two data files are created. Data input by the operator is output onto the first file, and the data obtained from the mesh generation program onto the second data file. The second data file is accessed by the programs described in Chapter 7 to provide the necessary input data. At the end of this section an example is given to provide a guide to data preparation. For details on mesh generation program consult Reference (67).

B.1.1 Input data

Data is input in the following order:

(a) Program Code

Code - Program classification number

Qort - Type of element used (1 - Quadrilateral,
Ø - Triangular)

Njob	-	Number of jobs to run
Nelemt	-	Number of elements
Nnode	-	Number of nodes
Cw	-	Number of nodes with prescribed w
Cx	-	Number of nodes with prescribed M_x
Cy	-	Number of nodes with prescribed M_y
C_{xy}	-	Number of nodes with prescribed M_{xy}
Nmat	-	Number of materials
Nskew	-	Number of nodes with coordinate transformation

(b) Control variables

Tnspds	-	Number of specified super nodes, i.e. not including standard generated nodes. If straight sided zone, only corner nodes are considered. If curve, mid-side nodes should also be included. Also if 2 super-nodes coincide only one is considered.
Pzone	-	Number of zones being used, ie. not including voids or generated zones
Vzone	-	Number of vertical zones (row of zones)
Hzone	-	Number of horizontal zones (column of zones)
Gh	-	Graphical output required? (1/Yes, Ø/No)

(c) Standard geometries

Ntip	-	Number of crack tips (it should be specified as Ø here)
Ngm	-	Number of generated sections. If > 0 then input the following parameters:
Nstart	-	Super-node number starting the core
Zns	-	Zone number starting the core
N1	-	Number of super-nodes on the core face

X1,Y1	-	Coordinates of the tip
R1,R2,R3	-	Radii for the inner core, grading node and outer node respectively
A	-	Starting angle
A1	-	Incremental angle
Dx,Dy	-	Zone's sub-divisions

(d) X and Y coordinates of specified super-nodes

Data sequence entered for each node

Q	-	Number of super-nodes occupying the position
Xcod, Ycod	-	X and Y coordinates
W	-	String of super-node numbers

(e) Defining zones

Zone	-	Number of like zones
Mn	-	Material number
Divx,Divy	-	Zone sub-divisions in x and y directions
p	-	string of like zone numbers

(f) Identifying closing sides

Nd	-	Number of closing faces. If > 0 then input the following parameters for each face:
Zn	-	Zone number
Side	-	Side of face to be joined (1,2,3 or 4)
Coin	-	Number of coinciding nodes. If > 0 then input the following parameters for each pair of nodes:
Nd	-	Node number retained

Cnd - Corresponding node number

(g) Boundary conditions - material properties

Sequence of nodes with prescribed W

Sequence of nodes with prescribed M_x

Sequence of nodes with prescribed M_y

Sequence of nodes with prescribed M_{xy}

Data sequence for each material ..

Thick, Density, E, ν, G, E, ν..

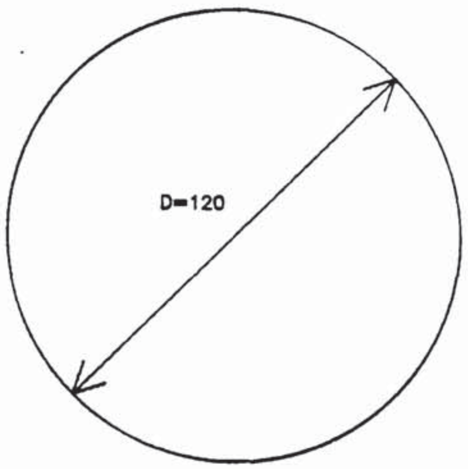
Data sequence for each skewed node ..

Nosk - Node number

Angsk - Angle of skew ....

B.1.2 Data input example.

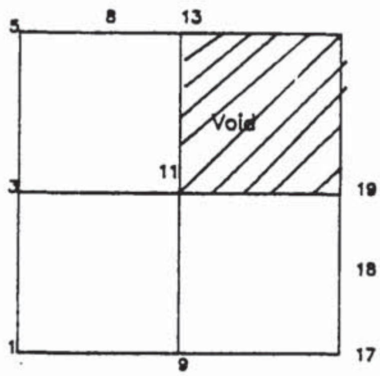
Simply supported circular plate.



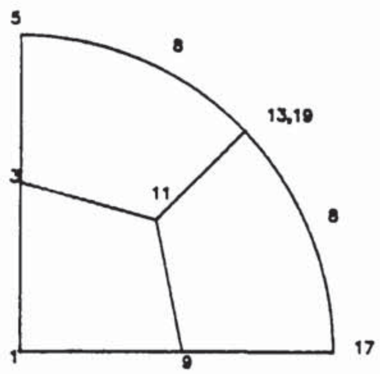
Material properties:
E=2.07x10^7 N/Cm^2
ν =.3
ρ=7.8x10^-3 Kg/Cm^3
h=1cm

The discretized plate

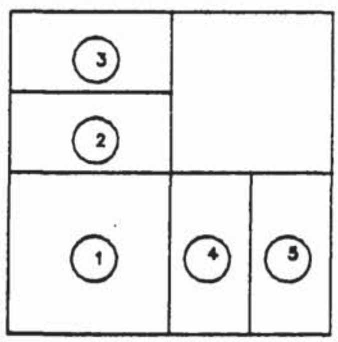
Due to symmetry only a quarter of the plate is required.

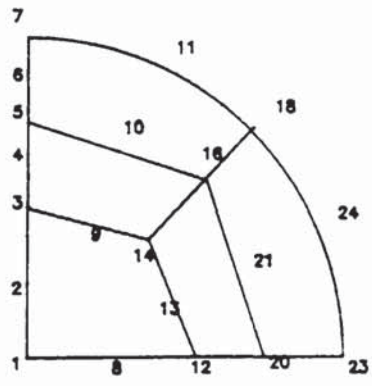


(a) "Chequer board"pattern



(b) Specified super nodes





(c) Element array with nodal numbering.

The input data

(a) 6, 1, 1, 5, 24, 5, 0, 3, 13, 1, 5

(b) 9, 3, 2, 2, 1

(c) 0, 0

(d) 1, 0, 0, 1   1, 0, 20, 3   1, 0, 60, 5   1, 20, 0, 9

1, $10\sqrt{2}$, $10\sqrt{2}$, 11   1, 60, 22.961, 55.4327, 8

2, 42.426, 42.426, 13, 19   1, 55.4327, 22.961, 18

1, 60, 0, 17

(e) 1, 1, 1, 1, 1

1, 1, 1, 2, 2

1, 1, 2, 1, 3

(f) 1, 2, 3, 0

(g) c_w: 7, 11, 18, 24, 23

$c_y(M_n)$: 11, 18, 24

c_{xy}: 1, 2, 3, 4, 5, 6, 7, 8, 12, 19, 20, 22, 23

Material properties: 1, 7.8E-3

2.07E7, .3, $\frac{2.07E7}{2.6}$, 2.07E7, .3

Nosk, Angsk: 11, 67.5°

18, 45°

24, 22.5°

APPENDIX C

Listing of computer programs:

MBRSP5	(Mixed beam response analysis)
RFPLT1	(Plate free vibration)
RFPLT2	(Plate forced vibration by mode superposition)
RFPLT3	(Plate forced vibration by Direct integration)

```
10    REM ******************PROGRAM MBRSP5*****************
20    REM *                    Based on                      *
30    REM *               Reissner principle                 *
40    REM * Version#1:This program calculates the dynamic    *
50    REM * displacements and stresses of a beam by means of *
60    REM * mixed finite element method.A direct integration *
70    REM * method known as Wilson thta is used.The method is *
80    REM * unconditionally stable.Damping is assumed to be  *
90    REM * viscous and proportional.                        *
100   REM * A complete damping matrix is thus derived ,based *
110   REM * on the orthogonality relations.A 3 node deflection*
120   REM * and a 3 node moment mixed element is used.       *
130   REM * Version#2:Mode superposition method is used to   *
140   REM * solve the equations.Modal dampings can be directly*
150   REM * employed in each uncoupled equation.             *
160   REM ************************************************
170   OPTION BASE 1
180   PRINTER IS 16
190   PRINT "Dynamic analysis of beams by mixed formulation"
200   PRINT
210   DIM Xcod(120),Leg(50),E(50),Ro(50),A(50,2),Mi(50,2)
220   DIM Th(50,2),P(45),V(4),Kode(100,2),Ge(3,3),He(6,6)
230   DIM Me(3,3),U(6),H(20,20),M(20,20),G(43,43),K(43,43)
240   DIM Vec(43,43),Eval(43),Apfo(30,1),F0(20),D(43),Offd(43)
250   DIM Offd2(43,1),Dl(43),Fo(43),C(20,20),A0(9),Dratio(50)
260   DIM I$[160],T$[80],A$[20],B$(20),Wcnt(3001),Mcnt(3001)
270   INTEGER N,Sol,Type,R,Pw,Pm,Neq,Nmod,Wpos,Mpos,Wplt,Mplt
280   MAT K=ZER
290   MAT M=ZER
300   MAT G=ZER
310   DISP "Before running the program for response analysis the"
320   DISP
330   DISP "following data files should be created on the current"
340   DISP
350   DISP "mass storage unit:"
360   DISP
370   DISP "1-Data file (Initil) to be used for recording the "
380   DISP
390   DISP "initial conditions.                              "
400   DISP
410   DISP "2-Data file (Eqn)  to be used for excitation "
420   DISP
430   DISP "functions.The program is halted.Create the"
440   DISP
450   DISP "files and press CONT"
460   PAUSE
470   INPUT "Choose the printer,0 for paper 16 for CRT ",Printer
480   PRINTER IS Printer
490   INPUT "What is your mass storage unit for data files?",Data$
500   DISP "Structural data[Geometric and material]"
510   INPUT "Type of the problem[S-S,etc]",Type$
520   INPUT "Number of elements",Nelemt
530   DISP "Number of elements=",Nelemt
540   Nnode=2*Nelemt+1
550   REM ----Nodal coordinates
560   Z=-1
570   FOR M=1 TO Nelemt+1
580   Z=Z+2
590   DISP "X-coordinate of node";Z;""
600   INPUT Xcod(Z)
610   IF M>1 THEN Leg(M-1)=Xcod(Z)-Xcod(Z-2)
620   NEXT M
630   FOR M=1 TO Nelemt
640   Xcod(2*M)=(Xcod(2*M+1)-Xcod(2*M-1))/2+Xcod(2*M-1)
650   NEXT M
```

```
660   FOR M=1 TO Nnode
670   DISP "X-coordinate of node";M;"=";Xcod(M)
680   NEXT M
690   DISP "press CLEAR then CONT"
700   PAUSE
710   REDIM Xcod(Nnode),Leg(Nelemt),E(Nelemt),Ro(Nelemt)
720   REDIM A(Nelemt,2)
730   REDIM Mi(Nelemt,2),Th(Nelemt,2),Kode(Nnode,2),Fo(Nnode)
740   DISP "Element details"
750   INPUT "How many groups of like elements?(Mat)",Glik
760   FOR G=1 TO Glik
770   DISP "Number of like elements in group";G;"",
780   INPUT K
790   INPUT "Elasticity modulus?",E
800   INPUT "Mass density?",Ro
810   DISP "String of like elements in group";G;""
820   FOR M=1 TO K
830   INPUT N
840   E(N)=E
850   Ro(N)=Ro
860   NEXT M
870   DISP "Material properties for group";G;""
880   DISP "Elasticity modulus=";E
890   DISP "Mass density=";Ro
900   NEXT G
910   INPUT "Is the problem one of damped or undamped?1/0",Damp
920   IF NOT Damp THEN GOTO 990
930   INPUT "Number of modes with proportional damping?",Nmod
940   REDIM Dratio(Nmod)
950   FOR I=1 TO Nmod
960   DISP "Damping ratio in mode";I;"?"
970   INPUT Dratio(I)
980   NEXT I
990   DISP "Uniform cross section?if yes input the area,0 to "
1000  DISP " indicate nonuniform"
1010  INPUT T
1020  IF T=0 THEN GOTO 1100
1030  INPUT "Moment of inertia?",T1,"Extreme fiber location?",T2
1040  FOR I=1 TO Nelemt
1050  A(I,1)=A(I,2)=T
1060  Mi(I,1)=Mi(I,2)=T1
1070  Th(I,1)=Th(I,2)=T2
1080  NEXT I
1090  GOTO 1170
1100  FOR M=1 TO Nelemt
1110  DISP "Input the following for element";M;""
1120  DISP "Area?,Moment of inertia?,Extreme fiber location? At1"
1130  INPUT A(M,1),Mi(M,1),Th(M,1)
1140  DISP "Area?,Moment of inertia?,Extreme fiber location At2"
1150  INPUT A(M,2),Mi(M,2),Th(M,2)
1160  NEXT M
1170  DISP "press CLEAR then CONT"
1180  PAUSE
1190  MAT Kode=ZER
1200  DISP "Introduction of prescribed freedoms[m,w] "
1210  INPUT "Number of nodes with prescribed moments?",Pm
1220  FOR I=1 TO Pm
1230  INPUT "Node number ",M
1240  Kode(M,1)=1
1250  NEXT I
1260  INPUT "Number of nodes with prescribed deflection?",Pw
1270  FOR I=1 TO Pw
1280  INPUT "Node number?",M
1290  Kode(M,2)=1
1300  NEXT I
1310  Pm=A=Nnode-Pm
```

PAGE NUMBERS CUT OFF IN ORIGINAL

```
1320  Pw=B=Nnode-Pw
1330  REDIM G(A,A),H(A,B),M(B,B),Apfo(B,1),F0(B),C(B,B)
1340  A=B
1350  REDIM K(A,A),Vec(A,A),Eval(A),D(A),D1(A),Offd2(A,1)
1360  REDIM P(Pm),Offd(A)
1370  A=B=0
1380  DISP "press CLEAR then CONT"
1390  PAUSE
1400  PRINT SPA(14),"Results of finite element by Reissner pri
      nciple"
1410  PRINT USING "K";"Response analysis by Wilson Theta Method"
1420  PRINT USING "K,16X,K";"Type of problem",Type$
1430  PRINT
1440  PRINT USING "K,12X,2D";"Number of elements",Nelemt
1450  PRINT
1460  PRINT USING "K,15X,2D";"Number of nodes",Nnode
1470  PRINT
1480  PRINT "Element properties:"
1490  PRINT LIN(2),"Elemt";SPA(2);"Modulus";SPA(3);"Area 1";SP
      A(5);"Area 2";SPA(3);"Moment";SPA(5);"Moment";SPA(5);"Le
      ngth";SPA(4);"Density"
1500  PRINT "number";SPA(1);"elastic";SPA(22);"Inertia1";SPA(3
      );"Inertia2"
1510  PRINT
1520  FOR M=1 TO Nelemt
1530  PRINT USING "2D,2X,7(MD.3DE,X)";M,E(M),A(M,1),A(M,2),Mi(
      M,1),Mi(M,2),Leg(M),Ro(M)
1540  NEXT M
1550  PRINT
1560  PRINT "Boundary conditions",LIN(2),"Node";SPA(4);"Xcoord
      ";SPA(6);"Moment";SPA(5);"Deflection"
1570  FOR M=1 TO Nnode
1580  PRINT USING "3D,3X,MD.4DE,5X,2(D,9X)";M,Xcod(M),Kode(M,1
      ),Kode(M,2)
1590  NEXT M
1600  INPUT "What is the loading type?1-for concentrated,2-for
      distributed,3-for both",Lcon
1610  IF (Lcon=1) OR (Lcon=3) THEN GOSUB Conf
1620  IF (Lcon=2) OR (Lcon=3) THEN GOSUB Disf
1630  G=0
1640  FOR I=1 TO Nnode
1650  IF Kode(I,2)=1 THEN 1680
1660  G=G+1
1670  F0(G)=Fo(I)+F0(G)
1680  NEXT I
1690  PRINT "Total nodal forces"
1700  PRINT "Node-coords";SPA(10);"Force"
1710  FOR I=1 TO Nnode
1720  IF Kode(I,2) THEN PRINT USING "MD.4DE,10X,K";Xcod(I),"Fixed"
1730  IF NOT Kode(I,2) THEN PRINT USING "2(MD.4DE,10X)";Xcod(I
      ),Fo(I)
1740  NEXT I
1750  PRINT "Response analysis data "
1760  REM Input excitation as a function of time.
1770  LINK "EXCITE",5500
1780  CALL Excitn(Neq)
1790  CALL Wtipt(Wcnt(*),Mcnt(*),Kode(*),Time,Delta,A0(*),Thet
      a,#1,Nnode,Pw)
1800  REM Assembly of [G],[H],[mass],[C] matrices.
1810  MAT Me=ZER
1820  MAT M=ZER
1830  FOR Z=1 TO Nelemt!Mass matrix assembly
1840  GOSUB Matme
1850  GOSUB Masemb
1860  NEXT Z
1870  MAT U=ZER
```

```
1880  FOR Z=1 TO Nelemt!Assembly of [G] matrix
1890  GOSUB Matge
1900  GOSUB Gasemb
1910  NEXT Z
1920  FOR Z=1 TO Nelemt!Assembly of [H] matrix
1930  C=1/Leg(Z)
1940  He(1,2)=He(2,1)=He(5,6)=He(6,5)=7/3
1950  He(1,4)=He(4,1)=He(2,3)=He(3,2)=He(3,6)=He(6,3)=-8/3
1960  He(4,5)=He(5,4)=-8/3
1970  He(1,6)=He(6,1)=He(2,5)=He(5,2)=1/3
1980  He(3,4)=He(4,3)=16/3
1990  MAT He=(C)*He
2000  GOSUB Hasemb
2010  NEXT Z
2020  REM Response analysis starts here.
2030  LINK "FDAMP",5500
2040  CALL Eqsolv(H(*),G(*),K(*),P(*),Pw,Pm)
2050  REM Damping matrix evaluation
2060  IF NOT Damp THEN GOTO 2080
2070  CALL Dampmat(C(*),Vec(*),Eval(*),M(*),K(*),Dratio(*),D(*
      ),Offd(*),Offd2(*),Dl(*),Nmod,Pw,Type,Sol)
2080  MAT Vec=M
2090  LINK "FINITL",5500
2100  CALL Eqsolv1(M(*),Dl(*),1,Pw)
2110  REM Initial acceleration is calculated and printed on fi
      le#1.
2120  CALL Initial(D(*),Apfo(*),F0(*),K(*),C(*),Offd2(*),Offd(
      *),M(*),Dl(*),Delta,1,Pw,Neq,#1)
2130  FOR K1=1 TO Neq!Loop round the number of forces.
2140  PRINT "Force set";K1;""
2150  ASSIGN #1 TO "Initil"
2160  MAT READ #1;D
2170  MAT READ #1;Offd
2180  FOR I=1 TO K1
2190  MAT READ #1;Offd2
2200  NEXT I
2210  MAT P=H*D
2220  INPUT "Node number to plot the displacements for?",Wplt
2230  FOR I=1 TO Nnode
2240  IF Kode(I,2)=1 THEN 2290
2250  G=G+1
2260  IF Wplt<>I THEN 2290
2270  Wpos=G
2280  GOTO 2300
2290  NEXT I
2300  Wcnt(1)=D(Wpos)
2310  INPUT "Node number to plot the moments for ?",Mplt
2320  IF Kode(Mplt,1)=0 THEN 2360
2330  BEEP
2340  DISP "Node";Mplt;" is free .Try again"
2350  GOTO 2310
2360  ! Determines position of Mplt
2370  G=0
2380  FOR I=1 TO Nnode
2390  IF Kode(I,1)=1 THEN 2440
2400  G=G+1
2410  IF Mplt<>I THEN 2440
2420  Mpos=G
2430  GOTO 2450
2440  NEXT I
2450  Mcnt(1)=P(Mpos)
2460  REM Calculation of response by Wilson theta method.
2470  LINK "WILSN1",5500
2480  LINK "Eqn",7250
2490  CALL Wilsns1(K(*),Vec(*),C(*),Apfo(*),F0(*),D(*),Offd(*)
      ,Offd2(*),Dl(*),A0(*),Time,Delta,Theta,K1,P(*),H(*),Wcnt
      (*),Mcnt(*),Mpos,Wpos,Pw,Pm)
```

```
2500  A$="DEFLECTION RESPONSE "
2510  B$="MOMENT RESPONSE"
2520  REM Response plots.
2530  LINK "FPLOT",5190
2540  CALL Plot(Wcnt(*),A$,Time,Delta,Wplt)
2550  CALL Plot(Mcnt(*),B$,Time,Delta,Mplt)
2560  LINPUT "File name for displacements?",Y$
2570  LINPUT "File name for moments?",M$
2580  ASSIGN #2 TO Y$
2590  ASSIGN #3 TO M$
2600  MAT PRINT #2;Wcnt
2610  MAT PRINT #3;Mcnt
2620  ASSIGN #2 TO *
2630  ASSIGN #3 TO *
2640  NEXT K1
2650  ASSIGN * TO #1
2660  BEEP
2670  PRINT "Execution terminated"
2680  END!OF PROGRAM
2690 Matge: ! SUBROUTINE TO EVALUATE [Ge]MATRIX
2700  Zeta(1)=.774596669241
2710  Zeta(2)=-Zeta(1)
2720  Zeta(3)=0
2730  MAT Ge=ZER(3,3)
2740  Y(1)=Y(2)=.555555555555
2750  Y(3)=.888888888888
2760  A=(Mi(Z,1)+Mi(Z,2))/2
2770  B=(Mi(Z,2)-Mi(Z,1))/2
2780  C=Leg(Z)/(2*E(Z))
2790  FOR I=1 TO 3
2800  Ge(1,1)=Ge(1,1)+Y(I)*(1/4*Zeta(I)^2*(-1+Zeta(I))^2)
2810  Ge(1,1)=Ge(1,1)/(A+B*Zeta(I))
2820  Ge(1,3)=Ge(1,3)-Y(I)*(-1/4*Zeta(I)^2*(Zeta(I)^2-1))
2830  Ge(1,3)=Ge(3,1)=Ge(1,3)/(A+B*Zeta(I))
2840  Ge(1,2)=Ge(1,2)-Y(I)*(-1/2*Zeta(I)*(Zeta(I)-1)*(1-Zeta(I
      )^2))
2850  Ge(1,2)=Ge(2,1)=Ge(1,2)/(A+B*Zeta(I))
2860  Ge(3,3)=Ge(3,3)+Y(I)*(1/4*Zeta(I)^2*(1+Zeta(I))^2)
2870  Ge(3,3)=Ge(3,3)/(A+B*Zeta(I))
2880  Ge(2,3)=Ge(2,3)+Y(I)*(1/2*Zeta(I)*(Zeta(I)+1)*(1-Zeta(I)^2))
2890  Ge(2,3)=Ge(3,2)=Ge(2,3)/(A+B*Zeta(I))
2900  Ge(2,2)=Ge(2,2)+Y(I)*(1-Zeta(I)^2)^2/(A+B*Zeta(I))
2910  NEXT I
2920  MAT Ge=(C)*Ge
2930  RETURN
2940 Matme:! Subroutine to evaluate [Me] matrix
2950  Zeta(1)=.774596669241
2960  Zeta(2)=-Zeta(1)
2970  Zeta(3)=0
2980  MAT Me=ZER(3,3)
2990  Y(1)=Y(2)=.55555555555
3000  Y(3)=.888888888888
3010  A=(A(Z,1)+A(Z,2))/2
3020  B=(A(Z,2)-A(Z,1))/2
3030  C=Leg(Z)/2*Ro(Z)
3040  FOR I=1 TO 3
3050  Me(1,1)=Me(1,1)+Y(I)*(1/4*Zeta(I)^2*(-1+Zeta(I))^2)
3060  Me(1,1)=Me(1,1)*(A+B*Zeta(I))
3070  Me(1,3)=Me(1,3)-Y(I)*(-1/4*Zeta(I)^2*(Zeta(I)^2-1))
3080  Me(1,3)=Me(3,1)=Me(1,3)*(A+B*Zeta(I))
3090  Me(1,2)=Me(1,2)-Y(I)*(-1/2*Zeta(I)*(Zeta(I)-1)*(1-Zeta(I
      )^2))
3100  Me(1,2)=Me(2,1)=Me(1,2)*(A+B*Zeta(I))
3110  Me(3,3)=Me(3,3)+Y(I)*(1/4*Zeta(I)^2*(1+Zeta(I))^2)
3120  Me(3,3)=Me(3,3)*(A+B*Zeta(I))
3130  Me(2,3)=Me(2,3)+Y(I)*(1/2*Zeta(I)*(1+Zeta(I))*(1-Zeta(I)^2))
```

```
3140  Me(2,3)=Me(3,2)=Me(2,3)*(A+B*Zeta(I))
3150  Me(2,2)=Me(2,2)+Y(I)*(1-Zeta(I)^2)^2*(A+B*Zeta(I))
3160  NEXT I
3170  MAT Me=(C)+Me
3180  RETURN
3190 Gasemb:  ! Construction of [G]
3200  IF Z>=2 THEN GOTO 3320
3210  MAT U=ZER
3220  Gp=0
3230  IF Kode(Z,1)<>0 THEN 3260
3240  U(1)=1
3250  Gp=Gp+1
3260  U(2)=Gp+1
3270  Gp=U(2)
3280  IF Kode(2*Z+1,1)<>0 THEN 3390
3290  U(3)=Gp+1
3300  Gp=U(3)
3310  GOTO 3390
3320  U(1)=U(3)
3330  U(2)=Gp+1
3340  Gp=U(2)
3350  U(3)=0
3360  IF Kode(2*Z+1,1)<>0 THEN GOTO 3390
3370  U(3)=Gp+1
3380  Gp=U(3)
3390  GOSUB Gs
3400  RETURN
3410 Masemb:  ! Construction of [G] matrice in reduced square
3420          ! form.
3430  IF Z>=2 THEN GOTO 3550
3440  MAT U=ZER
3450  Gp=0
3460  IF Kode(Z,2)<>0 THEN 3490
3470  U(1)=1
3480  Gp=Gp+1
3490  U(2)=Gp+1
3500  Gp=U(2)
3510  IF Kode(2*Z+1,2)<>0 THEN 3620
3520  U(3)=Gp+1
3530  Gp=U(3)
3540  GOTO 3620
3550  U(1)=U(3)
3560  U(2)=Gp+1
3570  Gp=U(2)
3580  U(3)=0
3590  IF Kode(2*Z+1,2)<>0 THEN GOTO 3620
3600  U(3)=Gp+1
3610  Gp=U(3)
3620  GOSUB Ms
3630  RETURN
3640 Hasemb:  ! Construction of [H] matrix in reduced rectangular
3650          ! form.
3660  IF Z>=2 THEN GOTO 3910
3670  U(1)=U(2)=1
3680  U(3)=U(4)=2
3690  U(5)=U(6)=3
3700  IF Kode(Z,1)=0 THEN 3760
3710  U(1)=0
3720  U(3)=U(3)-1
3730  U(5)=U(5)-1
3740  V(1)=V(1)+1
3750  V(3)=V(3)+1
3760  IF Kode(Z,2)=0 THEN GOTO 3820
3770  U(2)=0
3780  U(4)=U(4)-1
3790  U(6)=U(6)-1
```

```
3800  V(2)=V(2)+1
3810  V(4)=V(4)+1
3820  IF Kode(2*Z+1,1)=0 THEN GOTO 3860
3830  U(5)=0
3840  V(1)=V(1)+1
3850  V(3)=V(3)+1
3860  IF Kode(2*Z+1,2)=0 THEN 3900
3870  U(6)=0
3880  V(2)=V(2)+1
3890  V(4)=V(4)+1
3900  GOTO 4070
3910  U(1)=U(5)
3920  U(2)=U(6)
3930  U(3)=2*Z-V(1)
3940  U(4)=2*Z-V(2)
3950  IF Kode(2*Z+1,1)=0 THEN 4000
3960  U(5)=0
3970  V(1)=V(1)+1
3980  V(3)=V(3)+1
3990  GOTO 4010
4000  U(5)=2*Z+1-V(3)
4010  IF Kode(2*Z+1,2)=0 THEN 4060
4020  U(6)=0
4030  V(2)=V(2)+1
4040  V(4)=V(4)+1
4050  GOTO 4070
4060  U(6)=2*Z+1-V(4)
4070  GOSUB Hs
4080  RETURN
4090 Gs: FOR I=1 TO 3
4100  FOR J=1 TO 3
4110  G=U(I)
4120  L=U(J)
4130  IF (G=0) OR (L=0) THEN 4150
4140  G(G,L)=G(G,L)+Ge(I,J)
4150  NEXT J
4160  NEXT I
4170  RETURN !Ms
4180 Ms: FOR I=1 TO 3
4190  FOR J=1 TO 3
4200  G=U(I)
4210  L=U(J)
4220  IF (G=0) OR (L=0) THEN 4240
4230  M(G,L)=M(G,L)+Me(I,J)
4240  NEXT J
4250  NEXT I
4260  RETURN !As1
4270 Hs: FOR I=1 TO 5 STEP 2
4280  FOR J=2 TO 6 STEP 2
4290  G=U(I)
4300  L=U(J)
4310  IF (G=0) OR (L=0) THEN 4330
4320  H(G,L)=H(G,L)+He(I,J)
4330  NEXT J
4340  NEXT I
4350  RETURN !Hs
4360 Conf: ! Subroutine for concentrated loads
4370  DISP "Details of loading"
4380  INPUT "Number of nodes with concentrated loads?",W
4390  FOR I=1 TO W
4400  INPUT "Node number?",M,"Load value?",Fo
4410  Fo(M)=Fo
4420  NEXT I
4430  INPUT "Number of nodes with concentrated moments?",W
4440  FOR I=1 TO W
4450  INPUT "Node number?",M,"Concentrated moment?",Mo
```

```
4460  IF (M=1) OR (M=Nnode) THEN 4510
4470  Fo(M-1)=Fo(M-1)-1/Leg(M-1)*Mo
4480  Fo(M+1)=Fo(M+1)+1/Leg(M)*Mo
4490  Fo(M)=Fo(M)+Mo/Leg(M-1)-Mo/Leg(M)
4500  GOTO 4570
4510  IF M=1 THEN 4550
4520  Fo(M-1)=Fo(M-1)-1/Leg(M-1)*Mo
4530  Fo(M)=Fo(M)+Mo/Leg(M-1)
4540  GOTO 4570
4550  Fo(M+1)=Fo(M+1)+Mo/Leg(1)
4560  Fo(M)=Fo(M)-Mo/Leg(1)
4570  NEXT I
4580  RETURN
4590 Disf: ! Subroutine to evaluate for distributed loadings
4600  ! Load is assumed to vary linearly
4610  DISP "Details of distributed loading"
4620  PRINT
4630  DISP "Element number subjected to loading?"
4640  INPUT M
4650  INPUT "Load intensity at position 1",P1,"Load at 2?",P2
4660  Fo(2*M-1)=Fo(2*M-1)+(2*P1+P2)*(Leg(M)/6)
4670  Fo(2*M+1)=Fo(2*M+1)+(2*P2+P1)*(Leg(M)/6)
4680  INPUT "More loaded elements?if yes input the number ,else
      0",M
4690  IF M THEN GOTO 4650
4700  DISP "Press CLEAR then CONT"
4710  PAUSE
4720  PRINT "Distributed loading information"
4730  PRINT
4740  PRINT "Node-coords";SPA(10);"Equivalent force"
4750  FOR N=1 TO Nnode
4760  PRINT USING "2(MD.4DE,10X)";Xcod(N),Fo(N)
4770  NEXT N
4780  RETURN
4790  SUB Wtipt(Wcnt(*),Mcnt(*),Kode(*),Time,Delta,A0(*),T,#1,
      Nnode,INTEGER Pw)
4800  OPTION BASE 1
4810  REM Integration constants!Wilson method input subprogram
4820  DIM D(43),Offd(43)
4830  REDIM D(Pw),Offd(Pw)
4840  DISP "Integration constants"
4850  INPUT "What is the time duration?",Time
4860  INPUT "What is the time interval?",Delta
4870  PRINT
4880  PRINT USING "K,6X,MD.4DE";"Response duration",Time
4890  PRINT
4900  PRINT USING "K,7X,MD.4DE";"Time incremental",Delta
4910  REDIM Wcnt(INT(Time/Delta)+1),Mcnt(INT(Time/Delta)+1)
4920  INPUT "Select Theta[usually 1.4]?",T
4930  A0(1)=6/(T*Delta)^2
4940  A0(2)=3/(T*Delta)
4950  A0(3)=2*A0(2)
4960  A0(4)=T*Delta/2
4970  A0(5)=A0(1)/T
4980  A0(6)=-A0(3)/T
4990  A0(7)=1-3/T
5000  A0(8)=Delta/2
5010  A0(9)=Delta^2/6
5020  REM Initial displacement-velocity input
5030  DISP "Initial conditions"
5040  INPUT "If zero initial conditions press 0 otherwise 1
      ",In
5050  IF In<>0 THEN 5090
5060  MAT D=ZER
5070  MAT Offd=ZER
5080  GOTO 5230
```

```
5090  A=0
5100  FOR I=1 TO Nnode
5110  IF Kode(I,2) THEN GOTO 5150
5120  A=A+1
5130  DISP "Initial displacement of node";I;""
5140  INPUT D(A)
5150  NEXT I
5160  A=0
5170  FOR I=1 TO Nnode
5180  IF Kode(I,2) THEN 5220
5190  A=A+1
5200  DISP "Initial velocity of node";I;""
5210  INPUT Offd(A)
5220  NEXT I
5230  ASSIGN #1 TO "Initil:F"
5240  MAT PRINT #1;D
5250  MAT PRINT #1;Offd
5260  SUBEND
```

```
10    REM ******************************************
20    REM * Free vibration analysis of thin plates by *
30    REM * Reissner's method.                         *
40    REM *                                            *
50    REM * Program name:"RFPLT1:F8"                   *
60    REM *                [Reissner's Plate Vibration] *
70    REM *                                            *
80    REM * Version#1 Free vibration analysis.         *
90    REM ******************************************
100   OPTION BASE 1
110   PRINTER IS 16
120   PRINT PAGE,SPA(10),"PROGRAM INTRODUCTION#1",LIN(2)
130   PRINT "Thin plate vibration by mixed formulation:"
140   PRINT "The program is based on a mixed variational"
150   PRINT "principle known as Hellinger-Reissner's   "
160   PRINT "principle.An 8 node quadrilateral finite  "
170   PRINT "element is used for discretisation of the "
180   PRINT "plate.Lateral deflection is assumed to vary"
190   PRINT "parabolically inside the element.Bending and"
200   PRINT "twisting moments are also assumed to vary  "
210   PRINT "parabolically.Latch PRT ALL press CONT"
220   PAUSE
230   DIM J(2,2),Ge(24,24),He(24,8),A$[30],Vec(27,27)
240   DIM Eval(27),D(27),Xx(65),Yy(65),Offd(27)
250   DIM Offd2(27,1),Me(8,8),C(10,6),Th(8),H(95,27),G(95,95)
260   DIM Dens(8),M(43,43),Dl(43),Cr(27,27),Ar(27,27),Bm(2,8)
270   DIM Sf(8),Sfm(8,8),W(9,4),Angsk(20),K(27,79)
280   INTEGER Nodc(65,4),Node(16,9),Nosk(20)
290   INTEGER Nmat,Matno,I,K,Jo,Z,Sol,N,Op,Cw,Cx,R,Type,Cy
300   INTEGER Cxy,Nskew,Fw,Fm,Nnode,Nelemt
310   REM Gause points and weights for numerical.integration.
320   A=.774596669241
330   B=0
340   C=.55555555555
350   D=.888888888888
360   W(1,1)=W(1,2)=W(2,2)=W(3,2)=W(4,1)=W(7,1)=-A
370   W(2,1)=W(4,2)=W(5,1)=W(5,2)=W(6,2)=W(8,1)=0
380   W(3,1)=W(6,1)=W(7,2)=W(8,2)=W(9,1)=W(9,2)=A
390   W(1,3)=W(1,4)=W(6,4)=W(2,3)=W(7,3)=W(7,4)=C
400   W(9,3)=W(9,4)=W(3,3)=W(3,4)=W(8,3)=W(4,3)=C
410   W(6,3)=W(2,4)=W(8,4)=W(4,4)=W(5,3)=W(5,4)=D
420   A=B=C=D=0
430   PRINT
440   DISP "Type in name of the input data file?,press CONT"
450   INPUT Data$
460   ASSIGN #1 TO Data$,C
470   IF NOT C THEN GOTO 520
480   BEEP
490   DISP "File not found.Try again"
500   WAIT 2000
510   GOTO 440
520   INPUT "What is the printer 16/0?",Printer
530   PRINTER IS Printer
540   INPUT "Type of the solution?1 for deflection,2 for m
      oment eigenvectors",Type
550   PRINT ,SPA(1);"Vibration analysis of thin plates"
560   PRINT ,SPA(1);"---------------------------------"
570   PRINT
580   LINPUT "Type in name of the job.Not more than 30 charact
      ers",A$
590   PRINT "Job name........";A$;""
600   READ #1;Njob,Nelemt,Nnode
610   PRINT LIN(2)
620   PRINT "Element selected:"
630   PRINT "8-node quadrilateral"
```

```
640   PRINT LIN(3)
650   PRINT "Number of elements........";Nelemt,LIN(2)
660   PRINT "Number of nodes    ........";Nnode
670   READ #1;Cw,Cx,Cy,Cxy,Nmat,Nskew
680   Fm=3*Nnode-(Cx+Cy+Cxy)
690   Fw=Nnode-Cw
700   REDIM Xx(Nnode),Yy(Nnode),C(Nmat,6),Th(Nmat)
710   REDIM Dens(Nmat),G(Fm,Fm),H(Fm,Fw),M(Fw,Fw),Angsk(Nskew+1)
720   REDIM Node(Nelemt,9),Nodc(Nnode,4),Nosk(Nskew+1)
730   IF Type=2 THEN GOTO 770
740   REDIM Vec(Fw,Fw),Eval(Fw),D(Fw),Offd(Fw)
750   REDIM Offd2(Fw,1),D1(Fw),K(Fw,Fw)
760   GOTO 790
770   REDIM Vec(Fm,Fm),Eval(Fm),D(Fm),Offd(Fm)
780   REDIM Offd2(Fm,1),D1(Fm),K(Fm,Fm)
790   LINK "INPPLT",1740
800   CALL Feinpt(Xx(*),Yy(*),#1,Nnode,Nelemt,Cw,Cx,Cy,Cxy,Nod
      e(*),Nodc(*))
810   FOR Matno=1 TO Nmat
820   CALL Cmatrx(C(*),Th(*),Dens(*),#1,Matno)
830   NEXT Matno
840   IF Nskew=0 THEN GOTO 880
850   FOR I=1 TO Nskew
860   READ #1;Nosk(I),Angsk(I)
870   NEXT I
880   REM Generation of mixed matrices [Ge] AND [He]
890   LINK "HEPLT",1740
900   IF Nskew=0 THEN GOTO 1000
910   PRINT
920   LINK "TRNPLT",3800
930   PRINT "Nodal transformation."
940   PRINT
950   PRINT "Node number",SPA(6),"N-X angle(DEG)"
960   PRINT
970   FOR I=1 TO Nskew
980   PRINT USING "(3D,22X,MD.4DE)";Nosk(I),Angsk(I)
990   NEXT I
1000  FOR Z=1 TO Nelemt
1010  MAT He=ZER
1020  MAT Sfm=ZER
1030  FOR U=1 TO 9
1040  CALL Qaux(W(U,1),W(U,2),Xx(*),Yy(*),Detj,Bm(*),Sf(*),Be,
      Ds,Z,Node(*),6)
1050  FOR I=1 TO 8
1060  FOR J=I TO 8
1070  Sfm(I,J)=Sfm(I,J)+Sf(I)*Sf(J)*Detj*W(U,3)*W(U,4)
1080  NEXT J
1090  NEXT I
1100  CALL Heform(He(*),Bm(*),Detj,W(U,3),W(U,4))
1110  NEXT U
1120  CALL Geform(Ge(*),Sfm(*),C(*),Node(Z,9))
1130  CALL Mnsws(He(*),Xx(*),Yy(*),Bm(*),Sf(*),Be,Ds,Node(*),Z,K)
1140  IF Nskew=0 THEN GOTO 1210
1150  FOR I=1 TO Nskew
1160  FOR J=1 TO 8
1170  IF Nosk(I)<>Node(Z,J) THEN GOTO 1190
1180  CALL Transf(Ge(*),He(*),Angsk(I),J)
1190  NEXT J
1200  NEXT I
1210  CALL Ghasemb(G(*),Ge(*),H(*),He(*),Fm,Z,Node(*),Nodc(*))
1220  NEXT Z
1230  FOR I=1 TO Fm
1240  FOR J=I TO Fm
1250  G(I,J)=G(J,I)
1260  NEXT J
1270  NEXT I
```

```
1280  REM Generation of mass matrix[M].
1290  LINK "ME1PLT",1740
1300  FOR Z=1 TO Nelemt
1310  MAT Me=ZER
1320  CALL Meform(Dens(*),Th(*),Me(*),Xx(*),Yy(*),W(*),Detj,Sf
      (*),Bm(*),Node(*),Node(Z,9),Z)
1330  CALL Masemb(M(*),Me(*),Fw,Z,Node(*),Nodc(*))
1340  NEXT Z
1350  FOR I=1 TO Fw
1360  FOR J=I TO Fw
1370  M(I,J)=M(J,I)
1380  NEXT J
1390  NEXT I
1400 ! LINK "Eqsolv",1740 Choleski decompositon
1410 ! LINK "Gauss",1740  Simple Gaussian elimination
1420  LINK "PGauss",1740!  Triangular decomposition with partial
      pivoting.
1430  IF Type=1 THEN GOSUB Weigen
1440  IF Type=2 THEN GOSUB Meigen
1450  ASSIGN #1 TO *
1460  END!Of the program main routine.
1470 Weigen: ! Sub program for eigenvectors of (W).
1480  CALL Eqsolv(G(*),H(*),K(*),Fw,Fm,Type)
1490  LINK "EGNIPT",1740
1500  CALL Eigninpt(M1,M2,Lb,Ub,Fw,Sol)
1510  LINK "TRANS",1740
1520  CALL Trans(M(*),K(*),Vec(*),Eval(*),M1,M2,Lb,Ub,D(*),Off
      d(*),Offd2(*),Dl(*),Ar(*),Cr(*),Type,Fw,Sol)
1530  IF Sol>3 THEN 1560
1540  LINK "EIGEN",1740
1550  CALL Eigen(M(*),K(*),Vec(*),Eval(*),M1,M2,Lb,Ub,D(*),Off
      d(*),Offd2(*),Dl(*),Ar(*),Cr(*),Type,Fw,Sol)
1560  LINK "EVLV",1740
1570  CALL Evlv(M(*),K(*),Vec(*),Eval(*),M1,M2,Lb,Ub,D(*),Offd
      (*),Offd2(*),Dl(*),Ar(*),Cr(*),Type,Fw,Sol)
1580  RETURN !End of sub W
1590 Meigen: ! Sub program for eigenvectors of (M).
1600  CALL Eqsolv(H(*),M(*),K(*),Fm,Fw,Type)
1610  LINK "EGNIPT",1640
1620  CALL Eigninpt(M1,M2,Lb,Ub,Fm,Sol)
1630  LINK "TRANS",1640
1640  CALL Trans(G(*),K(*),Vec(*),Eval(*),M1,M2,Lb,Ub,D(*),Off
      d(*),Offd2(*),Dl(*),Ar(*),Cr(*),Type,Fm,Sol)
1650  IF Sol>3 THEN GOTO 1680
1660  LINK "EIGEN",1740
1670  CALL Eigen(G(*),K(*),Vec(*),Eval(*),M1,M2,Lb,Ub,D(*),Off
      d(*),Offd2(*),Dl(*),Ar(*),Cr(*),Type,Fm,Sol)
1680  LINK "EVLV",1740
1690  CALL Evlv(G(*),K(*),Vec(*),Eval(*),M1,M2,Lb,Ub,D(*),Offd
      (*),Offd2(*),Dl(*),Ar(*),Cr(*),Type,Fm,Sol)
1700  RETURN !End of sub M.
```

```
10    REM *******************************************
20    REM *Forced vibration analysis of thin plates by *
30    REM *Reissner's method.                           *
40    REM *Program name is:"RFPLT2:F"                   *
50    REM *                                             *
60    REM *Version#1:Free vibration analysis.           *
70    REM *Version#2:Forced vibration response analysis*
80    REM *by mode superposition method.                *
90    REM *Version#3:Forced vibration response analysis*
100   REM *by direct integration method.                *
110   REM *******************************************
120   OPTION BASE 1
130   PRINTER IS 16
140   PRINT PAGE,SPA(24),"PROGRAM INTRODUCTION #2",LIN(1)
150   PRINT "Thin plate vibration by mixed formulation:"
160   PRINT "The program is based on a mixed variational"
170   PRINT "principle known as Hellinger-Reissner's principle."
180   PRINT "An 8 node quadrilateral finite element is "
190   PRINT "used for discretisation of the plate."
200   PRINT "Lateral deflection is assumed to vary parabolicaly"
210   PRINT "inside the element:"
220   PRINT SPA(18);"W=a1+a2X+a3Y+a4XY+a5XY^2+a6YX^2+a7Y^2+a8X
      ^2",LIN(1)
230   PRINT "Bending and twisting moments also vary parabolicaly"
240   PRINT "inside the element:",LIN(1)
250   PRINT SPA(10);"Mx,My,Mxy=b1+b2X+b3Y+b4XY+b5XY^2+b6YX^2+b
      7Y^2+b8X^2",LIN(1)
260   PRINT "Changes in input include:
                  "
270   PRINT "1-Nodal connections,2-Material number,"
280   PRINT "3-Element properties and 4-Element thickness."
290   PRINT "Orthotrpic & isotropic materials may be used."
300   PRINT "In version#2 of this program,mode superposition "
310   PRINT "method is used in order to calculate time       "
320   PRINT "response history of the plate displacements and"
330   PRINT "moments under the action of external loads.Press
      CONT  "
340   PAUSE
350   PRINT PAGE,SPA(15),"Data files required are",LIN(1)
360   PRINT
370   PRINT "Before running the program the following data "
380   PRINT
390   PRINT "files should be created:"
400   PRINT
410   PRINT "1-Data file[Initil:F]to be used for recording "
420   PRINT
430   PRINT "the initial conditions."
440   PRINT
450   PRINT "2-Data file[Eqn:F] to be used in order to "
460   PRINT
470   PRINT "print the excitation forces on."
480   PRINT "Create the data files,Latch PRT ALL and press
      CONT."
490   PAUSE
500   DIM J(2,2),Ge(24,24),He(24,8),A$[20],Vec(26,26)
510   DIM Eval(26),D(26),Xx(65),Yy(65),Offd(26),Th(8)
520   DIM Offd2(26,1),Me(8,8),Cons(10,6),H(95,26),G(95,95)
530   DIM Dens(8),M(26,26),D1(26),Bm(2,8),K(27,96),Wt$(1)[20]
540   DIM Vect(16,16),P1(45),Angsk(20),Mcnt(1001),Initil(5)
550   DIM P(96),Mt$(5)[20],Wcnt(1000),I$[160],T$[80]
560   DIM Sf(8),Sfm(8,8),W(9,4),Apfo(30),F0(20),Dratio(20)
570   INTEGER Nodc(65,4),Node(16,9),Nosk(20),Type,R,Cv
580   INTEGER I,K,Jo,Z,Sol,N,Op,Cw,Cx
590   INTEGER Fw,Fm,Nnode,Nelemt,Nmat,Matno
600   INTEGER Cxy,Nskew,J2,Nmode,Wplt,Mplt,Neq,Njob,Ndmode
```

```
610   REM Gauss points and weights for numerical integration.
620   A=.774596669241
630   B=0
640   C=.55555555555
650   D=.888888888888
660   W(1,1)=W(1,2)=W(2,2)=W(3,2)=W(4,1)=W(7,1)=-A
670   W(2,1)=W(4,2)=W(5,1)=W(5,2)=W(6,2)=W(8,1)=0
680   W(3,1)=W(6,1)=W(7,2)=W(8,2)=W(9,1)=W(9,2)=A
690   W(1,3)=W(1,4)=W(6,4)=W(2,3)=W(7,3)=W(7,4)=C
700   W(9,3)=W(9,4)=W(3,3)=W(3,4)=W(8,3)=W(4,3)=C
710   W(6,3)=W(2,4)=W(8,4)=W(4,4)=W(5,3)=W(5,4)=D
720   A=B=C=D=0
730   PRINT
740   DISP "Type in name of the input data file?,press CONT"
750   INPUT Data$
760   ASSIGN #1 TO Data$,C
770   IF NOT C THEN GOTO 820
780   BEEP
790   DISP "File not found.Try again"
800   WAIT 4000
810   GOTO 740
820   DISP "What is the printing device?16/0"
830   INPUT P
840   PRINTER IS P
850   P=0
860   PRINT ,SPA(1);"Vibration analysis of thin plates"
870   PRINT ,SPA(1);"---------------------------------"
880   PRINT
890   LINPUT "Type in name of the job.Not more than 20 charact
      ers",A$
900   PRINT "Job name........";A$;""
910   LINK "INPPLT",2860
920   CALL Feinpt(Xx(*),Yy(*),#1,Nnode,Nelemt,Njob,Cw,Cx,Cy,Cx
      y,Nmat,Nskew,Node(*),Nodc(*))
930   Fm=3*Nnode-(Cx+Cy+Cxy)
940   Fw=Nnode-Cw
950   REDIM Xx(Nnode),Yy(Nnode),Cons(Nmat,6),Th(Nmat),Node(Nel
      emt,9)
960   REDIM Dens(Nmat),G(Fm,Fm),H(Fm,Fw),M(Fw,Fw),Apfo(Fw),F0(Fw)
970   REDIM Vec(Fw,Fw),Eval(Fw),D(Fw),Offd(Fw),Offd2(Fw,1),D1(
      Fw),k(Fw,Fw)
980   REDIM P(Fm),Nodc(Nnode,4),P1(Fw),Vect(Fw,Fw),Angsk(Nskew
      +1),Nosk(Nskew+1)
990   FOR Matno=1 TO Nmat
1000  CALL Cmatrx(Cons(*),Th(*),Dens(*),#1,Matno)
1010  NEXT Matno
1020  IF Nskew=0 THEN GOTO 1060
1030  FOR I=1 TO Nskew
1040  READ #1;Nosk(I),Angsk(I)
1050  NEXT I
1060  REM Loading conditions.
1070  LINK "FLOAD",2860
1080  CALL Loadap(F0(*),W(*),Xx(*),Yy(*),Detj,Bm(*),Sf(*),Node
      (*),Nodc(*),Nnode,Nelemt)
1090  PRINT
1100  PRINT "Response analysis data:"
1110  REM Input excitation as a function of time.
1120  LINK "EXCITE",2860
1130  CALL Excitn(Neq)
1140  REM Input information concerning the forced vibration of
      plate.
1150  LINK "RFINPT",2860
1160  CALL Rspipt(Wcnt(*),Mcnt(*),Offd(*),D(*),Time,Delta,Init
      il(*),Mt$(*),Wt$(*),Dratio(*),Neq,Nnode,Fw,J2,Nmode,Wplt
      ,Mplt,Nodc(*),Ndmode)
1170  REM Generation of mixed matrices [Ge] AND [He]
```

```
1180  LINK "HEPLT",2860
1190  IF Nskew=0 THEN GOTO 1290
1200  PRINT
1210  LINK "TRNPLT",6090
1220  PRINT "Nodal transformation."
1230  PRINT
1240  PRINT "Node number";SPA(7);"N-X angle(DEG)"
1250  PRINT
1260  FOR I=1 TO Nskew
1270  PRINT USING "(3D,22X,MD.4DE)";Nosk(I),Angsk(I)
1280  NEXT I
1290  FOR Z=1 TO Nelemt
1300  MAT He=ZER
1310  MAT Sfm=ZER
1320  FOR U=1 TO 9
1330  CALL Qaux(W(U,1),W(U,2),Xx(*),Yy(*),Detj,Bm(*),Sf(*),Be,
      Ds,Z,Node(*),6)
1340  FOR I=1 TO 8
1350  FOR J=I TO 8
1360  Sfm(I,J)=Sfm(I,J)+Sf(I)*Sf(J)*Detj*W(U,3)*W(U,4)
1370  NEXT J
1380  NEXT I
1390  CALL Heform(He(*),Bm(*),Detj,W(U,3),W(U,4))
1400  NEXT U
1410  CALL Geform(Ge(*),Sfm(*),Cons(*),Node(Z,9))
1420  CALL Mnsws(He(*),Xx(*),Yy(*),Bm(*),Sf(*),Be,Ds,Node(*),Z,K)
1430  IF Nskew=0 THEN GOTO 1500
1440  FOR I=1 TO Nskew
1450  FOR J=1 TO 8
1460  IF Nosk(I)<>Node(Z,J) THEN GOTO 1480
1470  CALL Transf(Ge(*),He(*),Angsk(I),J)
1480  NEXT J
1490  NEXT I
1500  CALL Ghasemb(G(*),Ge(*),H(*),He(*),Fm,Z,Node(*),Nodc(*))
1510  NEXT Z
1520  FOR I=1 TO Fm
1530  FOR J=I TO Fm
1540  G(I,J)=G(J,I)
1550  NEXT J
1560  NEXT I
1570  REM Generation of mass matrix[M].
1580  LINK "ME1PLT",2860
1590  FOR Z=1 TO Nelemt
1600  MAT Me=ZER
1610  CALL Meform(Dens(*),Th(*),Me(*),Xx(*),Yy(*),W(*),Detj,Sf
      (*),Bm(*),Node(*),Node(Z,9),Z)
1620  CALL Masemb(M(*),Me(*),Fw,Z,Node(*),Nodc(*))
1630  NEXT Z
1640  FOR I=1 TO Fw
1650  FOR J=I TO Fw
1660  M(I,J)=M(J,I)
1670  NEXT J
1680  NEXT I
1690  ASSIGN #1 TO *!To close the finite element input data file.
1700  REM Response analysis starts here
1710  LINK "MODAL",2860
1720  CALL Eqsolv(H(*),G(*),K(*),P(*),Fw,Fm)
1730  CALL Modal(Vec(*),Eval(*),M(*),K(*),Dratio(*),D(*),Offd(
      *),Offd2(*),D1(*),Nmode,Fw,1,Sol,Ndmode)
1740  MAT Vect=TRN(Vec)
1750  REM Load vector transformation.
1760  MAT P1=Vect*F0
1770  MAT F0=P1
1780  REM Loop round the number of excitation forces.
1790  ASSIGN #1 TO "Initil:F"!To read non zero initial conditions.
1800  LINK "DUHAML",2860
```

```
1810  LINK "Eqn",5250
1820  FOR K1=1 TO Neq!Loop round the number of forces
1830  PRINT "Force set";K1;""
1840  IF Initil(K1)>0 THEN GOTO 1900
1850  MAT K=ZER!The 1st two columns of [K] are used as initial
      conditions
1860  !           in transformed coordinates.
1870  Wcnt(1)=0
1880  Mcnt(1)=0
1890  GOTO 2100
1900  MAT READ #1;D
1910  MAT READ #1;Offd
1920  REM Initial bending moments are calculated.
1930  Pi=0
1940  FOR J=1 TO Fw
1950  P(Nodc(Mplt,J2))=Pi+H(Nodc(Mplt,J2),J)*D(J)
1960  Pi=P(Nodc(Mplt,J2))
1970  NEXT J
1980  Wcnt(1)=D(Nodc(Wplt,4))
1990  Mcnt(1)=P(Nodc(Mplt,J2))
2000  MAT P1=M*D
2010  MAT D=Vect*P1
2020  MAT P1=M*Offd
2030  MAT Offd=Vect*P1
2040  FOR I=1 TO Fw
2050  K(I,1)=D(I)
2060  K(I,2)=Offd(I)
2070  NEXT I
2080  MAT D=ZER
2090  MAT Offd=ZER
2100  REM Loop round the integration points
2110  Cnt=1
2120  T=0
2130  Npts=INT(Time/Delta)+1
2140  FOR Count=1 TO Npts-1
2150  T=T+Delta
2160  Cnt=Cnt+1
2170  CALL Eqn(T,F,K1)
2180  FOR Deg=1 TO Nmode
2190  F0=F*F0(Deg)
2200  Nf=SQR(Eval(Deg))
2210  IF Deg>Ndmode THEN GOTO 2240
2220  Ze=Dratio(Deg)
2230  GOTO 2250
2240  Ze=0
2250  Dnf=Nf*SQR(1-Ze^2)
2260  CALL Duhammel(T,Nf,Dnf,Ze,Delta,F0,D1(Deg),Offd(Deg),D(D
      eg),Offd2(Deg,1),Y,K(Deg,1),K(Deg,2),Initil(*),K1)
2270  Apfo(Deg)=1/Dnf*Y
2280  NEXT Deg
2290  REM Back transformation to system coordinates.
2300  FOR I=1 TO Fw
2310  A=0
2320  FOR J=1 TO Nmode
2330  P1(I)=A+Vec(I,J)*Apfo(J)
2340  A=P1(I)
2350  NEXT J
2360  NEXT I
2370  Pi=0
2380  FOR J=1 TO Fw
2390  P(Nodc(Mplt,J2))=Pi+H(Nodc(Mplt,J2),J)*P1(J)
2400  Pi=P(Nodc(Mplt,J2))
2410  NEXT J
2420  Wcnt(Cnt)=P1(Nodc(Wplt,4))
2430  Mcnt(Cnt)=P(Nodc(Mplt,J2))
2440  MAT P1=ZER
```

```
2450  NEXT Count
2460  REM Response plots.
2470  CALL Plot(Wcnt(*),Wt$(1),Time,Delta,Wplt)
2480  CALL Plot(Mcnt(*),Mt$(J2),Time,Delta,Mplt)
2490  LINPUT "File name to print displacements on?",Fdisp$
2500  LINPUT "File name to print moments on?",Fstrs$
2510  ASSIGN #2 TO Fdisp$
2520  ASSIGN #3 TO Fstrs$
2530  MAT PRINT #2;Wcnt
2540  MAT PRINT #3;Mcnt
2550  ASSIGN #2 TO *
2560  ASSIGN #3 TO *
2570  NEXT K1
2580  ASSIGN * TO #1!To close file Initil
2590  BEEP
2600  PRINT "Execution terminated"
2610  END!Of program
```

```
10    REM *********************************************
20    REM *Forced vibration analysis of thin plates by   *
30    REM *                                             *
40    REM *Reissner's method Program name is:"RFPLT3:F"  *
50    REM *                                             *
60    REM *Version#1:Free vibration analysis.           *
70    REM *Version#2:Forced vibration response analysis. *
80    REM *by mode superposition method.                *
90    REM *Version#3:Forced vibration response analysis. *
100   REM *by direct integration method.                *
110   REM *********************************************
120   OPTION BASE 1
130   PRINTER IS 16
140   PRINT PAGE,SPA(24),"PROGRAM INTRODUCTION #3",LIN(1)
150   PRINT "Thin plate vibration by mixed formulation:The"
160   PRINT "program is based on a mixed variational principle"
170   PRINT "known as Hellinger-Reissner's principle."
180   PRINT "An 8 node quadrilateral finite element is used"
190   PRINT "for discretisation of the plate.Lateral deflection"
200   PRINT "is assumed to vary parabolicaly inside the element"
210   PRINT SPA(18);"W=a1+a2X+a3Y+a4XY+a5XY^2+a6YX^2+a7Y^2+a8X
      ^2",LIN(1)
220   PRINT "Bending and twisting moments also vary parabolicaly"
230   PRINT "inside the element:",LIN(1)
240   PRINT SPA(10);"Mx,My,Mxy=b1+b2X+b3Y+b4XY+b5XY^2+b6YX^2+b
      7Y^2+b8X^2",LIN(1)
250   PRINT "Changes in input include:                    "
260   PRINT "1-Nodal connections,2-Material number,"
270   PRINT "3-Element properties and 4-Element thickness."
280   PRINT "Orthotrpic & isotropic materials may be used."
290   PRINT "In version#3 of this program,an unconditionally"
300   PRINT "stable direct integration method known as "
310   PRINT "Wilson theta is used in order to calculate time "
320   PRINT "response history of the plate displacement and
      "
330   PRINT "moments under external loads.Press CONT.
       "
340   PAUSE
350   PRINT PAGE,SPA(15),"Data files required by the programs
      ",LIN(1)
360   PRINT
370   PRINT "Before running the program the following data   "
380   PRINT
390   PRINT "files should be created:                        "
400   PRINT
410   PRINT "1-Data file[Initil:F]to be used for recording "
420   PRINT
430   PRINT "the initial conditions.                         "
440   PRINT
450   PRINT "2-Data file[Eqn:F] to be used in order to print "
460   PRINT
470   PRINT "the excitation forces on."
480   PRINT "Create the data files,Latch PRT ALL and press
      CONT."
490   PAUSE
500   DIM J(2,2),Ge(24,24),He(24,8),A$[20],Vec(27,27)
510   DIM Eval(27),D(27),Xx(65),Yy(65),Offd(27),Offd2(27,1)
520   DIM Me(8,8),Cons(10,6),Th(8),H(96,27),G(96,96),Dens(8)
530   DIM M(27,27),Dl(27),Bm(2,8),K(27,96),C(27,27)
540   DIM P(96),Mt$(5)[20],Wcnt(1000),Angsk(20),Wt$(1)[20]
550   DIM Sf(8),Sfm(8,8),W(9,4),Apfo(30,1),F0(27)
560   DIM Dratio(20),Mcnt(1000),A0(9)
570   INTEGER Nodc(65,4),Node(16,9),Nosk(20),Fw,Fm,Nnode,Cy
580   INTEGER Nelemt,Nmat,Matno,I,K,Jo,Z,Sol,N,Op,Cw,Cx,R,Type
590   INTEGER Cxy,Nskew,J2,Nmode,Wplt,Mplt,Neq,Njob
```

```
600   REM Gause points and weights for numerical integration.
610   A=.774596669241
620   B=0
630   C=.55555555555
640   D=.888888888888
650   W(1,1)=W(1,2)=W(2,2)=W(3,2)=W(4,1)=W(7,1)=-A
660   W(2,1)=W(4,2)=W(5,1)=W(5,2)=W(6,2)=W(8,1)=0
670   W(3,1)=W(6,1)=W(7,2)=W(8,2)=W(9,1)=W(9,2)=A
680   W(1,3)=W(1,4)=W(6,4)=W(2,3)=W(7,3)=W(7,4)=C
690   W(9,3)=W(9,4)=W(3,3)=W(3,4)=W(8,3)=W(4,3)=C
700   W(6,3)=W(2,4)=W(8,4)=W(4,4)=W(5,3)=W(5,4)=D
710   A=B=C=D=0
720   PRINT
730   DISP "Type in name of the input data file?,press CONT"
740   INPUT Data$
750   ASSIGN #1 TO Data$,C
760   IF NOT C THEN GOTO 810
770   BEEP
780   DISP "File not found.Try again"
790   WAIT 4000
800   GOTO 730
810   DISP "What is the printing device?16/0"
820   INPUT P
830   PRINTER IS P
840   P=0
850   PRINT ,SPA(1);"Vibration analysis of thin plates"
860   PRINT ,SPA(1);"----------------------------------"
870   PRINT
880   LINPUT "Type in name of the job.Not more than 20 charact
      ers",A$
890   PRINT "Job name........";A$;""
900   LINK "INPPLT",2300
910   CALL Feinpt(Xx(*),Yy(*),#1,Nnode,Nelemt,Njob,Cw,Cx,Cy,Cx
      y,Nmat,Nskew,Node(*),Nodc(*))
920   PRINT LIN(2)
930   PRINT "Element selected:"
940   PRINT "8-node quadrilateral"
950   PRINT LIN(3)
960   PRINT "Number of elements........";Nelemt,LIN(2)
970   PRINT "Number of nodes   ........";Nnode
980   Fm=3*Nnode-(Cx+Cy+Cxy)
990   Fw=Nnode-Cw
1000  REDIM Dens(Nmat),G(Fm,Fm),H(Fm,Fw),M(Fw,Fw),C(Fw,Fw)
1010  REDIM Vec(Fw,Fw),Eval(Fw),D(Fw),Offd(Fw),Offd2(Fw,1)
1020  REDIM P(Fm),Angsk(Nskew+1),Nosk(Nskew+1),Cons(Nmat,6)
1030  REDIM Apfo(Fw,1),F0(Fw),D1(Fw),K(Fw,Fw),Th(Nmat)
1040  FOR Matno=1 TO Nmat
1050  CALL Cmatrx(Cons(*),Th(*),Dens(*),#1,Matno)
1060  NEXT Matno
1070  IF Nskew=0 THEN GOTO 1110
1080  FOR I=1 TO Nskew
1090  READ #1;Nosk(I),Angsk(I)
1100  NEXT I
1110  REM Loading conditions.
1120  LINK "FLOAD",2300
1130  CALL Loadap(F0(*),W(*),Xx(*),Yy(*),Detj,Bm(*),Sf(*),Node
      (*),Nodc(*),Nnode,Nelemt)
1140  PRINT
1150  PRINT "Response analysis data:"
1160  REM Input excitation as a function of time.
1170  LINK "EXCITE",2300
1180  CALL Excitn(Neq)
1190  REM Input information concerning the forced vibration of
      plate.
1200  LINK "FINPUT",2300
1210  CALL Rspipt(Wcnt(*),Mcnt(*),D(*),Offd(*),Mt$(*),Wt$(*),T
      ime,Delta,Dratio(*),A0(*),Theta,#2,Neq,Nnode,Fw,J2,Nmode
      ,Wplt,Mplt,Nodc(*))
```

```
1220  REM Generation of mixed matrices [Ge] AND [He]
1230  LINK "HEPLT",2300
1240  IF Nskew=0 THEN GOTO 1340
1250  PRINT
1260  LINK "TRNPLT",6290
1270  PRINT "Nodal transformation."
1280  PRINT
1290  PRINT "Node number";SPA(7);"N-X angle(DEG)"
1300  PRINT
1310  FOR I=1 TO Nskew
1320  PRINT USING "(3D,22X,MD.4DE)";Nosk(I),Angsk(I)
1330  NEXT I
1340  FOR Z=1 TO Nelemt
1350  MAT He=ZER
1360  MAT Sfm=ZER
1370  FOR U=1 TO 9
1380  CALL Qaux(W(U,1),W(U,2),Xx(*),Yy(*),Detj,Bm(*),Sf(*),Be,
      Ds,Z,Node(*),6)
1390  FOR I=1 TO 8
1400  FOR J=I TO 8
1410  Sfm(I,J)=Sfm(I,J)+Sf(I)*Sf(J)*Detj*W(U,3)*W(U,4)
1420  NEXT J
1430  NEXT I
1440  CALL Heform(He(*),Bm(*),Detj,W(U,3),W(U,4))
1450  NEXT U
1460  CALL Geform(Ge(*),Sfm(*),Cons(*),Node(Z,9))
1470  CALL Mnsws(He(*),Xx(*),Yy(*),Bm(*),Sf(*),Be,Ds,Node(*),Z,K)
1480  IF Nskew=0 THEN GOTO 1550
1490  FOR I=1 TO Nskew
1500  FOR J=1 TO 8
1510  IF Nosk(I)<>Node(Z,J) THEN GOTO 1530
1520  CALL Transf(Ge(*),He(*),Angsk(I),J)
1530  NEXT J
1540  NEXT I
1550  CALL Ghasemb(G(*),Ge(*),H(*),He(*),Fm,Z,Node(*),Nodc(*))
1560  NEXT Z
1570  FOR I=1 TO Fm
1580  FOR J=I TO Fm
1590  G(I,J)=G(J,I)
1600  NEXT J
1610  NEXT I
1620  REM Generation of mass matrix[M].
1630  LINK "ME1PLT",2300
1640  FOR Z=1 TO Nelemt
1650  MAT Me=ZER
1660  CALL Meform(Dens(*),Th(*),Me(*),Xx(*),Yy(*),W(*),Detj,Sf
      (*),Bm(*),Node(*),Node(Z,9),Z)
1670  CALL Masemb(M(*),Me(*),Fw,Z,Node(*),Nodc(*))
1680  NEXT Z
1690  FOR I=1 TO Fw
1700  FOR J=I TO Fw
1710  M(I,J)=M(J,I)
1720  NEXT J
1730  NEXT I
1740  ASSIGN #1 TO *!To close the finite element input data file.
1750  REM Response analysis starts here
1760  LINK "FDAMP",2300
1770  CALL Eqsolv(H(*),G(*),K(*),P(*),Fw,Fm)
1780  REM Damping matrix evaluation
1790  IF Nmode=0 THEN GOTO 1810
1800  CALL Dampmat(C(*),Vec(*),Eval(*),M(*),K(*),Dratio(*),D(*
      ),Offd(*),Offd2(*),D1(*),Nmode,Fw,Type,Sol)
1810  LINK "FINITL",2300
1820  MAT Vec=M
1830  CALL Eqsolv1(M(*),D1(*),1,Fw)
1840  REM Initial acceleration is calculated and printed on file
      #2.
```

```
1850  CALL Initial(D(*),Apfo(*),F0(*),K(*),C(*),Offd2(*),Offd(
      *),M(*),Dl(*),Delta,1,Fw,Neq,#2)
1860  FOR K1=1 TO Neq!Loop round the number of forces
1870  PRINT "Force set";K1;""
1880  ASSIGN #1 TO "Initil"
1890  MAT READ #1;D
1900  MAT READ #1;Offd
1910  FOR I=1 TO K1
1920  MAT READ #1;Offd2
1930  NEXT I
1940  REM Calculation of displacement & bending moment at time
      0
1950  Pi=0
1960  FOR J=1 TO Fw
1970  P(Nodc(Mplt,J2))=Pi+H(Nodc(Mplt,J2),J)*D(J)
1980  Pi=P(Nodc(Mplt,J2))
1990  NEXT J
2000  Wcnt(1)=D(Nodc(Wplt,4))
2010  Mcnt(1)=P(Nodc(Mplt,J2))
2020  REM Calculation of response by Wilson theta method.
2030  LINK "WILSNT",2300
2040  LINK "Eqn",4450
2050  CALL Wilsnsol(K(*),Vec(*),C(*),Apfo(*),F0(*),D(*),Offd(*
      ),Offd2(*),Dl(*),A0(*),Time,Delta,Theta,K1,P(*),H(*),Wcn
      t(*),Mcnt(*),Nodc(*),Mplt,Wplt,Fw,Fm,J2)
2060  REM Response plots.
2070  LINK "FPLOT",2300
2080  CALL Plot(Wcnt(*),Wt$(1),Time,Delta,Wplt)
2090  CALL Plot(Mcnt(*),Mt$(J2),Time,Delta,Mplt)
2100  LINPUT "File name to print displacements on?",Fdisp$
2110  LINPUT "File name to print stresses on?",Fstrs$
2120  ASSIGN #2 TO Fdisp$
2130  ASSIGN #3 TO Fstrs$
2140  MAT PRINT #2;Wcnt
2150  MAT PRINT #3;Mcnt
2160  ASSIGN #2 TO *
2170  ASSIGN #3 TO *
2180  NEXT K1
2190  ASSIGN * TO #1!To close file Initil
2200  BEEP
2210  PRINT "Execution terminated"
2220  END!Of program
```

```
PROGRAMME STORED IN FILE :     INPPLT          Page   1
              LISTED ON :     17/6/83

10     SUB Feinpt(X(*),Y(*),#1,INTEGER N,N1,Njb,Cw,Cx,Cy,Cxy,Nm
       at,Nskw,N(*),Ndc(*))
20     OPTION BASE 1
30     ! Nodal connection matrix is evaluated.
40     DIM K(32,4)
50     READ #1;Njb,N1,N,Cw,Cx,Cy,Cxy,Nmat,Nskw
60     REDIM X(N),Y(N),N(N1,9),Ndc(N,4)
70     Big=Cx
80     IF Cy>=Big THEN Big=Cy
90     IF Cxy>=Big THEN Big=Cxy
100    IF Cw>=Big THEN Big=Cw
110    IF Big<>0 THEN REDIM K(Big,4)
120    FOR I=1 TO N
130    READ #1;X(I),Y(I)
140    NEXT I
150    READ #1;N(*)
160    PRINT LIN(4)
170    PRINT "Nodal point data:",LIN(2)
180    PRINT "Node";SPA(4);"X-coord";SPA(5);"Y-coord"
190    FOR I=1 TO N
200    PRINT USING "3D,3X,2(MD.4DE,2X)";I,X(I),Y(I)
210    NEXT I
220    PRINT LIN(4)
230    PRINT "Element Data:";LIN(2)
240    PRINT "Element";SPA(18);"Nodal connections";SPA(19);"Mat
       erial"
250    FOR W=1 TO N1
260    PRINT USING "3D,7X,8(3D,2X),13X,2D";W,N(W,1),N(W,2),N(W,
       3),N(W,4),N(W,5),N(W,6),N(W,7),N(W,8),N(W,9)
270    NEXT W
280    MAT Ndc=ZER
290    FOR J=1 TO Cw
300    READ #1;K(J,1)
310    NEXT J
320    FOR J=1 TO Cx
330    READ #1;K(J,2)
340    NEXT J
350    FOR J=1 TO Cy
360    READ #1;K(J,3)
370    NEXT J
380    FOR J=1 TO Cxy
390    READ #1;K(J,4)
400    NEXT J
410    A=0
420    FOR I=1 TO N
430    FOR J=1 TO Cx
440    IF K(J,2)=I THEN My
450    NEXT J
460    Ndc(I,1)=A+1
470    A=Ndc(I,1)
480 My:  FOR J=1 TO Cy
490    IF K(J,3)=I THEN Mxy
500    NEXT J
510    Ndc(I,2)=A+1
520    A=Ndc(I,2)
530 Mxy:  FOR J=1 TO Cxy
540    IF K(J,4)=I THEN GOTO 580
550    NEXT J
560    Ndc(I,3)=A+1
570    A=Ndc(I,3)
580    NEXT I
590    FOR I=1 TO N
600    Ndc(I,4)=0
610    NEXT I
620    A=0
```

```
630    FOR I=1 TO N
640    FOR J=1 TO Cw
650    IF K(J,1)=I THEN GOTO 690
660    NEXT J
670    Ndc(I,4)=A+1
680    A=Ndc(I,4)
690    NEXT I
700    DISP "Nodal connection matrix is[for reducing [K] and [M]"
710    DISP Ndc(*);
720    SUBEND!End of feinput
730    SUB Cmatrx(Z(*),Th(*),D(*),#1,INTEGER Mat)
740    REM Calculation of elastic constants for plate element
750       ! when several materials are present.
760    OPTION BASE 1
770    DIM A(5)
780    READ #1;Th(Mat),D(Mat)
790    READ #1;A(*)
800    IF Mat<>1 THEN GOTO 840
810    PRINT LIN(3)
820    PRINT "Material and elastic properties",LIN(1)
830    PRINT "Mat.no";SPA(2);"Mat.Dens";SPA(3);"Elemt.thick";SP
       A(3);"Exx            Eyy          Vxy           Gxy",LIN(1)
840    PRINT USING "2D,4X,6(MD.4DE, X)";Mat,D(Mat),Th(Mat),A(1)
       ,A(4),A(2),A(3)
850    C0=12/Th(Mat)^3
860    C11=C0/A(1)
870    C12=-C0*A(2)/A(4)
880    C13=0
890    C22=C0/A(4)
900    C23=0
910    C33=C0/A(3)
920    Z(Mat,1)=C11
930    Z(Mat,2)=C12
940    Z(Mat,3)=C13
950    Z(Mat,4)=C22
960    Z(Mat,5)=C23
970    Z(Mat,6)=C33
980    SUBEND !End of Cmatrx.
```

```
10     SUB Qaux(L1,L2,X(*),Y(*),U,Pm(*),Sf(*),Be,Ds,INTEGER Z,N
       (*),Jo)
20     OPTION BASE 1
30     DEFAULT ON
40     DIM J(2,2)
50     REM This sub program evaluates the jacobian J,
60     ! its determinant U ,and shape function
70     ! derivatives of W and M.
80     Dn1=1/4*(1-L2)*(2*L1+L2)
90     Dn2=1/4*(1-L1)*(2*L2+L1)
100    Dn3=1/4*(1-L2)*(2*L1-L2)
110    Dn4=-1/4*(1+L1)*(L1-2*L2)
120    Dn5=1/4*(1+L2)*(2*L1+L2)
130    Dn6=1/4*(1+L1)*(2*L2+L1)
140    Dn7=-1/4*(1+L2)*(-2*L1+L2)
150    Dn8=1/4*(1-L1)*(2*L2-L1)
160    Dn9=-L1*(1-L2)
170    Dn10=-1/2*(1-L1^2)
180    Dn11=1/2*(1-L2^2)
190    Dn12=-L2*(1+L1)
200    Dn13=-L1*(1+L2)
210    Dn14=-Dn10
220    Dn15=-Dn11
230    Dn16=-L2*(1-L1)
240    REM Shape functions
250    Sf(1)=1/4*(1-L1)*(1-L2)*(-L1-L2-1)
260    Sf(2)=1/4*(1+L1)*(1-L2)*(L1-L2-1)
270    Sf(3)=1/4*(1+L1)*(1+L2)*(L1+L2-1)
280    Sf(4)=1/4*(1-L1)*(1+L2)*(-L1+L2-1)
290    Sf(5)=1/2*(1-L1^2)*(1-L2)
300    Sf(6)=1/2*(1+L1)*(1-L2^2)
310    Sf(7)=1/2*(1-L1^2)*(1+L2)
320    Sf(8)=1/2*(1-L1)*(1-L2^2)
330    J(1,1)=Dn1*X(N(Z,1))+Dn3*X(N(Z,2))+Dn5*X(N(Z,3))+Dn7*X(N
       (Z,4))
340    J(1,1)=J(1,1)+Dn9*X(N(Z,5))+Dn11*X(N(Z,6))+Dn13*X(N(Z,7)
       )+Dn15*X(N(Z,8))
350    U1=J(1,1)
360    J(1,2)=Dn1*Y(N(Z,1))+Dn3*Y(N(Z,2))+Dn5*Y(N(Z,3))+Dn7*Y(N
       (Z,4))
370    J(1,2)=J(1,2)+Dn9*Y(N(Z,5))+Dn11*Y(N(Z,6))+Dn13*Y(N(Z,7)
       )+Dn15*Y(N(Z,8))
380    U2=J(1,2)
390    J(2,1)=Dn2*X(N(Z,1))+Dn4*X(N(Z,2))+Dn6*X(N(Z,3))+Dn8*X(N
       (Z,4))
400    J(2,1)=J(2,1)+Dn10*X(N(Z,5))+Dn12*X(N(Z,6))+Dn14*X(N(Z,7
       ))+Dn16*X(N(Z,8))
410    U3=J(2,1)
420    J(2,2)=Dn2*Y(N(Z,1))+Dn4*Y(N(Z,2))+Dn6*Y(N(Z,3))+Dn8*Y(N
       (Z,4))
430    J(2,2)=J(2,2)+Dn10*Y(N(Z,5))+Dn12*Y(N(Z,6))+Dn14*Y(N(Z,7
       ))+Dn16*Y(N(Z,8))
440    U4=J(2,2)
450    REM U replaces DETJ
460    U=U1*U4-U2*U3
470    IF Jo=5 THEN SUBEXIT
480    J(1,1)=J(2,2)/U
490    J(1,2)=-J(1,2)/U
500    J(2,1)=-J(2,1)/U
510    J(2,2)=U1/U
520    ! Determination of p[2,8],derivatives of W shape functions.
530    FOR I=1 TO 2
540    Pm(I,1)=Dn1*J(I,1)+Dn2*J(I,2)
550    Pm(I,2)=Dn3*J(I,1)+Dn4*J(I,2)
560    Pm(I,3)=Dn5*J(I,1)+Dn6*J(I,2)
```

```
570    Pm(I,4)=Dn7*J(I,1)+Dn8*J(I,2)
580    Pm(I,5)=Dn9*J(I,1)+Dn10*J(I,2)
590    Pm(I,6)=Dn11*J(I,1)+Dn12*J(I,2)
600    Pm(I,7)=Dn13*J(I,1)+Dn14*J(I,2)
610    Pm(I,8)=Dn15*J(I,1)+Dn16*J(I,2)
620    NEXT I
630    IF Jo=5 THEN SUBEXIT
640    ON Jo GOSUB Side1,Side2,Side3,Side4
650    DEFAULT OFF
660    SUBEXIT
670 Side1:       Be=ATN(ABS(U1/U2))
680    Ds=SQR(U1^2+U2^2)
690    IF U2<0 THEN Be=PI+Be
700    IF U2>=0 THEN Be=-Be
710    RETURN
720 Side3:       Be=ATN(ABS(U1/U2))
730    Ds=SQR(U1^2+U2^2)
740    U1=-1*U1
750    U2=-1*U2
760    IF U2<0 THEN Be=PI-Be
770    RETURN
780 Side2:       Be=ATN(ABS(U3/U4))
790    Ds=SQR(U3^2+U4^2)
800    IF (U4>0) AND (U3>0) THEN Be=-Be
810    RETURN
820 Side4:       Be=ATN(ABS(U3/U4))
830    Ds=SQR(U3^2+U4^2)
840    U3=-U3
850    U4=-U4
860    IF (U4<0) AND (U3>0) THEN Be=PI+Be
870    IF (U4>=0) OR (U3<=0) THEN Be=PI-Be
880    RETURN
890    SUBEND!End of Qaux.
```

```
PROGRAMME STORED IN FILE :    EXCITE          Page   1
                 LISTED ON :    17/6/83

10     SUB Excitn(INTEGER Neq)
20     OPTION BASE 1
30     DIM I$[160],T$[80]
40     DISP "Excitation as a function of time"
50     REM File "Eqn:F" is opened to input the forces.
60     INPUT "Is this a re-run 1/yes,0/no?",Re
70     IF NOT Re THEN GOTO 110
80     BEEP
90     INPUT "How many equations?",Neq
100    GOTO 450
110    ASSIGN #2 TO "Eqn:F"
120    T$="SUB EXIT"
130    J=0
140    FOR N=2010 TO 2100 STEP 10
150    J=J+1
160    I$[1,5]=VAL$(N)
170    I$[6,10]="L"&VAL$(J)&":"
180    LINPUT "Equation is?[e.g type F=SIN(10*T)]",I$[11]
190    PRINT "Excitation function(";J;") is:";I$[11]
200    PRINT #2;I$
210    BEEP
220    DISP "Any more statements concerning",I$[11]
230    INPUT "1 or 0",A
240    IF NOT A THEN 310
250    N=N+1
260    I$[1,5]=VAL$(N)
270    LINPUT "Type in the statement",I$[6]
280    PRINT #2;I$
290    PRINT SPA(23),I$[6]
300    GOTO 210
310    I$[1,5]=VAL$(N+1)
320    I$[6]=T$
330    PRINT #2;I$
340    INPUT "Any more equations?1/0",More
350    IF More THEN GOTO Nextn
360    Neq=J
370    I$[1,5]=VAL$(2110)
380    I$[6]="SUB END"
390    PRINT #2;I$
400    GOTO Ed
410 Nextn:  PRINT
420    NEXT N
430 Ed:  PRINT #2;END
440    ASSIGN #2 TO *
450    SUBEND
```

```
10    SUB Pspipt(Wc(*),Mc(*),D(*),Of(*),Mt$(*),Wt$(*),Time,Del
      ta,Dr(*),A0(*),T,#1,INTEGER Neq,Nnode,Fw,J2,Nmod,Wp,Mp,N
      dc(*))
20    OPTION BASE 1
30    REM Input information regarding Wilson theta
40       ! direct integration method for forced
50       ! vibration analysis of plates.
60    DISP "Integration constants"
70    INPUT "What is the time duration?",Time
80    INPUT "What is the time intertval?",Delta
90    PRINT
100   PRINT USING "K,1X,MD.4DE";"Response duration",Time
110   PRINT
120   PRINT USING "K,2X,MD.4DE";"Time incremental",Delta
130   REDIM Wc(INT(Time/Delta)+1),Mc(INT(Time/Delta)+1)
140   INPUT "Select theta[usually 1.4]",T
150   A0(1)=6/(T*Delta)^2
160   A0(2)=3/(T*Delta)
170   A0(3)=2*A0(2)
180   A0(4)=T*Delta/2
190   A0(5)=A0(1)/T
200   A0(6)=-A0(3)/T
210   A0(7)=1-3/T
220   A0(8)=Delta/2
230   A0(9)=Delta^2/6
240   REM Initial displacement-velocity input
250   DISP "Initial conditions"
260   INPUT "If initial conds are zero press 0 otherwise 1",In
270   IF In<>0 THEN GOTO 310
280   MAT D=ZER
290   MAT Of=ZER
300   GOTO 410
310   FOR I=1 TO Nnode
320   IF Ndc(I,4)=0 THEN GOTO 350
330   DISP "Initial displ.of node";I;"?"
340   INPUT D(Ndc(I,4))
350   NEXT I
360   FOR I=1 TO Nnode
370   IF Ndc(I,4)=0 THEN GOTO 400
380   DISP "Initial veloc.of node";I;"?"
390   INPUT Of(Ndc(I,4))
400   NEXT I
410   ASSIGN #1 TO "Initil:F"
420   MAT PRINT #1;D
430   MAT PRINT #1;Of
440   DISP "Information regarding damping"
450   INPUT "Is the damping significant?1/0",Damp
460   IF NOT Damp THEN GOTO 530
470   INPUT "Number of modes with damping?",Nmod
480   REDIM Dr(Nmod)
490   FOR I=1 TO Nmod
500   DISP "Damping ratio in mode";I;"?"
510   INPUT Dr(I)
520   NEXT I
530   BEEP
540   DISP "Displacement/moment time history plot"
550   INPUT "Node number to plot the displacements for?",Wp
560   INPUT "Node number to plot moment for?",Mp
570   INPUT "Code?[1 for Mx-2 for My-3 for Mxy]",J2
580   Mt$(1)="BENDING MOMENT-X"
590   Mt$(2)="BENDING MOMENT-Y"
600   Mt$(3)="TWISTING MOMENT-XY"
610   Wt$(1)="DEFLECTION-Z"
620   SUBEND
```

```
PROGRAMME STORED IN FILE :    LOAD            Page   1
                LISTED ON :    17/6/83


10    SUB Loadap(R(*),W(*),X(*),Y(*),Detj,Bm(*),Sf(*),INTEGER
      N(*),Ndc(*),Nnode,Nelemt)
20    OPTION BASE 1
30    DIM P(8),Re(8),El(20)
40    INTEGER El
50    REM Equivalent nodal forces due to concentrated or
60       ! distributed loading conditions are determined
70    DISP "Load information"
80    DISP "The following load cases can be accomodated:"
90    DISP "1) Concentrated nodal forces consisting of "
100   DISP "    loads acting in Z directn."
110   DISP "2) Constantly distributed load acting normal to pl
      ate."
120   DISP "3) Varying distributed load acting normal to plate."
130   DISP "Such loading is converted into equivalent nodal fo
      rces."
140   INPUT "Load type?1 for conc.2 for constant distrd.3
      for varying distrd",Type
150   IF Type=1 THEN GOSUB Conc
160   IF Type=2 THEN GOSUB Cdis
170   IF Type=3 THEN GOSUB Vdis
180   BEEP
190   DISP "Press 1 if more loading and 0 to stop loading"
200   INPUT M
210   IF NOT M THEN Printout
220   GOTO 140
230 Conc:  INPUT "Number of nodes with concentrated loads",N
240   FOR I=1 TO N
250   INPUT "Node number?",S1,"Value of load?",Val
260   IF Ndc(S1,4)<>0 THEN GOTO 300
270   BEEP
280   DISP "Made a mistake.Try again"
290   GOTO 250
300   R(Ndc(S1,4))=R(Ndc(S1,4))+Val
310   NEXT I
320   RETURN
330 Cdis: INPUT "Number of elements with loading",Nel
340   IF Nel<=Nelemt THEN GOTO 370
350   BEEP
360   GOTO 330
370   REDIM El(Nel)
380   INPUT "Load per unit area?",P
390   FOR I=1 TO 8
400   P(I)=P
410   NEXT I
420   IF Nel=Nelemt THEN 480
430   DISP "Input elements under pressure one by one.Each time
      press CONT"
440   FOR Z=1 TO Nel
450   INPUT "Element number?",El(Z)
460   NEXT Z
470   GOTO 510
480   FOR Z=1 TO Nelemt
490   El(Z)=Z
500   NEXT Z
510   FOR Z=1 TO Nel
520   El=El(Z)
530   GOSUB Calc
540   MAT Pe=ZER
550   NEXT Z
560   RETURN
570 Vdis: INPUT "Number of elements with loading",Nel
580   FOR Z=1 TO Nel
590   INPUT "Element number?",El
600   FOR I=1 TO 8
```

```
610   DISP "Load intensity at station";I;"?"
620   INPUT P(I)
630   NEXT I
640   GOSUB Calc
650   MAT Re=ZER
660   NEXT Z
670   RETURN
680 Calc:  FOR U=1 TO 9
690   CALL Qaux(W(U,1),W(U,2),X(*),Y(*),Detj,Bm(*),Sf(*),Be,Ds
      ,El,N(*),5)
700   FOR I=1 TO 8
710   FOR J=1 TO 8
720   Re(I)=Re(I)+Sf(I)*Sf(J)*P(J)*Detj*W(U,3)*W(U,4)
730   NEXT J
740   NEXT I
750   NEXT U
760   FOR I=1 TO 8
770   S1=Ndc(N(El,I),4)
780   IF S1=0 THEN GOTO 800
790   R(S1)=R(S1)+Re(I)
800   NEXT I
810   RETURN
820 Printout:  PRINT
830   PRINT "Equivalent nodal forces"
840   PRINT
850   PRINT "Node number";SPA(7);"Applied loads"
860   FOR I=1 TO Nnode
870   IF Ndc(I,4)=0 THEN GOTO 900
880   PRINT USING "3D,15X,MD.4DE";I,R(Ndc(I,4))
890   GOTO 910
900   PRINT USING "3D,15X,MD.4DE";I,0
910   NEXT I
920   SUBEXIT
930   RETURN
940   SUBEND
```

```
10    SUB Heform(He(*),Bm(*),Detj,T1,T2)
20    ! [He] matrix construction.
30    OPTION BASE 1
40    FOR I=1 TO 8
50    FOR J=1 TO 4
60    He(3*I-2,2*J-1)=He(3*I-2,2*J-1)+Bm(1,I)*Bm(1,2*J-1)*Detj
      *T1*T2
70    He(3*I-2,2*J)=He(3*I-2,2*J)+Bm(1,I)*Bm(1,2*J)*Detj*T1*T2
80    He(3*I-1,2*J-1)=He(3*I-1,2*J-1)+Bm(2,I)*Bm(2,2*J-1)*Detj
      *T1*T2
90    He(3*I-1,2*J)=He(3*I-1,2*J)+Bm(2,I)*Bm(2,2*J)*Detj*T1*T2
100   He(3*I,2*J-1)=He(3*I,2*J-1)+(Bm(2,I)*Bm(1,2*J-1)+Bm(1,I)
      *Bm(2,2*J-1))*Detj*T1*T2
110   He(3*I,2*J)=He(3*I,2*J)+(Bm(2,I)*Bm(1,2*J)+Bm(1,I)*Bm(2,
      2*J))*Detj*T1*T2
120   NEXT J
130   NEXT I
140   SUBEND!End of Heform.
150   SUB Geform(Ge(*),A(*),C(*),INTEGER Matno)
160   MAT Ge=ZER
170   FOR I=1 TO 8
180   FOR S=I TO 8
190   A=A(I,S)
200   IF I=S THEN GOTO 290
210   Ge(3*I-2,3*S-2)=A*C(Matno,1)
220   Ge(3*I-2,3*S-1)=A*C(Matno,2)
230   Ge(3*I-1,3*S-2)=A*C(Matno,2)
240   Ge(3*I-1,3*S-1)=A*C(Matno,4)
250   Ge(3*I,3*S-2)=A*C(Matno,3)
260   Ge(3*I,3*S-1)=A*C(Matno,5)
270   Ge(3*I,3*S)=A*C(Matno,6)
280   GOTO 330
290   Ge(3*I-2,3*S-2)=A*C(Matno,1)
300   Ge(3*I-2,3*S-1)=A*C(Matno,2)
310   Ge(3*I-1,3*S-1)=A*C(Matno,4)
320   Ge(3*I,3*S)=A*C(Matno,6)
330   NEXT S
340   NEXT I
350   FOR I=1 TO 24
360   FOR J=I TO 24
370   Ge(J,I)=Ge(I,J)
380   NEXT J
390   NEXT I
400   SUBEND!End of Geform.
410   SUB Mnsws(He(*),X(*),Y(*),Bm(*),Sf(*),Be,Ds,INTEGER N(*)
      ,Z,K)
420   OPTION BASE 1
430   DIM C(24,1),D(1,8),Zn(2,8)
440   A=.577350269
450   FOR I=1 TO 2
460   FOR J=2 TO 7 STEP 5
470   Zn(I,J)=-1
480   NEXT J
490   FOR J=3 TO 6 STEP 3
500   Zn(I,J)=1
510   NEXT J
520   NEXT I
530   Zn(1,1)=Zn(1,4)=Zn(1,5)=Zn(1,8)=-A
540   Zn(2,1)=Zn(2,4)=Zn(2,5)=Zn(2,8)=A
550   FOR K=1 TO 4
560   FOR I=1 TO 2
570   CALL Gaus(Zn(I,2*K-1),Zn(I,2*K),X(*),Y(*),Detj,Bm(*),Sf(
      *),Be,Ds,Z,N(*),K)
580   Be=Be*360/(2*PI)
590   DEG
```

```
600   L1=-COS(Be)*SIN(Be)
610   L2=COS(Be)*SIN(Be)
620   L3=COS(Be)^2-SIN(Be)^2
630   L4=-SIN(Be)
640   L5=COS(Be)
650   FOR S=1 TO 8
660   C(3*S-2,1)=L1*Sf(S)
670   C(3*S-1,1)=L2*Sf(S)
680   C(3*S,1)=L3*Sf(S)
690   NEXT S
700   FOR S=1 TO 8
710   D(1,S)=L4*Bm(1,S)+L5*Bm(2,S)
720   NEXT S
730   IF (K=1) OR (K=3) THEN Ds=-Ds
740   MAT D=(Ds)*D
750   FOR S=1 TO 24
760   FOR J=1 TO 8
770   He(S,J)=He(S,J)+C(S,1)*D(1,J)
780   NEXT J
790   NEXT S
800   NEXT I
810   NEXT K
820   SUBEND!End of Mnsws
830   SUB Ghasemb(K(*),Ke(*),H(*),He(*),INTEGER Fm,Z,N(*),Ndc(*))
840   ! Assembly of coefficient matrices [G] and[H].
850   FOR I=1 TO 8
860   FOR J=1 TO 8
870   FOR V=2 TO 0 STEP -1
880   S1=Ndc(N(Z,I),1)
890   S3=Ndc(N(Z,I),2)
900   S5=Ndc(N(Z,I),3)
910   IF V=2 THEN S2=Ndc(N(Z,J),1)
920   IF V=1 THEN S2=Ndc(N(Z,J),2)
930   IF V=0 THEN S2=Ndc(N(Z,J),3)
940   IF S2=0 THEN L1
950   IF (S1=0) OR (S1<S2) THEN L2
960   K(S1,S2)=K(S1,S2)+Ke(3*I-2,3*J-V)
970 L2: IF S3=0 THEN GOTO L3
980   IF S3<S2 THEN L3
990   k(S3,S2)=K(S3,S2)+Ke(3*I-1,3*J-V)
1000 L3: IF S5=0 THEN L1
1010  IF S5<S2 THEN L1
1020  k(S5,S2)=K(S5,S2)+ke(3*I,3*J-V)
1030 L1: NEXT V
1040  NEXT J
1050  NEXT I
1060  FOR J=1 TO 8! [H] Assembly
1070  S4=Ndc(N(Z,J),4)
1080  IF S4=0 THEN Nexj
1090  FOR I=1 TO 3
1100  FOR V=0 TO 2
1110  S1=Ndc(N(Z,I),V+1)
1120  IF S1=0 THEN Nexv
1130  H(S1,S4)=H(S1,S4)+He(3*I-2+V,J)
1140 Nexv:NEXT V
1150  NEXT I
1160 Nexj:NEXT J
1170  SUBEND!End ofGhasemb
```

```
10    SUB Transf(Ge(*),He(*),Be,W)
20    OPTION BASE 1
30    DIM B(3,3)
40    GOSUB Cosd
50    FOR I=1 TO 24
60    FOR J=I TO 24
70    Ge(J,I)=Ge(I,J)
80    NEXT J
90    NEXT I
100   GOTO 300
110 Cosd: ! Transformation of coordinates.
120   DEG
130   B(1,1)=B(2,2)=SIN(Be)^2
140   B(1,2)=B(2,1)=COS(Be)^2
150   B(1,3)=-2*SIN(Be)*COS(Be)
160   B(2,3)=-B(1,3)
170   B(3,1)=B(1,3)/2
180   B(3,2)=-B(3,1)
190   B(3,3)=COS(Be)^2-SIN(Be)^2
200   MAT B=INV(B)
210   FOR T=1 TO W-1 STEP 1
220   CALL Matmult2(T,W,Ge(*),B(*))
230   NEXT T
240   FOR V=W+1 TO 8 STEP 1
250   CALL Matmult1(W,V,Ge(*),B(*))
260   NEXT V
270   CALL Matmult(W,W,Ge(*),B(*))
280   CALL Matmult3(W,1,He(*),B(*))
290   RETURN
300   SUBEND
310   SUB Matmult1(T,V,Ge(*),B(*))
320   OPTION BASE 1
330   DIM C(3,3)
340   REM [B]t*[C]
350   C(1,1)=Ge(3*T-2,3*V-2)
360   C(1,2)=Ge(3*T-2,3*V-1)
370   C(1,3)=Ge(3*T-2,3*V)
380   C(2,1)=Ge(3*T-1,3*V-2)
390   C(2,2)=Ge(3*T-1,3*V-1)
400   C(2,3)=Ge(3*T-1,3*V)
410   C(3,1)=Ge(3*T,3*V-2)
420   C(3,2)=Ge(3*T,3*V-1)
430   C(3,3)=Ge(3*T,3*V)
440   FOR I=1 TO 3
450   FOR J=1 TO 3
460   A=0
470   FOR R=1 TO 3
480   Ge(3*T-3+I,3*V-3+J)=A+B(R,I)*C(R,J)
490   A=Ge(3*T-3+I,3*V-3+J)
500   NEXT R
510   NEXT J
520   NEXT I
530   SUBEND!END OF Matmult1
540   SUB Matmult2(T,V,Ge(*),B(*))
550   OPTION BASE 1
560   DIM C(3,3)
570   REM [C]*[B]
580   C(1,1)=Ge(3*T-2,3*V-2)
590   C(1,2)=Ge(3*T-2,3*V-1)
600   C(1,3)=Ge(3*T-2,3*V)
610   C(2,1)=Ge(3*T-1,3*V-2)
620   C(2,2)=Ge(3*T-1,3*V-1)
630   C(2,3)=Ge(3*T-1,3*V)
640   C(3,1)=Ge(3*T,3*V-2)
650   C(3,2)=Ge(3*T,3*V-1)
```

```
660   C(3,3)=Ge(3*T,3*V)
670   FOR I=1 TO 3
680   FOR J=1 TO 3
690   A=0
700   FOR R=1 TO 3
710   Ge(3*T-3+I,3*V-3+J)=A+C(I,R)*B(R,J)
720   A=Ge(3*T-3+I,3*V-3+J)
730   NEXT R
740   NEXT J
750   NEXT I
760   SUBEND!End of Matmult2
770   SUB Matmult(T,V,Ge(*),B(*))
780   OPTION BASE 1
790   DIM C(3,3)
800   REM [B]t*[C]*[B]
810   C(1,1)=Ge(3*T-2,3*V-2)
820   C(1,2)=Ge(3*T-2,3*V-1)
830   C(1,3)=Ge(3*T-2,3*V)
840   C(2,2)=Ge(3*T-1,3*V-1)
850   C(2,3)=Ge(3*T-1,3*V)
860   C(3,3)=Ge(3*T,3*V)
870   FOR I=1 TO 3
880   FOR J=I TO 3
890   C(J,I)=C(I,J)
900   NEXT J
910   NEXT I
920   FOR R=1 TO 3
930   FOR S=1 TO 3
940   A=0
950   FOR I=1 TO 3
960   FOR J=1 TO 3
970   Ge(3*T-3+R,3*V-3+S)=A+B(I,R)*C(I,J)*B(J,S)
980   A=Ge(3*T-3+R,3*V-3+S)
990   NEXT J
1000  NEXT I
1010  NEXT S
1020  NEXT R
1030  SUBEND!End of Matmult
1040  SUB Matmult3(T,V,He(*),B(*))
1050  OPTION BASE 1
1060  DIM C(3,8),V(3)
1070  REM [B]t*[He]
1080  V(1)=2
1090  V(2)=1
1100  V(3)=0
1110  FOR I=1 TO 3
1120  FOR J=1 TO 8
1130  C(I,J)=He(3*T-V(I),J)
1140  NEXT J
1150  NEXT I
1160  FOR I=1 TO 3
1170  FOR J=1 TO 8
1180  A=0
1190  FOR R=1 TO 3
1200  He(3*T-3+I,J)=A+B(R,I)*C(R,J)
1210  A=He(3*T-3+I,J)
1220  NEXT R
1230  NEXT J
1240  NEXT I
1250  SUBEND !End of Matmult3
```

```
10     SUB Meform(D(*),Th(*),Me(*),X(*),Y(*),W(*),Detj,Sf(*),Bm
       (*),INTEGER N(*),Matno,Z)
20     REM Determination of consistant mass matrix for
30     !   plate element.
40     FOR U=1 TO 9
50     CALL Gauss(W(U,1),W(U,2),X(*),Y(*),Detj,Bm(*),Sf(*),Be,Ds
       ,Z,N(*),5)
60     FOR I=1 TO 8
70     FOR J=I TO 8
80     Me(I,J)=Me(I,J)+Sf(I)*Sf(J)*Detj*W(U,3)*W(U,4)
90     NEXT J
100    NEXT I
110    NEXT U
120    MAT Me=(D(Matno)*Th(Matno))*Me
130    FOR I=1 TO 8
140    FOR J=I TO 8
150    Me(J,I)=Me(I,J)
160    NEXT J
170    NEXT I
180    SUBEND!End of Meform.
190    SUB Masemb(M(*),Me(*),INTEGER Fw,Z,N(*),Ndc(*))
200    ! Mass matrix assembly
210    FOR I=1 TO 8
220    S1=Ndc(N(Z,I),4)
230    IF S1=0 THEN GOTO 290
240    FOR J=1 TO 8
250    S2=Ndc(N(Z,J),4)
260    IF (S2=0) OR (S1>S2) THEN GOTO 280
270    M(S1,S2)=M(S1,S2)+Me(I,J)
280    NEXT J
290    NEXT I
300    SUBEND !End of Masemb.
```

```
10    SUB Dampmat(C(*),Vec(*),Eval(*),M(*),K(*),Zeta(*),D(*),O
      ffd(*),Offd2(*),D1(*),INTEGER P,N,Type,Sol)
20    REM Evaluation of a full damping matrix with
30    !   known damping ratios.
40    OPTION BASE 1
50    DIM Theta(20)
60    REDIM Theta(N)
70    PRINT "Evaluation of normal modes of vibration "
80    PRINT "----------------------------------------"
90    M1=1
100   M2=P
110   Sol=2
120   LINK "TRANS:F",9200
130   CALL Trans(M(*),K(*),Vec(*),Eval(*),Zeta(*),M1,M2,Lb,Ub,
      D(*),Offd(*),Offd2(*),D1(*),Type,N,Sol,P)
140   LINK "EIGEN:F",9200
150   CALL Eigen(M(*),K(*),Vec(*),Eval(*),Zeta(*),M1,M2,Lb,Ub,
      D(*),Offd(*),Offd2(*),D1(*),Type,N,Sol,P)
160   FOR R=1 TO P
170   Mr=0
180   ! Finds Mass of mode r
190   Br=2*Zeta(R)*SQR(Eval(R))
200   Vec=0
210   MAT Theta=ZER
220   FOR I=1 TO N
230   FOR J=1 TO N
240   Theta(I)=M(I,J)*Vec(J,R)+Theta(I)
250   NEXT J
260   NEXT I
270   FOR I=1 TO N
280   FOR J=I TO N
290   C(I,J)=Br*Theta(I)*Theta(J)+C(I,J)
300   C(J,I)=C(I,J)
310   NEXT J
320   NEXT I
330   NEXT R
340   MAT Offd=ZER
350   MAT Offd2=ZER
360   MAT D=ZER
370   MAT Vec=ZER
380   MAT D1=ZER
390   SUBEND
400   SUB Eqsolv(H(*),A(*),K(*),P(*),INTEGER R,N)
410   OPTION BASE 1
420   DIM B(89,10)
430   REDIM B(N,R)
440   MAT B=H
450   D1=1
460   D2=0
470   FOR I=1 TO N
480   FOR J=1 TO N
490   X=A(I,J)
500   FOR K=I-1 TO 1 STEP -1
510   X=X-A(J,K)*A(I,K)
520   NEXT K
530   IF J<>I THEN 680
540   D1=D1*X
550   IF X<>0 THEN L1
560   D2=0
570   GOTO Fail
580 L1:  IF ABS(D1)<1 THEN L2
590   D1=D1*.0625
600   D2=D2+4
610   GOTO L1
620 L2:  IF ABS(D1)>=.0625 THEN 660
```

```
630   D1=D1+16
640   D2=D2-4
650  GOTO L2
660   IF X<0 THEN GOTO Fail
670   P(I)=1/SQR(X)
680   IF J<>I THEN A(J,I)=X*P(I)
690   NEXT J
700   NEXT I
710   FOR J=1 TO R
720   REM SOLUTION OF LY=B
730   FOR I=1 TO N
740   Z=B(I,J)
750   FOR K=I-1 TO 1 STEP -1
760   Z=Z-A(I,K)*B(K,J)
770   NEXT K
780   B(I,J)=Z*P(I)
790   NEXT I
800   REM SOLUTION OF UX=Y
810   FOR I=N TO 1 STEP -1
820   Z=B(I,J)
830   FOR K=I+1 TO N
840   Z=Z-A(K,I)*B(K,J)
850   NEXT K
860   B(I,J)=Z*P(I)
870   NEXT I
880   NEXT J
890   FOR I=1 TO R
900   FOR J=1 TO R
910   A=0
920   FOR S=1 TO N
930   K(I,J)=A+H(S,I)*B(S,J)
940   A=K(I,J)
950   NEXT S
960   NEXT J
970   NEXT I
980   MAT H=B
990   GOTO 1020
1000 Fail: DISP "PROGRAM FAILED IN EQSOLV SUBPROGRAM. COMPUTATION
     STOPED"
1010  STOP
1020  SUBEND
```

```
PROGRAMME STORED IN FILE :    FINITL           Page   1
                LISTED ON :    17/6/83


10     SUB Eqsolv1(A(*),P(*),INTEGER R,N)
20     OPTION BASE 1
30     MAT P=ZER
40     REM  [A]=[L]*[U] Triangularization of [A]
50     D1=1
60     D2=0
70     FOR I=1 TO N
80     FOR J=1 TO N
90     X=A(I,J)
100    FOR K=I-1 TO 1 STEP -1
110    X=X-A(J,K)*A(I,K)
120    NEXT K
130    IF J<>I THEN 280
140    D1=D1*X
150    IF X<>0 THEN L1
160    D2=0
170    GOTO Fail
180 L1:  IF ABS(D1)<1 THEN L2
190    D1=D1*.0625
200    D2=D2+4
210    GOTO L1
220 L2:  IF ABS(D1)>=.0625 THEN 260
230    D1=D1*16
240    D2=D2-4
250   GOTO L2
260    IF X<0 THEN GOTO Fail
270    P(I)=1/SQR(X)
280    IF J<>I THEN A(J,I)=X*P(I)
290    NEXT J
300    NEXT I
310    GOTO 340
320 Fail:  DISP "PROGRAM FAILED IN EQSOLV SUBPROGRAM. COMPUTATION
    STOPED"
330    STOP
340    SUBEND
350    SUB Eqsolv2(B(*),A(*),P(*),INTEGER R,N)
360    OPTION BASE 1
370          FOR J=1 TO R
380    REM SOLUTION OF LY=B
390    FOR I=1 TO N
400    Z=B(I,J)
410    FOR K=I-1 TO 1 STEP -1
420    Z=Z-A(I,K)*B(K,J)
430    NEXT K
440    B(I,J)=Z*P(I)
450    NEXT I
460    REM Solution of UX=Y
470    FOR I=N TO 1 STEP -1
480    Z=B(I,J)
490    FOR K=I+1 TO N
500    Z=Z-A(K,I)*B(K,J)
510    NEXT K
520    B(I,J)=Z*P(I)
530    NEXT I
540    NEXT J
550    SUBEND
560    SUB Initial(D(*),Apfo(*),F0(*),K(*),C(*),Offd2(*),Offd(*
    ),M(*),P(*),Delta,INTEGER R,N,Neq,#1)
570    OPTION BASE 1
580    ! This subprogram evaluates the acceleration vector
590    ! for different forcing functions.
600    READ #1,1
610    MAT READ #1;D
620    MAT READ #1;Offd
630    LINK "Eqn",9200,640
```

```
640    FOR K1=1 TO Neq
650    T=0
660    CALL Eqn(T,F,K1)
670    FOR I=1 TO N
680    Apfo(I,1)=F*F0(I)
690    NEXT I
700    FOR I=1 TO N
710    FOR J=1 TO N
720    Offd2(I,1)=-K(I,J)*D(J)-C(I,J)*Offd(J)+Apfo(I,1)
730    NEXT J
740    Apfo(I,1)=Offd2(I,1)
750    NEXT I
760    CALL Eqsolv2(Offd2(*),M(*),P(*),1,N)
770    MAT PRINT #1;Offd2
780    NEXT K1
790    ASSIGN * TO #1
800    SUBEND
810    SUB Eqn(T,F,K1)
820    OPTION BASE 1
830    ON K1 GOTO L1,L2,L3,L4,L5
```

```
PROGRAMME STORED IN FILE :    WILSON          Page   1
                LISTED ON :   17/6/83


10    SUB Wilsnsol(K(*),M(*),C(*),Apfo(*),F0(*),D(*),D1(*),D2(
      *),D1(*),A0(*),Tm,De,Th,K1,P(*),H(*),Wcnt(*),Mcnt(*),INT
      EGER Ndc(*),Mp,Wp,N,R,J2)
20    ! Direct numerical integration by Wilson theta.
30    OPTION BASE 1
40    REM Effective K,M,C matrices
50    ! [K]=[K]+A0*[M]+A1*[C]
60    FOR I=1 TO N
70    FOR J=I TO N
80    K(I,J)=K(I,J)+A0(1)*M(I,J)+A0(2)*C(I,J)
90    K(J,I)=K(I,J)
100   NEXT J
110   NEXT I
120   REM Matrix K the effective stiffness matrix is
130   ! triangularized.
140   CALL Eqsolv1(K(*),D1(*),R,N)
150   REM Loop round the integration points
160   Cnt=1
170   T=0
180   Npts=INT(Tm/De)+1
190   FOR Count=1 TO Npts-1
200   CALL Eqn(T,F,K1)
210   Cnt=Cnt+1
220   FOR I=1 TO N
230   Apfo(I,1)=F*F0(I)
240   NEXT I
250   T=T+De
260   CALL Eqn(T,F,K1)
270   FOR I=1 TO N
280   Apfo(I,1)=(1-Th)*Apfo(I,1)+Th*F*F0(I)
290   NEXT I
300   FOR I=1 TO N
310   FOR J=1 TO N
320   Apfo(I,1)=(A0(1)*M(I,J)+A0(2)*C(I,J))*D(J)+(A0(3)*M(I,J)
      +2*C(I,J))*D1(J)+(2*M(I,J)+A0(4)*C(I,J))*D2(J,1)+Apfo(I,
      1)
330   NEXT J
340   NEXT I
350   CALL Eqsolv2(Apfo(*),K(*),D1(*),1,N)
360   FOR I=1 TO N
370   Apfo(I,1)=A0(5)*(Apfo(I,1)-D(I))+A0(6)*D1(I)+A0(7)*D2(I,1)
380   NEXT I
390   FOR I=1 TO N
400   D(I)=D(I)+De*D1(I)+A0(9)*(Apfo(I,1)+2*D2(I,1))
410   NEXT I
420   FOR I=1 TO N
430   D1(I)=D1(I)+A0(8)*(Apfo(I,1)+D2(I,1))
440   NEXT I
450   MAT D2=Apfo
460   Pi=0
470   ! Calculation of bending moment for time T.
480   FOR J=1 TO N
490   P(Ndc(Mp,J2))=Pi+H(Ndc(Mp,J2),J)*D(J)
500   Pi=P(Ndc(Mp,J2))
510   NEXT J
520   Wcnt(Cnt)=D(Ndc(Wp,4))
530   Mcnt(Cnt)=P(Ndc(Mp,J2))
540   NEXT Count
550   SUBEND
```

PROGRAMME STORED IN FILE : DUHML Page 1
LISTED ON : 17/6/83

```
10    SUB Duhammel(T,Nf,Dnf,Ze,Delta,F0,Abar,Bbar,Abar1,Bbar1,
      Y,Ynlt,Dynlt,In(*),K1)
20    OPTION BASE 1
30    REM Evaluates Duhammel integral by trapezoidal rule.
40    IF (T<>0) AND (T<>Delta) THEN GOTO 120
50    IF T<>0 THEN GOTO 90
60    Abar=0
70    Abar1=F0
80    GOTO 140
90    Abar=(Abar+Abar1)*EXP(-Ze*Nf*Delta)+F0*COS(Dnf*Delta)
100   Abar1=F0*COS(Dnf*Delta)
110   GOTO 140
120   Abar=(Abar+Abar1)*EXP(-Ze*Nf*Delta)+F0*COS(Dnf*T)
130   Abar1=F0*COS(Dnf*T)
140   IF (T<>0) AND (T<>Delta) THEN GOTO 210
150   IF T<>0 THEN GOTO 180
160   Bbar=0
170   GOTO 230
180   Bbar=(Bbar+Bbar1)*EXP(-Ze*Nf*Delta)+F0*SIN(Dnf*Delta)
190   Bbar1=F0*SIN(Dnf*Delta)
200   GOTO 230
210   Bbar=(Bbar+Bbar1)*EXP(-Ze*Nf*Delta)+F0*SIN(Dnf*T)
220   Bbar1=F0*SIN(Dnf*T)
230   Y=Delta/2*(Abar*SIN(Dnf*T)-Bbar*COS(Dnf*T))
240   IF In(K1)=0 THEN SUBEXIT
250   Y0=(Dynlt+Ynlt*Ze*Nf)/Dnf*SIN(Dnf*T)+Ynlt*COS(Dnf*T)
260   Y0=Y0*EXP(-Ze*Nf*T)
270   Y=Y0+Y
280   SUBEND
290   SUB Modal(Vec(*),Eval(*),M(*),K(*),Zeta(*),D(*),Offd(*),
      Offd2(*),D1(*),INTEGER P,N,Type,Sol,Ndm)
300   REM Evaluation of system modal characteristics.
310   OPTION BASE 1
320   DIM Ar(16,16),Cr(16,16)
330   REDIM Ar(N,N),Cr(N,N)
340   PRINT "--------------------------------------------------"
350   Sol=1
360   M1=1
370   M2=P
380   LINK "TRANS:F",15570
390   CALL Trans(M(*),K(*),Vec(*),Eval(*),Zeta(*),M1,M2,Lb,Ub,
      D(*),Offd(*),Offd2(*),D1(*),Ar(*),Cr(*),Type,N,Sol,Ndm)
400   LINK "EIGEN:F",15570
410   CALL Eigen(M(*),K(*),Vec(*),Eval(*),Zeta(*),M1,M2,Lb,Ub,
      D(*),Offd(*),Offd2(*),D1(*),Ar(*),Cr(*),Type,N,Sol,Ndm)
420   MAT Offd=ZER
430   MAT Offd2=ZER
440   MAT D=ZER
450   MAT D1=ZER
460   SUBEND
```

```
10    SUB Plot(Drw(*),C$,Time,Delta,INTEGER Nd1)
20    OPTION BASE 1
30    REM This sub program plots the graphs for
40    !   displacements and stresses against time.
50    BEEP
60    DISP "Choose the plotter.5 for incremental,13 for CR
      T.7 for 9827A"
70    INPUT Pltr
80    ! Finds Max & Min of Drw(*)
90    Max=Min=Drw(1)
100   FOR I=2 TO Time/Delta
110   IF Drw(I)>Max THEN Max=Drw(I)
120   IF Drw(I)<Min THEN Min=Drw(I)
130   NEXT I
140   IF ABS(Max)>ABS(Min) THEN PRINT USING "K,4X,MD.4DE";"Max
      imum response is:",Max
150   IF ABS(Min)>ABS(Max) THEN PRINT USING "K,4X,MD.4DE";"Max
      imum response is:",Min
160   IF Pltr=13 THEN GOSUB Plt2
170   IF Pltr=5 THEN GOSUB Plt1
180   IF Pltr=7 THEN GOSUB Plt3
190   GOTO 340
200 Plt1:  DISP "Set the plotter then press CONT"
210   PAUSE
220   PLOTTER IS 5,"INCREMENTAL"
230   LIMIT 20,920,10,600
240   RETURN
250 Plt2:  PLOTTER IS 13,"GRAPHICS"
260   DISP "Do you need a hard copy?"
270   INPUT Dump
280   GRAPHICS
290   RETURN
300 Plt3:  DISP "Set the plotter then press CONT"
310   PAUSE
320   PLOTTER IS 7,5,"9872A"
330   RETURN
340   FRAME
350   LOCATE 11,105,6,95
360   MOVE 4,29
370   CSIZE 2,1
380   LDIR PI/2
390   LABEL "";C$;" AT NODE";Nd1;""
400   LDIR 0
410   Xmax=Time
420   CALL Axes(0,Xmax,Min,Max,1)
430   MOVE 0,Drw(1)
440   FOR I=2 TO INT(Time/Delta)
450   DRAW Delta*(I-1),Drw(I)
460   NEXT I
470   IF Dump THEN DUMP GRAPHICS
480   GCLEAR
490   EXIT GRAPHICS
500   SUBEND
510   SUB Axes(Xmin,Xmax,Ymin,Ymax,Scay)
520   OPTION BASE 1
530   X3=(Xmax-Xmin)/8
540   Y3=(Ymax-Ymin)/8
550   Jx=5*10^(INT(LGT(X3))-1)
560   Jy=5*10^(INT(LGT(Y3))-1)
570   X3=J *INT(X3/Jx+.5)
580   Y3=Jy*INT(Y3/Jy+.5)
590   Sxmin=X3*INT(Xmin/X3)
600   Symin=Y3*INT(Ymin/Y3)
610   Sxmax=-X3*INT(-Xmax/X3)
620   Symax=-Y3*INT(-Ymax/Y3)
```

```
630   X4=(Sxmax-Sxmin)/17
640   Y4=(Symax-Symin)/17
650   SCALE Sxmin-X4,Sxmax+X4,Symin-Y4,Symax+Y4
660   Xint=Yint=0
670   IF SGN(Sxmax)*SGN(Sxmin)>0 THEN Xint=Sxmin
680   IF SGN(Symax)*SGN(Symin)>0 THEN Yint=Symin
690   AXES X3,Y3,Xint,Yint,2,2,Scay
700   CSIZE 2.5*Scay
710   LDIR ATN(0)
720   LORG 6
730   IF Sxmax<=0 THEN 760
740   P=INT(LGT(Sxmax))
750   GOTO 770
760   P=INT(LGT(-Sxmin))
770   J=0
780   IF (P<-1) OR (P>2) THEN J=1
790   FOR Lx=Sxmin TO Sxmax STEP X3
800   IF Lx=Xint THEN Next
810   MOVE Lx,Yint-Y3*.1
820   LABEL USING "K";" "&VAL$(Lx/10^(J*P))&" "
830 Next:  NEXT Lx
840   IF J=0 THEN Skipx
850   MOVE Sxmax+X3/2,Yint-Y3*.1
860   LORG 3
870   LABEL USING "K";" x 10^",P
880 Skipx:  LDIR 0
890   LORG 8
900   IF Symax<=0 THEN 930
910   P=INT(LGT(Symax))
920   GOTO 940
930   P=INT(LGT(-Symin))
940   J=0
950   IF (P<-1) OR (P>2) THEN J=1
960   LORG 2
970   MOVE Sxmax,Yint-.8*Y3
980   LABEL "TIME(Sec)"
990   LORG 8
1000  FOR Ly=Symin TO Symax STEP Y3
1010  IF Ly=Yint THEN 1040
1020  MOVE Xint,Ly
1030  LABEL USING "K";VAL$(Ly/10^(J*P))&" "
1040  NEXT Ly
1050  IF J=0 THEN Skip
1060  LORG 2
1070  MOVE Xint,Symax
1080  LABEL USING "K";" x 10^",P
1090 Skip: PENUP
1100  CSIZE 3.3*Scay
1110  Xmin=Symin
1120  Ymin=Symin
1130  Xmax=Sxmax
1140  Ymax=Symax
1150  SUBEND
```